

NON-TRIVIAL WANDERING DOMAINS FOR HETERODIMENSIONAL CYCLES

SHIN KIRIKI, YUSHI NAKANO, AND TERUHIKO SOMA

Dedicated to Professor Iku Nemoto on his 70th birthday.

ABSTRACT. We present a sufficient condition for three-dimensional diffeomorphisms having heterodimensional cycles to be approximated arbitrarily well by diffeomorphisms with non-trivial contracting wandering domains via several perturbations. The key idea is to show that diffeomorphisms with heterodimensional cycles associated with saddle points with non-real eigenvalues can be approximated by diffeomorphisms with generalized homoclinic tangencies presented by Tatjer. The generalized homoclinic tangency is an organizing center including a Bogdanov-Takens bifurcation, by which one can obtain non-trivial contracting wandering domains together with a Denjoy-like construction.

1. INTRODUCTION

1.1. **Main result.** In this paper we study the existence of non-trivial wandering domains in nonhyperbolic dynamics. Here a *non-trivial wandering domain* for a given map f on a manifold M means a non-empty connected open set $D \subset M$ which satisfies the following conditions:

- $f^i(D) \cap f^j(D) = \emptyset$ for every $i, j \geq 0$ with $i \neq j$;
- the union of the ω -limit sets of points in D for f , denoted by $\omega(D, f)$, is not equal to a single periodic orbit.

See [13, p. 36] for the original definition in the one-dimensional case. A wandering domain D is called *contracting* if the diameter of $f^n(D)$ converges to zero as $n \rightarrow \infty$. In the early 20th century, Bohl [4] and Denjoy [10] constructed examples of C^1 diffeomorphisms on a circle which have contracting wandering domains in which the union of the ω -limit sets of points is a Cantor set. Following these results, similar phenomