

Topology of the regular part for infinitely renormalizable quadratic polynomials

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Abstract

In this paper we describe the well studied process of renormalization of quadratic polynomials from the point of view of their natural extensions. In particular, we describe the topology of the inverse limit of infinitely renormalizable quadratic polynomials and prove that when they satisfy *a priori* bounds, the topology is rigid modulo combinatorial equivalence.

1 Introduction and basic theory

There are some dynamical systems that can be interpreted by geometric objects. Sullivan constructed a lamination by Riemann surfaces associated to expanding maps on the circle, by using its inverse limit [20]. These laminations play a crucial role in his proof of the universality of infinitely renormalizable unimodal maps. Later on in [16], Lyubich and Minsky generalized this construction to every rational map on the sphere. They proved a rigidity theorem for critically non-recurrent rational maps without parabolic cycles, by means of hyperbolic 3-laminations associated to these maps.

For a given rational map, Lyubich and Minsky's construction of the hyperbolic 3-lamination is essentially based on the construction of a Riemann surface lamination associated to the map. In their setting, they also used inverse limits. However, the construction of the Riemann surface lamination is more involved than Sullivan's, since the presence of critical orbits forces to exclude some "irregular points" from the inverse limit. The rest is called the *regular part* (or *regular leaf space*), which has nice analytic properties. Indeed, we may regard the regular part as the desired Riemann surface lamination when the critical orbits behave nicely.

Part of the program presented by Lyubich and Minsky was to investigate the properties of the regular part of quadratic maps $f_c(z) = z^2 + c$ ([16, §10]). Here are two possible problems:

- Describe the topological structure of the regular part for a quadratic map; and
- Classify the quadratic maps by the topologies of their regular parts.

The authors dealt with these problems when f_c has a (super)attracting or parabolic cycle and perhaps we have reasonable solutions [2, 3, 9, 10]. However, needless to say, the most important class to consider is the family of infinitely renormalizable quadratic maps.

Main results. Let us roughly summarize the main results of this paper. (The precise definitions and statements will be given later.) Let f_c be an infinitely renormalizable quadratic map. Such a map uniquely determines an invariant called the *combinatorics*, which is represented by a sequence of parameters $\{s_0, s_1, \dots\}$ with superattracting f_{s_i} . We say f_c has *a priori bounds* when each level of renormalization is separated by an annulus of definite modulus.

Under the assumption of *a priori bounds*, the regular part of f_c is a lamination under the topology induced from its inverse limit. Our results reveals the relations between the topology of the regular part \mathcal{R}_c and the combinatorics $\{s_0, s_1, \dots\}$ of f_c with *a priori bounds*. The first theorem is:

Structure Theorem. *The regular part \mathcal{R}_c of f_c with a priori bounds is decomposed into blocks that are homeomorphic to the regular part of f_{s_i} for s_i in the combinatorics of f_c . In particular, the configuration of the blocks perfectly reflect the nest of the renormalizations.*

Next we consider the classification of such an f_c by the topology of regular parts. We will prove:

Main Theorem. *If two non-real infinitely renormalizable maps f_c and $f_{c'}$ with a priori bounds admit an orientation preserving homeomorphism between their regular parts, then they have the same combinatorics. Moreover, if the Mandelbrot set is locally connected at the parameter c , we have $c = c'$.*

Thus the topology of the regular part of the infinitely renormalizable map with *a priori bounds* has rigidity up to combinatorial equivalence. In particular, it may even determine the original dynamics if the Mandelbrot set is locally connected. (cf. Mostow's rigidity for finite volume complete hyperbolic manifolds and corresponding Kleinian groups.) The assumption of 'non-real' is a technical condition for the proof, which we believe is unnecessary. On the other hand, the 'orientation-preserving' condition excludes trivial homeomorphisms that come from complex conjugation.

Outline of the paper. In the rest of this section we give a brief introduction to the theory of dynamics of quadratic maps and their renormalizations. In Section 2, we review the definition of the inverse limits and the regular parts generated by quadratic maps. Section 3 is devoted to the statement and the proof of the Structure Theorem (Theorem 4). Finally in Section 4, we prove the Main Theorem (Theorem 7) stated as above.

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1.1 Preliminaries

We start with notation on the dynamics of quadratic maps. Readers may refer to [4] and [13] for basic definitions on dynamics of quadratic maps.

Julia/Fatou sets, and external rays. For a quadratic map $f_c(z) = z^2 + c$ on the Riemann sphere $\overline{\mathbb{C}}$, we denote the *Julia set* by $J(f_c)$, and the *Fatou set* by $F(f_c)$. The *filled Julia set* is denoted by $K(f_c)$. The closure of the forward orbit of 0, denoted by $P(f_c)$, is called the *postcritical set*.

Throughout this paper we assume that $K(f_c)$ and $J(f_c)$ are both connected, i.e., the postcritical set $P(f_c)$ is bounded. Let $A_c := \overline{\mathbb{C}} \setminus K(f_c)$ denote the *basin of infinity* of f_c . The *Böttcher coordinate* $\psi_c : A_c \rightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is a unique Riemann map with $\psi_c(f_c(z)) = \psi_c(z)^2$.

For $r > 1$, the set $E_c(r) := \psi_c^{-1}(\{w \in \mathbb{C} : |w| = r\})$ is called the *equipotential curve* of level r . For $\theta \in \mathbb{R}/\mathbb{Z}$, the set $R_c(\theta) := \psi_c^{-1}(\{w \in \mathbb{C} \setminus \overline{\mathbb{D}} : \arg w = \theta\})$ is called the *external ray* of angle θ .

Ray portrait. (See Milnor's [19, §6].) Let $O = \{p_1, \dots, p_m\}$ be a repelling cycle of f_c . Let $\Theta(p_i)$ denote the set of angles of external rays landing at p_i . The collection $\text{rp}(O) = \{\Theta(p_1), \dots, \Theta(p_m)\}$ is called the *ray portrait* of O . A ray portrait is called *non-trivial*, if there are at least two rays landing at every point in O .

Superattracting quadratic maps. When $f_s(z) = z^2 + s$ has a cycle containing the critical point $z = 0$, we say s is a *superattracting parameter*. Let $\{\alpha_s(1), \dots, \alpha_s(m) = 0\}$ denote the superattracting cycle with $f_s(\alpha_s(i)) = \alpha_s(i+1)$, where we take indexes modulo m . Let D_s be the Fatou component containing 0. It is known that the dynamics of $f_s^m : \overline{D}_s \rightarrow \overline{D}_s$ is conjugate to $f_0 : \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$. Let $\Psi_s : \overline{D}_s \rightarrow \overline{\mathbb{D}}$ be this conjugacy. The *internal equipotential* $I_s(r)$ of level $r < 1$ is defined by $\Psi_s^{-1}(\{|w| = r\})$. We also denote $\Psi_s^{-1}(\{|w| < r\})$ by $D_s(r)$.

Let O_s be the repelling cycle in $\bigcup_{1 \leq i \leq m} f_s^i(\partial D_s)$ that is the orbit of the pull-back of $1 \in \partial \overline{\mathbb{D}}$ by Ψ_s . It is known that if $m \geq 2$, the ray portrait $\text{rp}(O_s)$ is non-trivial. Indeed, a result due to Milnor (see [19]) states that the map f_s is uniquely determined by $\text{rp}(O_s)$. Following Milnor's terminology, we call $\text{rp}(O_s)$ the *characteristic ray portrait* of f_s .

Quadratic-like maps. (See [13, 17, 18].) Let $g : U \rightarrow V$ be a *quadratic-like map*. Throughout this paper we will only consider quadratic-like maps $g : U \rightarrow V$ with connected *filled Julia set* $K(g)$. Let $J(g)$ and $P(g)$ denote the *Julia set* and the *postcritical set* of g . The β -*fixed point* of g is denoted by $\beta(g)$.

By Douady-Hubbard's Straightening Theorem [4], there exists a unique $c = c(g) \in \mathbb{C}$ and a quasiconformal map $h : V \rightarrow V'$ such that h conjugates $g : U \rightarrow V$ to $f_c : f_c^{-1}(V') \rightarrow V'$ where $\overline{\partial}h = 0$ a.e. on $K(g)$. The quadratic map f_c is called the *straightening* of g and h is called a *straightening map*. Although such h is not uniquely determined, we always assume that any quadratic-like map g comes together with one fixed straightening map $h = h_g$.

One can always find $r_g > 1$ such that if r satisfies $1 < r \leq r_g$ and $\theta \in \mathbb{R}/\mathbb{Z}$, the pulled-back equipotential

$$E_g(r) := h^{-1}(E_c(r))$$

and the external ray

$$R_g(\theta) := h^{-1}(\{\rho e^{2\pi i \theta} : 1 < \rho \leq r_g\})$$

are well-defined.

Renormalization of quadratic maps A quadratic-like map $g : U \rightarrow V$ is said to be *renormalizable*, if there exist a number $m > 1$, called the *order of renormalization*, and two open sets $U_1 \subset U$ and $V_1 \subset V$ containing the critical point of g , such that $g_1 = g^m|_{U_1} : U_1 \rightarrow V_1$ is again a quadratic-like map with connected Julia set $K(g_1)$. We say $g_1 : U_1 \rightarrow V_1$ is a *renormalization* of $g : U \rightarrow V$. We call $K_1 := K(g_1), g(K_1), \dots, g^{m-1}(K_1)$ the *little Julia sets*. We also assume that m is the minimal order with this property and that the renormalization is *non-crossing*. (See [17] or [18].)

Infinitely renormalizable maps. (See [15].) In this paper we only deal with quadratic-like maps which are restrictions of some iterated quadratic map. Given a quadratic map f_c and $r > 1$, let $U_c(r)$ denote the topological disk in \mathbb{C} enclosed by $E_c(r)$, then the restriction $f_c|_{U_c(\sqrt{r})} : U_c(\sqrt{r}) \rightarrow U_c(r)$ is a quadratic-like map. Set $g_0 = f_c, U_0 := U_c(\sqrt{r})$ and $V_0 := U_c(r)$. We say f_c is *infinitely renormalizable* if there exist an infinite sequence of numbers $1 = p_0 < p_1 < p_2 < \dots$ and two sequences of open sets $\{U_n\}$ and $\{V_n\}$ such that each $g_n = f_c^{p_n} : U_n \rightarrow V_n$ is a quadratic-like map, with the property that g_{n+1} is a renormalization of g_n of order $m_n := p_{n+1}/p_n > 1$. The index n of g_n is called the *level* of renormalization.

Combinatorics of renormalizable maps. (See Lyubich's [14] and [15].) From now on, f_c will denote an infinitely renormalizable quadratic map with renormalizations $\{g_n : U_n \rightarrow V_n\}$ as above. In order to describe the combinatorics of f_c , first we observe that the orbit under g_n of the β -fixed point of g_{n+1} is a repelling cycle O_n of g_n .

By our non-crossing assumption, the ray portrait $\text{rp}(h_n(O_n))$ is non-trivial and it determines a unique superattracting quadratic map $f_{s_n}(z) = z^2 + s_n$ with characteristic ray portrait $\text{rp}(h_n(O_n))$. We call the infinite sequence of superattracting parameters $\{s_0, s_1, s_2, \dots\}$ the *combinatorics* of f_c , this is a well-defined invariant if we take m_n minimal for each n .

A priori bounds. An infinitely renormalizable f_c is said to have *a priori bounds* if there exists $\epsilon > 0$ such that $\text{mod}(V_n \setminus U_n) > \epsilon$ for all $n \geq 0$. As we will see later, this condition insures nice properties of $P(f_c)$.

2 Inverse limits and regular parts

In this section we consider the inverse limit of a quadratic polynomial and define its *regular part*. In some sense, the regular part is the analytically well-behaved subspace of the inverse limit of a quadratic polynomial. Readers may refer to [16] and [8] for more details on the objects defined here.

2.1 Inverse limits and solenoidal cones

Inverse Limits. Consider $\{f_{-n} : X_{-n} \rightarrow X_{-n+1}\}_{n=1}^{\infty}$, a sequence of d -to-1 branched covering maps on the manifolds X_{-n} with the same dimension. The *inverse limit* of this sequence is defined as

$$\varprojlim(f_{-n}, X_{-n}) := \{\hat{x} = (x_0, x_{-1}, x_{-2}, \dots) \in \prod_{n \geq 0} X_{-n} : f_{-n}(x_{-n}) = x_{-n+1}\}.$$

The space $\varprojlim(f_{-n}, X_{-n})$ has a *natural topology* which is induced from the product topology in $\prod_{n \geq 0} X_{-n}$. The projection $\pi : \varprojlim(f_{-n}, X_{-n}) \rightarrow X_0$ is defined by $\pi(\hat{x}) := x_0$.

Example 1: Natural extensions of quadratic maps. When all the pairs (f_{-n}, X_{-n}) coincide with $(f_c, \overline{\mathbb{C}})$, following Lyubich and Minsky [16] we will denote $\varprojlim(f_c, \overline{\mathbb{C}})$ by \mathcal{N}_c . The set \mathcal{N}_c is called the *natural extension* of f_c . In this case, we denote the projection to the first coordinate by $\pi_c : \mathcal{N}_c \rightarrow \overline{\mathbb{C}}$. There is a natural homeomorphism $\hat{f}_c : \mathcal{N}_c \rightarrow \mathcal{N}_c$ given by $\hat{f}_c(z_0, z_{-1}, \dots) := (f_c(z_0), z_0, z_{-1}, \dots)$.

Let X be a forward invariant set. The *invariant lift* \hat{X} of X is the set of points $\hat{z} \in \mathcal{N}_c$ such that all coordinates of \hat{z} belong to X . In particular, $\hat{\infty} = (\infty, \infty, \dots)$.

Inverse limits appear naturally in dynamics, for instance, in the theory of dynamics of Hénon maps. See [5] for more details.

Example 2: The dyadic solenoid. A well-known example of an inverse limit is the *dyadic solenoid* $\mathcal{S}^1 := \varprojlim(f_0, \mathbb{S}^1)$, where $f_0(z) = z^2$ and \mathbb{S}^1 is the unit circle in \mathbb{C} . The dyadic solenoid is a connected set but is not path-connected. Since \mathbb{S}^1 is a topological group, \mathcal{S}^1 is a topological group acting on itself by translations: For any element $\tau = (\tau_0, \tau_{-1}, \dots)$ in \mathcal{S}^1 , the *left translation* $\tau : \mathcal{S}^1 \rightarrow \mathcal{S}^1$ is given by $\tau(z_0, z_{-1}, \dots) = (\tau_0 z_0, \tau_{-1} z_{-1}, \dots)$.

Solenoidal cones. A *solenoidal cone* is a space homeomorphic to $\varprojlim(f_0, \overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Let f_c be a map with connected $K(f_c)$, then the set $\hat{A}_c := \varprojlim(f_c, A_c)$ in \mathcal{N}_c is a solenoidal cone. To see this, consider the lift of the inverse ψ_c^{-1} of the Böttcher coordinate given by $\hat{\psi}_c^{-1} : (z_0, z_{-1}, \dots) \mapsto (\psi_c^{-1}(z_0), \psi_c^{-1}(z_{-1}), \dots)$. The set $\hat{A}_c \setminus \{\hat{\infty}\}$ is foliated by sets of the form $\mathcal{S}_c(r) := \pi_c^{-1}(E_c(r))$ with $r > 1$. For every r , $\mathcal{S}_c(r)$ is homeomorphic to the dyadic solenoid. In fact, there is a canonical homeomorphism $\phi_r = \phi_{c,r} : \mathcal{S}_c(r) \rightarrow \mathcal{S}^1$ defined by $\phi_r : (z_0, z_{-1}, \dots) \mapsto (\psi_c(z_0)/r, \psi_c(z_{-1})/r^{1/2}, \dots)$. We call such $\mathcal{S}_c(r)$ a *solenoidal equipotential*.

Let us give a few more examples of solenoidal cones. For $r > 1$, set $\mathbb{D}_r := \{|z| < r\}$. We denote the inverse limits associated with the backward dynamics

$$\dots \xrightarrow{f_0} \overline{\mathbb{C}} \setminus f_0^{-2}(\overline{\mathbb{D}}_r) \xrightarrow{f_0} \overline{\mathbb{C}} \setminus f_0^{-1}(\overline{\mathbb{D}}_r) \xrightarrow{f_0} \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_r$$

by $\hat{A}_0(r)$. This is a solenoidal cone compactly contained in $\hat{A}_0 \subset \mathcal{N}_0$. Similarly, the set $\hat{A}_c(r) := \hat{\psi}_c^{-1}(\hat{A}_0(r))$ is a solenoidal cone compactly contained in $\hat{A}_c \subset \mathcal{N}_c$. Note that the boundary of $\hat{A}_c(r)$ in \mathcal{N}_c is $\mathcal{S}_c(r)$. The union $\hat{A}_c(r) \cup \mathcal{S}_c(r)$ is called the *compact solenoidal cone at infinity* of radius r associated to f_c .

Let f_s be a superattracting quadratic map with critical orbit of size m . For all $r < 1$, the inverse limit given by the backward dynamics

$$\dots \xrightarrow{f_s^m} D_s(r^{1/4}) \xrightarrow{f_s^m} D_s(r^{1/2}) \xrightarrow{f_s^m} D_s(r)$$

is also a solenoidal cone. We denote it by $\varprojlim(f_s^m, D_s(r))$. We may consider $\varprojlim(f_s^m, D_s(r))$ as a subset of \mathcal{N}_s by the following embedding map: For $(x_0, x_{-1}, \dots) \in \varprojlim(f_s^m, D_s(r))$, define $\iota : (x_0, x_{-1}, \dots) \mapsto (y_0, y_{-1}, \dots) \in \mathcal{N}_s$ so that $x_{-k} = y_{-mk}$ for all $k \geq 0$. Then $\hat{D}_s(r) := \iota(\varprojlim(f_s^m, D_s(r)))$ is a solenoidal cone in \mathcal{N}_s . Note that $\partial\hat{D}_s(r)$ is a proper subset of $\pi_s^{-1}(I_s(r))$ unless $s = 0$. Now $\hat{D}_s(r), \hat{f}_s(\hat{D}_s(r)), \dots, \hat{f}_s^{m-1}(\hat{D}_s(r))$ are disjoint solenoidal cones in \mathcal{N}_s .

Quadratic-like inverse limits. Let $g : U \rightarrow V$ be a proper holomorphic map, and let $\varprojlim(g, V)$ denote the inverse limit for the sequence

$$\dots \rightarrow g^{-2}(V) \rightarrow g^{-1}(V) \rightarrow V.$$

Here we allow $U = V$. In some cases g will be taken as the restriction of a map defined on a bigger set, in these cases we will consider all branches of the inverse of g satisfying $g^{-n}(V) \subset U$.

We will use the following relation between inverse limits of quadratic-like maps and its straightening:

Proposition 1. *Let $g : U \rightarrow V$ be a quadratic-like map with straightening $f_c(z) = z^2 + c$. Then the inverse limit $\varprojlim(g, V)$ is homeomorphic to \mathcal{N}_c with a compact solenoidal cone at infinity removed.*

Proof. Any topological conjugacy between two given maps induces a homeomorphism between the inverse limits. By the Straightening Theorem, g is conjugated to a suitable restriction of f_c . By choosing the right conjugacy, g is conjugated to f_c restricted to the interior of some equipotential, let us say f_c restricted to $U_c(\sqrt{r})$. Now, the inverse limit $\varprojlim(f_c, U_c(r))$ is equal to \mathcal{N}_c with the compact solenoidal cone of radius r at infinity removed.

■

Remark. By construction, the homeomorphism in Proposition 1 can be chosen to be a leafwise quasiconformal map.

2.2 Regular parts and infinitely renormalizable maps

Regular parts of quadratic natural extensions. Let f_c be a quadratic map. A point $\hat{z} = (z_0, z_{-1}, \dots)$, in the natural extension $\mathcal{N}_c = \varprojlim(f_c, \overline{\mathbb{C}})$, is called *regular* if there exists a neighborhood U_0 of z_0 such that the pull-back of U_0 along \hat{z} is eventually univalent. The *regular part* (or *regular leaf space*) $\mathcal{R}_{f_c} = \mathcal{R}_c$ is the set of regular points in \mathcal{N}_c . Let $\mathcal{I}_{f_c} = \mathcal{I}_c$ denote the set of irregular points. Now we have $\mathcal{N}_c = \mathcal{R}_c \sqcup \mathcal{I}_c$.

The regular parts are analytically well-behaved subspaces of the natural extensions. More precisely, \mathcal{R}_c is the biggest subspace of \mathcal{N}_c whose path connected components, called *leaves*, admit Riemann surface structures. It is a fact, due to Lyubich and Minsky [16], that all leaves of \mathcal{R}_c are isomorphic to \mathbb{C} or \mathbb{D} . Moreover, \hat{f}_c sends leaves to leaves isomorphically.

Leaves are wildly foliated in the natural extension, indeed every leaf is dense in \mathcal{N}_c . See [16, §3] for more details.

Example: Regular part of superattracting maps. Relevant examples of regular parts are given by superattracting quadratic maps. Let f_s be a superattracting quadratic map with superattracting cycle $\{\alpha_s(1), \dots, \alpha_s(m) = 0\}$ as in the previous section. Under the homeomorphic action $\hat{f}_s : \mathcal{N}_s \rightarrow \mathcal{N}_s$, the points $\hat{\alpha}_s(i) := (\alpha_s(i), \alpha_s(i-1), \alpha_s(i-2), \dots)$ form a cycle of period m . In this case, the set \mathcal{I}_s of irregular points is $\{\hat{\infty}, \hat{\alpha}_s(1), \dots, \hat{\alpha}_s(m)\}$. Thus the regular part \mathcal{R}_s is \mathcal{N}_s minus these $m+1$ irregular points. Note that for any $r > 1$, $\hat{\infty} \in \hat{A}_s(r)$ and $\hat{\alpha}_s(i) \in \hat{f}_s^i(\hat{D}_s(1/r))$. Hence \mathcal{I}_s is contained in a disjoint union of solenoidal cones. Under the topology induced from \mathcal{N}_s , \mathcal{R}_s is a Riemann surface lamination with all leaves isomorphic to \mathbb{C} .

Regular part of infinitely renormalizable maps and *a priori* bounds. Let $\widehat{P}(f_c)$ be the invariant lift of the postcritical set, that is, the set of points $\hat{z} = (z_0, z_{-1}, \dots) \in \mathcal{N}_c$ with $z_{-n} \in P(f_c)$ for all $n \geq 0$.

Now suppose that f_c has *a priori* bounds. If K_n denotes the little Julia set of the n th renormalization, it follows that the postcritical set is given by

$$P(f_c) = \bigcap_{n \geq 0} \bigcup_{j \geq 0} f_c^j(K_n)$$

and homeomorphic to a Cantor set. Moreover, the map f_c restricted to $P(f_c)$ acts as a minimal \mathbb{Z} -action. See McMullen's [17, Theorems 9.4]. Hence we have the following:

Lemma 2. *If f_c is a quadratic polynomial with *a priori* bounds, then the set of irregular points in $\mathcal{N}_c - \{\hat{\infty}\}$ is $\widehat{P}(f_c)$. Moreover, the projection π_c restricted to $\widehat{P}(f_c)$ is a homeomorphism over $P(f_c)$. Thus, the irregular part \mathcal{I}_c is homeomorphic to a Cantor set together with the isolated point $\hat{\infty}$.*

We will also need the following fact, due to Kaimanovich and Lyubich ([8, Lemma 3.18]):

Theorem 3 (Kaimanovich-Lyubich). *If f_c has *a priori* bounds, then \mathcal{R}_c is a locally compact Riemann surface lamination, whose leaves are conformally isomorphic to \mathbb{C} .*

Theorem 3 is originally proved for f_c with *persistent recurrence*, which is a weaker condition than *a priori* bounds.

3 Structure Theorem

In this section we describe the topological structures of the natural extensions of infinitely renormalizable quadratic maps in detail.

Blocks for superattracting maps. We first define the blocks associated with superattracting quadratic maps. Let s be a superattracting parameter as in Section 1, with a superattracting cycle of period $m \geq 2$. For a fixed $r > 1$, we set

$$\mathcal{B}_s := \mathcal{N}_s \setminus \left(\overline{\hat{A}_s(r)} \cup \bigsqcup_{i=0}^{m-1} \overline{\hat{f}_s^i(\hat{D}_s(1/r))} \right)$$

and call it a *block* associated with f_s . That is, \mathcal{B}_s is the natural extension of f_s with compact solenoidal cones at each of the irregular points removed. Note that \mathcal{B}_s is an open set and has $m + 1$ boundary components which are all solenoidal equipotentials.

In addition, we also define

$$\mathcal{Q}_s := \mathcal{N}_s \setminus \left(\{\hat{\infty}\} \cup \bigsqcup_{i=0}^{m-1} \widehat{f_s^i(\hat{D}_s(1/r))} \right) = \mathcal{B}_s \sqcup \left(\widehat{A_s(r)} \setminus \{\hat{\infty}\} \right)$$

for later use.

Structure Theorem for infinitely renormalizable maps. Recall that the combinatorics of infinitely renormalizable maps are characterized by sequences of superattracting parameters. The following theorem shows that the natural extensions of infinitely renormalizable quadratic maps can be partially decomposed into blocks which are homeomorphic to the blocks of such superattracting parameters. We will establish:

Theorem 4 (Structure Theorem). *Let f_c be infinitely renormalizable with a priori bounds, and $\{g_n = f_c^{p_n} | U_n \rightarrow V_n\}_{n \geq 0}$ be the associated sequence of renormalizations with combinatorics $\{s_0, s_1, \dots\}$. Set $m_n := p_{n+1}/p_n$. Then there exist disjoint open subsets $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots$ of \mathcal{R}_c such that:*

- (1) *For $n = 0$, the set \mathcal{B}_0 is homeomorphic to \mathcal{Q}_{s_0} . Moreover, the union $\mathcal{B}_0 \cup \{\hat{\infty}\}$ forms a neighborhood of $\hat{\infty}$ with m_0 boundary components which are all homeomorphic to the dyadic solenoid.*
- (2) *For each $n \geq 1$, the set \mathcal{B}_n is homeomorphic to \mathcal{B}_{s_n} . Moreover, \mathcal{B}_n has $m_n + 1$ boundary components which are all homeomorphic to the dyadic solenoid.*
- (3) *For any $n \geq 1$, the sets $\mathcal{B}_n, \widehat{f_c}(\mathcal{B}_n), \dots, \widehat{f_c}^{p_n-1}(\mathcal{B}_n)$ have disjoint closures.*
- (4) *For $0 \leq n < n'$, the closures $\overline{\mathcal{B}_n}$ and $\overline{\mathcal{B}_{n'}}$ intersects iff $n' = n + 1$. In this case, for all $0 \leq i < m_n$ the closures $\widehat{f_c}^{ip_n}(\mathcal{B}_{n+1})$ and $\overline{\mathcal{B}_n}$ share just one of their solenoidal boundary components.*
- (5) *The natural extension of f_c and its regular part are given by*

$$\mathcal{N}_c = \{\hat{\infty}\} \sqcup \mathcal{R}_c \sqcup \widehat{P(f_c)} \quad \text{and} \quad \mathcal{R}_c = B_0 \sqcup \bigcup_{n=1}^{\infty} \bigsqcup_{i=0}^{p_n-1} \widehat{f_c^i(\mathcal{B}_n)}.$$

We call the open sets $\{\widehat{f_c^i(\mathcal{B}_n)}\}_{n,i}$ *blocks*. Theorem 4 implies that \mathcal{R}_c has a (locally finite) tree structure given by configuration of the blocks (Figure 3). Note that the block \mathcal{B}_n which we will construct may not be an invariant set of $f_c^{p_n}$.

3.1 Proof of the Structure Theorem

We construct the blocks by an iterative procedure. Let us state it as a lemma so that we can apply it to each level of the renormalization.

Let $g : U \rightarrow V$ be an infinitely renormalizable quadratic-like map with combinatorics $\{s = s_0, s_1, s_2, \dots\}$ and a renormalization $g_1 = g^m|_{U_1} \rightarrow V_1$. For $r > 1$, let $U_g(r)$ denote the topological disk in \mathbb{C} enclosed by $E_g(r)$. We may assume that $V = U_g(r)$ and $V_1 = U_{g_1}(r_1)$ for some $r, r_1 > 1$, and r_1 is sufficiently close to 1 such that

$$U_1 \Subset V_1 \Subset g^{-m}(V) \Subset U \Subset V.$$

Since $V_1 \Subset g^{-m}(V)$, the map $g^i|_{V_1}$ makes sense and $g^i(V_1) \subset V$ for all $1 \leq i \leq m$.

Set $\mathcal{X} := \varprojlim(g, V)$ and $\mathcal{X}_1 := \varprojlim(g_1, V_1)$. Then \mathcal{X}_1 is naturally embedded in \mathcal{X} as follows: For $\hat{x} = (x_0, x_{-1}, \dots) \in \mathcal{X}_1$, set $\iota(\hat{x}) := (x_0^*, x_{-1}^*, \dots) \in \mathcal{X}$ so that $x_{-k} = x_{-mk}^*$ for all $k \geq 0$. We denote the embedded image $\iota(\mathcal{X}_1)$ by \mathcal{X}_1^* . By Proposition 1, \mathcal{X}_1 (resp. \mathcal{X}_1^*) is homeomorphic to \mathcal{N}_{c_1} with a compact solenoidal cone at infinity removed, where $c_1 = c(g_1)$ is the straightening of infinitely renormalizable g_1 with combinatorics $\{s_1, s_2, \dots\}$. Recall that $s = s_0$ is a superattracting parameter, and its characteristic ray portrait $\text{rp}(O_s)$ is given by the cyclic orbit of $\beta(g_1)$ (the β -fixed point of g_1) by g . In principle, the set $\mathcal{X} \setminus \bigsqcup_{i=0}^{m-1} \widehat{g^i}(\mathcal{X}_1^*)$ is a prototype for the block homeomorphic to \mathcal{B}_s . However, the sets $\widehat{g^i}(\mathcal{X}_1^*)$ may not be disjoint, and this would contradict properties (2) and (3) of the theorem. This case occurs when the little Julia sets touch at the β -fixed point. To avoid this situation, we need deform \mathcal{X}_1^* to a smaller set \mathcal{Y}_1^* , such that \mathcal{X}_1^* and \mathcal{Y}_1^* are homotopically homeomorphic. More precisely, we claim:

Lemma 5. *There exists a topological disk $W_1 \subset V_1$ with the following properties:*

- (a) *The set $\mathcal{Y}_1 := \varprojlim(g_1, W_1) \subset \mathcal{X}_1$ is homeomorphic to \mathcal{X}_1 .*
- (b) *Set $\mathcal{Y}_1^* := \iota(\mathcal{Y}_1) \subset \mathcal{X}$. Then the sets $\widehat{g^i}(\mathcal{Y}_1^*)$ ($0 \leq i < m$) have disjoint closures.*
- (c) *The set $\mathcal{X} \setminus \bigsqcup_{i=0}^{m-1} \widehat{g^i}(\mathcal{Y}_1^*)$ is homeomorphic to the block \mathcal{B}_s .*

Proof of (a) and (b). We will modify the standard idea of *thickening* of puzzle pieces (see for instance [18, Lemmas 1.5 and 1.6]) into *thinning*. Set $\beta_1 := \beta(g_1)$ and $K_1 := K(g_1)$. In the pulled-back external rays landing at β_1 by the straightening map $h = h_g$, there are two of such rays R_1 and R_2 such that $R_1 \cup R_2$ separates any other rays landing at β_1 and $K_1 \setminus \{\beta_1\}$. Analogously, for the preimage $\beta_1^* := g_1^{-1}(\{\beta_1\}) \setminus \{\beta_1\}$, there are two rays R_3 and R_4 landing at β_1^* with the same property. The rays $\{R_1, R_2, R_3, R_4\}$ are called the *supporting rays of K_1* .

Next we take a small disk, centered at β_1 , so that its preimage by g_1 consists of two disks Δ and Δ^* around β_1 and β_1^* respectively. The union of Δ , Δ^* , and the supporting rays divide V into three topological disks. Let us take the one containing $K_1 \setminus (\overline{\Delta} \cup \overline{\Delta^*})$. By cutting the boundary by other external rays, we have a smaller domain W' shown as in Figure 1.

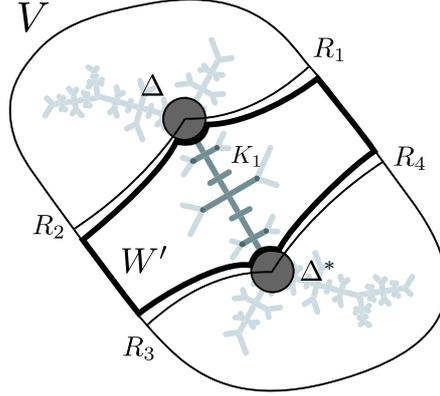


Figure 1: The heavy curves show the boundary of W' .

Let W_1 denote the topological disk that is the connected component of $W' \cap V_1$ containing the critical point of g_1 . Since $W_1 \subset V_1 \Subset g^{-m}(V)$, the sets $W_1, g(W_1), \dots, g^{m-1}(W_1)$ are all defined. In particular, they have disjoint closures by construction. Now the inverse limit of the family $\{g_1 : g_1^{-n-1}(W_1) \rightarrow g_1^{-n}(W_1)\}_{n \geq 0}$, which we denote by $\mathcal{Y}_1 := \varprojlim (g_1, W_1)$, is a proper subset of $\mathcal{X}_1 = \varprojlim (g_1, V_1)$.

Let us check that \mathcal{Y}_1 is homeomorphic to \mathcal{X}_1 . By definition $V_1 \setminus W_1$ consists of disjoint topological disks and does not intersect the postcritical set $P(g_1)$ since we take sufficiently small Δ and Δ^* . (See [17, Theorem 8.1] for example. This is the only part we use the *infinite* renormalizability.) Thus $g_1 : g_1^{-n-1}(V_1) \rightarrow g_1^{-n}(V_1)$ is isotopic to $g_1 : g_1^{-n-1}(W_1) \rightarrow g_1^{-n}(W_1)$ for each $n \geq 0$ and this isotopy gives a homeomorphism between the inverse limits.

Let \mathcal{Y}_1^* be the embedded image of \mathcal{Y}_1 by the map $\iota : \mathcal{X}_1 \rightarrow \mathcal{X}$. For all $0 \leq i < m$, the sets $\hat{g}^i(\mathcal{Y}_1^*)$ are defined and have disjoint closures since their projections $g^i(W_1)$ are defined and have disjoint closures. Hence we have (a) and (b) of the statement.

Proof of (c). Set $\mathcal{B} := \mathcal{X} \setminus \bigsqcup_{i=0}^{m-1} \hat{g}^i(\overline{\mathcal{Y}_1^*})$. Now it is enough to show that \mathcal{B} is homeomorphic to the block \mathcal{B}_s associated with f_s , that is,

$$\mathcal{B}_s = \mathcal{N}_s \setminus \left(\overline{\hat{A}_s(r)} \sqcup \bigsqcup_{i=0}^{m-1} \overline{\hat{f}_s^i(\hat{D}_s(1/r))} \right) = \pi_s^{-1}(U_s(r)) \setminus \bigsqcup_{i=0}^{m-1} \overline{\hat{f}_s^i(\hat{D}_s(1/r))}.$$

Here we take the same r as in the construction of $V = U_g(r)$. For later use we also set $V_s := U_s(r)$.

We first work with the dynamics downstairs. Set $B := V \setminus \bigsqcup_{i=0}^{m-1} g^i(\overline{W_1})$ and mark B with some arcs given as follows (See Figure 2, left): First join $g(\beta_1)$ and $\partial g(W_1)$ by an arc δ within $g(\Delta)$. Since $g : W_1 \rightarrow g(W_1)$ is a branched covering, the pull-back $g^{-1}(\delta)$ has two components in Δ and Δ^* . Now the markings are given by $g^{-1}(\delta), \delta, g(\delta), \dots, g^{m-2}(\delta)$ and all of the forward images of the supporting rays $\bigcup_{j=1}^4 R_j$. The markings decompose B into finitely many pieces that are all topological disks. Note that the boundary of each piece intersects the equipotential $E_g(r)$ and at least two external rays.

Correspondingly, set $B_s := V_s \setminus \bigsqcup_{i=1}^{m-1} \overline{f_s^i(D_s(1/r))}$, and complete the marking of B_s by taking all the forward images of supporting rays of D_s and small arcs from each point of the

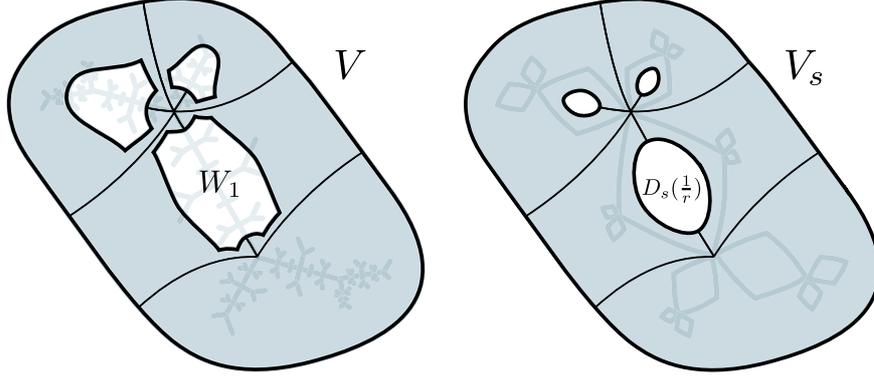


Figure 2: The shaded region show B and B_s with their markings drawn in.

landing points of such rays to the equipotentials $f_s^i(I_s(1/r))$ (Figure 2, right). The markings also decompose B_s into some pieces as in the case of B .

Clearly, there is a homeomorphism ϕ from B to B_s respecting the configuration of the markings which in particular sends the supporting external rays into the supporting external rays without changing angles. Since by construction B and B_s do not intersect the postcritical sets, it is not difficult to check that the map ϕ lifts to a homeomorphism $\hat{\phi} : \pi^{-1}(B) \rightarrow \pi_s^{-1}(B_s)$ respecting the backward orbits of the markings, and then extends to a homeomorphism $\hat{\phi} : \mathcal{B} \rightarrow \mathcal{B}_s$. \blacksquare (Lemma 5)

Proof of Theorem 4 (Structure Theorem). We first set $\mathcal{X}_0 = \lim_{\leftarrow}(g_0, V_0) \subset \mathcal{N}_c$ and apply Lemma 5 with $g = g_0$. Then we have a block $\mathcal{B}_0 := \mathcal{X}_0 \setminus \overline{\bigsqcup_{i=0}^{m_0-1} \hat{g}_0^i(\mathcal{Y}_1^*)}$ that is homeomorphic to \mathcal{B}_{s_0} . To get property (1), we need to replace \mathcal{B}_0 by $\mathcal{R}_c \setminus \overline{\bigsqcup_{i=0}^{m-1} \hat{g}^i(\mathcal{Y}_1^*)}$ to cover all the basin at infinity of f_c . By Proposition 1 again, \mathcal{B}_0 is homeomorphic to \mathcal{Q}_{s_0} .

Now assume that for $n \geq 1$ we applied Lemma 5 with $g = g_{n-1}$, and we have a topological disk $W_n \subset V_n$ and $\mathcal{Y}_n = \lim_{\leftarrow}(g_n, W_n) \subset \mathcal{X}_n = \lim_{\leftarrow}(g_n, V_n)$. For the next induction step, we apply the lemma with a slight modification. We may assume that $g_n : U_n \rightarrow V_n$ and $g_{n+1} : U_{n+1} \rightarrow V_{n+1}$ satisfy the original condition

$$U_{n+1} \Subset V_{n+1} \Subset g_n^{-m_n}(V_n) \Subset U_n \Subset V_n$$

in the lemma, and also

$$V_{n+1} \sqcup g_n(V_{n+1}) \sqcup \cdots \sqcup g_n^{m_n-1}(V_{n+1}) \Subset W_n.$$

In the same way as the lemma, we construct a topological disk $W_{n+1} \subset V_{n+1}$ such that $\mathcal{Y}_{n+1} = \lim_{\leftarrow}(g_{n+1}, W_{n+1})$ is naturally embedded into \mathcal{X}_n as $\iota(\mathcal{Y}_{n+1}) = \mathcal{Y}_{n+1}^*$, and the set $\mathcal{X}_n \setminus \overline{\bigsqcup_{i=0}^{m_n-1} \hat{g}_n^i(\mathcal{Y}_{n+1}^*)}$ is homeomorphic to \mathcal{B}_{s_n} . In addition, the set $\mathcal{B}'_n := \mathcal{Y}_n \setminus \overline{\bigsqcup_{i=0}^{m_n-1} \hat{g}_n^i(\mathcal{Y}_{n+1}^*)}$ is also homeomorphic to \mathcal{B}_{s_n} , since \mathcal{Y}_n is homeomorphic to \mathcal{X}_n and we have

$$\mathcal{Y}_{n+1}^* \sqcup \hat{g}_n(\mathcal{Y}_{n+1}^*) \sqcup \cdots \sqcup \hat{g}_n^{m_n-1}(\mathcal{Y}_{n+1}^*) \Subset \mathcal{Y}_n \subset \mathcal{X}_n$$

by the second condition above.

Finally we define the block \mathcal{B}_n in the theorem by the image of the iterated natural embeddings

$$\mathcal{B}'_n \hookrightarrow \varprojlim(g_{n-1}, W_{n-1}) \hookrightarrow \cdots \hookrightarrow \varprojlim(g_1, W_1) \hookrightarrow \varprojlim(g_0, V_0)$$

so that the block becomes a subset of the original natural extension \mathcal{N}_c (Figure 3). Now $\overline{\mathcal{B}_n}$

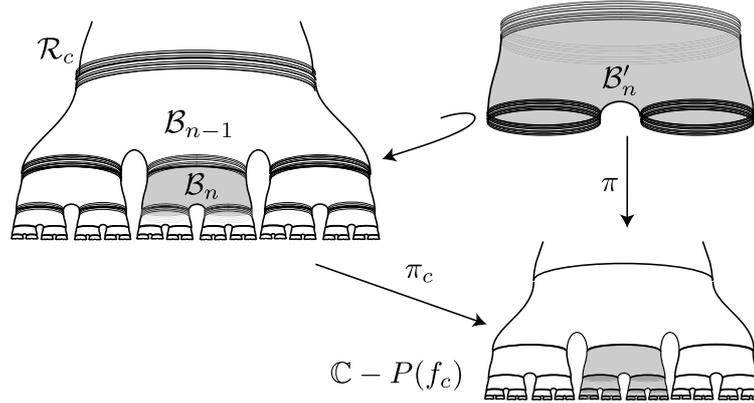


Figure 3: A caricature of the tree structure of \mathcal{R}_c .

contains no irregular point by this construction. Hence $\overline{\mathcal{B}_n} \subset \mathcal{R}_c$. Properties (2), (3), and (4) of the statement are clear, also by the construction.

Only for (5) we need the condition that f_c has *a priori* bounds: By Lemma 2, the set $\widehat{P(f_c)} \cup \{\infty\}$ consists of all the irregular points. Hence the regular part consists of all the backward orbits that do not remain in $P(f_c) \cup \{\infty\}$. Since such a backward orbit must be contained in either \mathcal{B}_0 or the closure of the block $\widehat{f_c^i(\mathcal{B}_n)}$ for some $n \geq 1$ and $0 \leq i < p_n$, we have $\mathcal{R}_c \subset B_0 \sqcup \bigcup_{n=1}^{\infty} \bigsqcup_{i=0}^{p_n-1} \widehat{f_c^i(\mathcal{B}_n)}$. The opposite inclusion holds since $\overline{\mathcal{B}_n} \subset \mathcal{R}_c$ for all $n \geq 0$, thus the second equality in (5) follows. Now the first equality is straightforward. ■

To end this section we show a proposition that is important for the arguments in the next section.

Buildings at finite levels. For f_c with a priori bounds, we define an open set

$$\mathcal{Q}_n := \mathcal{R}_c \setminus \bigcup_{k=n+1}^{\infty} \bigsqcup_{i=0}^{p_k-1} \widehat{f_c^i(\mathcal{B}_k)},$$

which consists of blocks up to n th level. Then any $\hat{z} \in \mathcal{R}_c$ is contained in \mathcal{Q}_n with sufficiently large n .

The proposition below claims that \mathcal{Q}_n can be embedded into the regular part of a superattracting quadratic map. For the combinatorics $\{s_0, s_1, \dots\}$ of f_c , its finite subsequence $\{s_0, s_1, \dots, s_n\}$ determines a superattracting parameter σ_n . More precisely, for β -fixed point $\beta(g_{n+1})$ of $g_{n+1} = f_c^{p_{n+1}}$, its forward orbit O_{n+1} by f_c forms a repelling periodic cycle. Then

its ray portrait $\text{rp}(O_{n+1})$ uniquely determines a superattracting quadratic map f_{σ_n} . Now we have:

Proposition 6. *For infinitely renormalizable f_c , let \mathcal{Q}_n be the set defined as above. Then we have a homeomorphism h_n between \mathcal{Q}_n and \mathcal{Q}_{σ_n} .*

Proof. The proof is almost straightforward by Lemma 5. In fact, we can apply the same argument by setting $g := g_0$ and $g_1 := g_{n+1}$. ■

4 Rigidity

In this section we prove the Main Theorem of the paper which is the following:

Theorem 7 (Main Theorem). *Let f_c and $f_{c'}$ be infinitely renormalizable maps with a priori bounds. Assume $\text{Im}(c) \neq 0$. If $h : \mathcal{R}_c \rightarrow \mathcal{R}_{c'}$ is an orientation preserving homeomorphism, then c and c' belong to the same combinatorial class.*

There is a natural homeomorphism between \mathcal{N}_c and $\mathcal{N}_{\bar{c}}$, so we avoid this inconsistency by requiring h to be orientation preserving.

From the point of view of the parameter plane, it is known that c is combinatorially rigid if and only if the Mandelbrot set is locally connected (MLC) at c . In view of that, our main theorem has the following corollary.

Corollary 8. *Assume that $h : \mathcal{R}_c \rightarrow \mathcal{R}_{c'}$ and c are as in the Main Theorem. If in addition the Mandelbrot set is locally connected at c , then $c = c'$.*

In [15], Lyubich proved MLC for f_c with a priori bounds satisfying a secondary limb condition. In this direction, recent papers by Kahn [6] and Lyubich [7] give a priori bounds for infinitely renormalizable parameters with special combinatorics. In all these parameters, the Mandelbrot set is locally connected.

The strategy to prove the Main Theorem is to show that any homeomorphism given in the statement admits a homeomorphic extension to the natural extensions, satisfying the properties of the following general Theorem:

Theorem 9. *Let c and c' be any pair of parameters such that there exists $h : \mathcal{N}_c \rightarrow \mathcal{N}_{c'}$, an orientation preserving homeomorphism between natural extensions, such that:*

1. $h(\hat{\infty}) = \hat{\infty}$; and
2. h sends repelling leaves into repelling leaves.

Then f_c and $f_{c'}$ belong to the same combinatorial class.

A leaf L in \mathcal{R}_c is called *repelling*, if it contains a repelling periodic point of \hat{f}_c . Clearly, every repelling leaf is invariant under some iterate of \hat{f}_c , but the converse is not true in general. For instance, if f_c has a parabolic cycle there are invariant leaves of \mathcal{R}_c without periodic points.

We will see that, for quadratic polynomials with *a priori* bounds, repelling leaves have particular topological properties. Hence any homeomorphism as in the Main Theorem must send repelling leaves into repelling leaves.

This was the approach in [3] (see also [2]) to prove rigidity for hyperbolic and complex semi-hyperbolic maps.

4.1 Proof of Theorem 9

There are several models describing the combinatorics of quadratic polynomials, a comprehensive text can be found in [1]. In this paper, we are going to adopt the description given by rational laminations. Any quadratic polynomial f_c with c in the Mandelbrot set determines an equivalence relation in \mathbb{Q}/\mathbb{Z} , called the *rational lamination* of f_c . This equivalence relation is defined as follows: Given θ and θ' in \mathbb{Q}/\mathbb{Z} , we say that $\theta \sim \theta'$ if the external rays R_θ and $R_{\theta'}$ land at the same point in the Julia set $J(f_c)$.

Given any equivalence relation R in \mathbb{Q}/\mathbb{Z} we may ask whether R is the lamination associated to a quadratic polynomial. In [11], Kiwi gave necessary and sufficient conditions for which R is the rational lamination associated to a map f_c .

Two parameters c_1 and c_2 in the Mandelbrot set are *combinatorially equivalent*, if f_{c_1} and f_{c_2} determine the same rational lamination. Clearly, this is an equivalence relation in the Mandelbrot set. The induced partition also coincides with the combinatorial invariants given by ray portraits for superattracting maps f_s , or the sequence $\{s_1, s_2, \dots\}$ for infinitely renormalizable maps f_c .

We will need the following property of rational laminations:

Lemma 10. *Let R and R' be two rational laminations, assume that there is $\theta \in \mathbb{Q}/\mathbb{Z}$ such that each class in R' is obtained by rotating a class in R by angle θ . Then $\theta = 0 \pmod{1}$.*

In the dynamical plane, if p is a periodic point in the Julia set $J(f_c)$ then p is the landing point of external rays which are periodic under f_c (see [19]). Now, if the periodic lift \hat{p} belongs to the regular part, then there are periodic lifted external rays landing at \hat{p} in $L(\hat{p})$. Each of these lifted external rays will intersect a leaf of a solenoidal equipotential.

Let us consider the dyadic solenoid $\mathcal{S}^1 = \varprojlim (f_0, \mathbb{S}^1) \subset \mathcal{R}_0$. One can check that every repelling periodic point in \mathcal{R}_0 belongs to \mathcal{S}^1 . By restriction, let us call a leaf S in \mathcal{S}^1 repelling if S is contained in a repelling leaf in \mathcal{R}_0 . If a leaf $S \subset \mathcal{S}^1$ is invariant under some iterate of \hat{f}_0 , then S must be repelling. In other words, there is a one-to-one correspondence between repelling leaves in \mathcal{S}^1 and periodic points in \mathbb{S}^1 of f_0 . With this fact and the lift of Böttcher's coordinate we can carry information of repelling periodic points of f_c to repelling leaves in \mathcal{R}_c .

More precisely, let L be a repelling leaf in \mathcal{R}_c and let $\mathcal{S}_c(r)$ some solenoidal equipotential. Consider the intersection $L \cap \mathcal{S}_c(r)$, which may consist of several leaves in $\mathcal{S}_c(r)$. Let ϕ be the canonical homeomorphism of $\mathcal{S}_c(r)$ with \mathcal{S}^1 , then every leaf in $\phi(L \cap \mathcal{S}_c(r))$ is repelling in \mathcal{S}^1 . Hence, every leaf in $\phi(L \cap \mathcal{S}_c(r))$ contains a periodic point in \mathcal{S}^1 by \hat{f}_0 . In fact, the pullback to L of each of these periodic points is precisely the intersection of a periodic solenoidal external ray landing at the periodic point of L . As a consequence we have:

Lemma 11. *Let S_1 and S_2 be two leaves in $\mathcal{S}_c(r)$. Then S_1 and S_2 belong to the same repelling leaf L in \mathcal{R}_c if and only if they intersect at periodic lifted external rays landing at the same point in $\pi^{-1}(J(f_c)) \cap \mathcal{R}_c$.*

The proof of Theorem 9 is decomposed in three statements; Lemma 12 whose proof can be found in [3], Proposition 13 which is due to Kwapisz [12], and Lemma 14. The first starts by noting that the foliation of the solenoidal cone by solenoidal equipotentials defines a local base of neighborhoods at ∞ in \mathcal{N}_c . Hence, given a homeomorphism h as in Theorem 9, we can find a solenoidal equipotential $\mathcal{S}_c(r)$ whose image lies between two solenoidal equipotentials. Recall that a solenoidal equipotential $\mathcal{S}_c(r)$ has associated a canonical homeomorphism $\phi_r : \mathcal{S}_c(r) \rightarrow \mathcal{S}^1$, moreover, $\phi_{r^2} \circ \hat{f}_c \circ \phi_r^{-1} = \hat{f}_0$. So the map $h \circ \phi_r^{-1}$, maps \mathcal{S}^1 into a cylinder over \mathcal{S}^1 bounded by the two solenoidal equipotentials. Hence, we are in the following situation:

Lemma 12. *Let $e : \mathcal{S}^1 \rightarrow \mathcal{S}^1 \times (0, 1)$ be a topological embedding, then there is an embedding $e' : \mathcal{S}^1 \rightarrow \mathcal{S}^1 \times (0, 1)$ isotopic to e such that $e'(\mathcal{S}^1) = \mathcal{S}^1 \times \{1/2\}$.*

Then isotopy in Lemma 12 pulls back to an isotopy ψ defined on $\mathcal{S}_c(r)$. This isotopy induces a homeomorphism h' from $\mathcal{S}_c(r)$ to a solenoidal equipotential in \mathcal{N}_c . We can extend ψ to an isotopy defined globally in \mathcal{N}_c , so that this isotopy is the identity outside a neighborhood of $\mathcal{S}_c(r)$. With this construction, h' extends to a homeomorphism defined on \mathcal{N}_c and isotopic to h .

We now restrict our attention to the map $h'|_{\mathcal{S}_c(r)}$. We want to study the isotopic properties of $h'|_{\mathcal{S}_c(r)}$ but keeping in mind that every map isotopic to $h'|_{\mathcal{S}_c(r)}$ will induce a map isotopic to h' defined globally in \mathcal{R}_c . Using the canonical homeomorphism of solenoidal equipotentials to \mathcal{S}^1 , h' restricted to $\mathcal{S}_c(r)$ induces a self-homeomorphism of the dyadic solenoid \mathcal{S}^1 . Now using the group structure of \mathcal{S}^1 , as described by Kwapisz in [12], each homotopy class of homeomorphisms of \mathcal{S}^1 is uniquely represented by a map with a special form:

Proposition 13 (Kwapisz). *Let $\Phi : \mathcal{S}^1 \rightarrow \mathcal{S}^1$ be a homeomorphism of the dyadic solenoid, then there exist a unique n and an element $\tau \in \mathcal{S}^1$ such that Φ is isotopic to $\hat{z} \mapsto \tau \hat{f}_0^n(\hat{z})$.*

The number n is uniquely determined by the homotopy class of h' . The map \hat{f}_c^{-1} restricted to a solenoidal equipotential acts, under canonical homeomorphism, in \mathcal{S}^1 as \hat{f}_0^{-1} . Note that if $\tau \in \mathcal{S}^1$, then $\hat{f}_0^{-1}(\tau \hat{z}) = \hat{f}_0^{-1}(\tau) \hat{f}_0^{-1}(\hat{z})$. Thus, by post-composing h' with \hat{f}_c^{-n} , Proposition 13 implies that $\hat{f}_c^{-n} \circ h'$ is isotopic to a homeomorphism $h'' : \mathcal{N}_c \rightarrow \mathcal{N}_c$ that sends a solenoidal equipotential, say $\mathcal{S}_c(r)$ into a solenoidal equipotential. The restriction of h'' to $\mathcal{S}_c(r)$, under the canonical homeomorphism to \mathcal{S}^1 , is just some translation by $\tau' := \hat{f}_0^{-n}(\tau)$ of the dyadic solenoid \mathcal{S}^1 .

Isotopies defined in solenoids keep each leaf invariant. On the other hand, the map \hat{f}_c , sends repelling leaves into repelling leaves. So, if h is a homeomorphism that sends repelling leaves into repelling leaves, the map h'' , induced by Proposition 13, also sends repelling leaves to repelling leaves.

Lemma 14. *Let h'' be the homeomorphism above, then τ' is isotopic to the identity.*

Proof. Let us consider the restriction of h'' to the solenoidal equipotential $\mathcal{S}_c(r)$ such that $h''(\mathcal{S}_c(r))$ is also a solenoidal equipotential, under canonical homeomorphisms we can regard $H := h''|_{\mathcal{S}_c(r)}$ as a self-map of \mathcal{S}^1 . We assume that H has the form $\hat{z} \mapsto \tau' \hat{z}$. By Lemma 11, H sends repelling leaves into repelling leaves.

Let S be a periodic leaf in \mathcal{S}^1 with periodic point $\hat{\theta}$, and let $\hat{\theta}'$ be the periodic point in $H(S)$. By sliding \mathcal{S}^1 along $H(S)$ to send $H(\hat{\theta})$ to $\hat{\theta}'$, this operation induces a new map H' in the isotopy class of H , that satisfies $H'(\hat{\theta}) = \tau'' \hat{\theta} = \hat{\theta}$ for some $\tau'' \in \mathcal{S}^1$. Since $\hat{\theta}$ and $\hat{\theta}'$ are periodic in \mathcal{S}^1 , τ'' must be periodic as well. Hence, the map H' leaves the set of periodic points in \mathcal{S}^1 invariant.

Now, periodic points in \mathcal{S}^1 are determined by the first coordinate. The translation τ' induces a rotation in the set of periodic angles which extends to a rotation on the rational lamination. By Lemma 10 this implies that the rational laminations are the same, and that the translation τ'' is the identity. By construction τ' is isotopic to τ'' . ■

Now we are ready to proof Theorem 9.

Proof of Theorem 9. From the existence of the homeomorphism h , we obtained the homeomorphism h'' . In turn, Lemma 14 implies that h'' is isotopic to another homeomorphism. Such homeomorphism when restricted to a solenoidal equipotential is the identity, under canonical identifications. Any rational ray R_θ landing in $J(f_c)$ lifts to a periodic external ray in \mathcal{N}_c . Under canonical identification, the intersection of the lift of periodic external rays is a periodic point in \mathcal{S}^1 . Moreover, this periodic point in \mathcal{S}^1 is precisely the periodic lift of the angle θ in \mathcal{S}^1 . Finally, Lemma 11 implies that the rational laminations of f_c and $f_{c'}$ are the same. Thus c and c' belong to the same combinatorial class. ■

4.2 Ends of the regular part

A path $\gamma : [0, \infty) \rightarrow \mathcal{R}_c$ is said to *escape to infinity* if it leaves every compact set $K \subset \mathcal{R}_c$. We define an *end* of \mathcal{R}_c to be an equivalence class of paths escaping to infinity. Let γ and σ be two paths escaping to infinity. We say that γ and σ access the same end if for every compact set $K \subset \mathcal{R}_c$, the paths γ and σ eventually belong to the same connected component of $\mathcal{R}_c \setminus K$. Consider the set $End(\mathcal{R}_c)$ consisting of \mathcal{R}_c union with the abstract set of ends.

Let f_c be an infinitely renormalizable quadratic polynomial with *a priori* bounds. By Theorem 3, the regular part \mathcal{R}_c is locally compact. Thus $End(\mathcal{R}_c)$ is a compact set, which we will call the *end compactification* of \mathcal{R}_c .

Proposition 15. *Let f_c be an infinitely renormalizable quadratic polynomial with a priori bounds, then $End(\mathcal{R}_c)$ is homeomorphic to \mathcal{N}_c .*

Proof. First, we will show that there exists a bijection Φ between the set of irregular points and the set of ends. Let \hat{i} be an irregular point in \mathcal{N}_c , let $i_0 = \pi(\hat{i})$ and take any $z_0 \in \mathbb{C} \setminus P(f_c)$. Since $P(f_c)$ is a Cantor set, there is a path σ , connecting z_0 with i_0 , which intersects $P(f_c)$ only at i_0 . We can lift the path σ to \mathcal{N}_c to a path $\hat{\sigma}$ from a point in the fiber of z_0 connecting to \hat{i} . By construction, the path $\hat{\sigma}$ intersects \mathcal{I}_c at \hat{i} , then the restriction of $\hat{\sigma}$ to \mathcal{R}_c is a path escaping to infinity. Let $\Phi(\hat{i}) = [\hat{\sigma}]$, where $[\hat{\sigma}]$ is the end represented by $\hat{\sigma}$.

To check that Φ is well-defined, let $\hat{\sigma}$ and $\hat{\sigma}'$ be two paths in \mathcal{N}_c intersecting the irregular set only at the end point \hat{i} . These paths do not need to start at the same point or belong to the same leaf. Let L be the leaf containing $\sigma([0, 1))$ in \mathcal{R}_c . Since every leaf is dense in \mathcal{R}_c and is simply connected, we can construct a family of paths $\hat{\sigma}_n$ in L , ending at \hat{i} and such that $\hat{\sigma}_n \rightarrow \hat{\sigma}'$ pointwise. Let K be any compact set in \mathcal{R}_c , and U be a connected component of $\mathcal{R}_c \setminus K$ which eventually contains $\hat{\sigma}'$. Since U is open, there is an N such that $\hat{\sigma}_N$ also eventually belongs to U . But $\hat{\sigma}$ and $\hat{\sigma}_N$ belong to the same path connected component (same leaf), thus $\hat{\sigma}$ must also be eventually contained in U .

To see that Φ is injective, let \hat{i} and \hat{i}' be two different irregular points. By Lemma 2, the projection π is a homeomorphism between the set of irregular points and $P(f_c)$. Thus we have $\pi(\hat{i}) \neq \pi(\hat{i}')$. Hence, if σ and σ' are two paths escaping to \hat{i} and \hat{i}' respectively, then σ and σ' eventually belong to different components of some level of renormalization.

Finally, let us prove that Φ is surjective. Let e be an end of the regular part \mathcal{R}_c , and consider $\hat{\sigma}$ a path escaping to e . Let $V_0 = \overline{U_c(r)}$ for some r , where $U_c(r)$ is as in Section 1. For each level of renormalization n , let Z_n be the union of disjoint open neighborhoods of the little Julia sets of level n . If these Julia sets touch, we can shrink the domains a little to make them disjoint as in the proof of Theorem 4. Then $C_n = V_0 \setminus Z_n$ is a compact set in $\mathbb{C} \setminus P(f_c)$. Thus $\pi^{-1}(C_n)$ is compact in \mathcal{R}_c , by definition the path $\hat{\sigma}$ must eventually escape $\pi^{-1}(C_n)$. It follows that the projection $\pi(\hat{\sigma})$ eventually belongs either to a neighborhood of infinity, and then $\hat{\sigma}$ escapes to $\hat{\infty}$, or to a component of Z_n , say \mathcal{V}_n . Since the components of Z_n are disjoint, it is clear that \mathcal{V}_{n+1} is contained in \mathcal{V}_n . By *a priori* bounds, the domains $\{\mathcal{V}_n\}$ shrink to a point i_0 in $P(f_c)$. This process can be repeated for every coordinate of $\hat{\sigma}$ to get a sequence of points $\{i_n\}$ in $P(f_c)$ which are the coordinates of a point \hat{i} in $\widehat{P(f_c)}$. By Lemma 2, \hat{i} is irregular. Now, local neighborhoods of irregular points in \mathcal{N}_c intersect complements of compact sets in \mathcal{R}_c . So one can check that Φ induces the desired homeomorphism. ■

On the remaining part of the paper, h will denote a homeomorphism of the regular parts of two infinitely renormalizable quadratic polynomials, f_c and $f_{c'}$, with *a priori* bounds.

Corollary 16. *The map h admits an extension to a homeomorphism $\tilde{h} : \mathcal{N}_c \rightarrow \mathcal{N}_{c'}$ of the natural extensions. Moreover, $\tilde{h}(\hat{\infty}) = \hat{\infty}$.*

Proof. By Proposition 15 the map h extends to the natural extensions sending irregular points to irregular points, and by Lemma 2 the point $\hat{\infty}$ is the only isolated irregular point, hence $h(\hat{\infty}) = \hat{\infty}$. ■

4.3 Topology of periodic leaves

Since leaves are path connected components of \mathcal{R}_c , given a leaf $L \subset \mathcal{R}_c$ we can consider how many access to $\hat{\infty}$ the leaf has. That is, the number of path components of $L \setminus K$ that are connected to $\hat{\infty}$ in \mathcal{N}_c , for a suitable large compact set $K \subset \mathcal{R}_c$. Note that a leaf has access to a point in $P(f_c)$ if and only if it intersects infinitely many levels in the tree structure of \mathcal{R}_c . However, this is not the case for repelling leaves:

Lemma 17. *Let L be a repelling leaf, then there is a level n such that $L \subset \mathcal{Q}_n$. In this case, L has access only to $\hat{\infty}$.*

Proof. Let \hat{p} be the periodic point in L and let $p = \pi(\hat{p})$. Since f_c is infinitely renormalizable, p is repelling, and therefore it must belong to the Julia set $J(f_c)$. Moreover, the inverse of the classical Königs linearization coordinate around p provides a global uniformization coordinate for L . From this uniformization it follows that a point \hat{z} in \mathcal{R}_c belongs to L only if the coordinates of \hat{z} accumulate on the cycle of p .

Since the postcritical set is the intersection of the renormalization domains, we can find a level $n + 1$ of the renormalization such that the orbit, of renormalization domains of level $n + 1$, is outside a neighborhood of the cycle of p . By this choice, no point in L can intersect the level $n + 1$ of the tree structure of \mathcal{R}_c . The statement of the lemma now follows. ■

Let f_s be a superattracting map, every leaf L invariant under some iterate of \hat{f}_s must contain a repelling periodic point and hence L is repelling. In this case, there are no critical points in the Julia set $J(f_s)$ so the fiber $\pi^{-1}(J(f_s))$ is compact (see [16, Lemma 8.4]). For a periodic point p in $J(f_s)$, let \hat{p} be invariant lift of p in \mathcal{R}_s . From [3], we have the following:

Proposition 18. *Let f_s be a superattracting polynomial, and let L be a repelling leaf in \mathcal{R}_s . Then the number of access of L to $\hat{\infty}$ is equal to the number of external rays landing at p . Moreover, if L is a leaf which has at least three access to ends, then L must be repelling.*

Let us remark that Proposition 15 also holds when f_s is superattracting, however, repelling leaves may have access to other irregular points. Nevertheless, if some repelling leaf L has at least three access to $\hat{\infty}$ then by Proposition 18, the corresponding periodic point p has at least three external rays landing at p . If f_c is non-hyperbolic, then there exists a periodic point p with at least three rays landing at p if and only if $Im(c) \neq 0$.

Let us now go back to the case where f_c is infinitely renormalizable with *a priori* bounds:

Lemma 19. *Let f_c be infinitely renormalizable with a priori bounds. Let $L \subset \mathcal{R}_c$ be a leaf that has access to $\hat{\infty}$ but to no other irregular point, and the number of access to infinity is at least three, then L must be a repelling leaf.*

Proof. Since the only access to infinity of L is $\hat{\infty}$, there is a level n such that $L \subset \mathcal{Q}_n$. Remind, as in the discussion previous to Proposition 6, that the dynamics of \hat{f}_c on \mathcal{Q}_n determines a superattracting polynomial f_{σ_n} . By Proposition 6, \mathcal{Q}_n is homeomorphic to \mathcal{Q}_{σ_n} . Let L' be the leaf in \mathcal{Q}_{σ_n} corresponding to L . By Proposition 18, the leaf L' is repelling in \mathcal{Q}_{σ_n} under dynamics of \hat{f}_{σ_n} and has at least three accesses to $\hat{\infty}$. Let \hat{p}' be the periodic point in L' under \hat{f}_{σ_n} . Then $p' := \pi(\hat{p}')$ is a periodic point in $J(f_{\sigma_n})$ such that there are at least three rays landing at p' by the construction of the homeomorphism between \mathcal{Q}_n and \mathcal{Q}_{σ_n} . Now, there exists a periodic point p in $J(f_c)$ whose ray portrait is the same as p' . In particular, p has at least three rays landing. Let Z_n as in the proof of Proposition 15. Then $p \in \mathbb{C} \setminus Z_n$ and L contains a periodic lift of p . Thus L is repelling. ■

Now we are ready to prove the Main Theorem:

Proof of Main Theorem. By Corollary 16, the map h extends to a homeomorphism of natural extensions \tilde{h} , with $\tilde{h}(\hat{\infty}) = \hat{\infty}$. Since $Im(c) \neq 0$ then there exists a repelling

leaf L in \mathcal{N}_c such that L has at least three access to $\hat{\infty}$. This is a topological property, so $h(L)$ is also a leaf with at least three access to $\hat{\infty}$. By Lemma 19, $h(L)$ is also repelling and moreover $Im(c') \neq 0$. In this way, \tilde{h} sends a repelling leaf into a repelling leaf. By an isotopy argument similar to the one used in the proof of Lemma 14, we can see that this implies that h sends repelling leaves into repelling leaves. Hence, \tilde{h} satisfies the conditions of Theorem 9, which implies that f_c and $f_{c'}$ belong to the same combinatorial class. ■

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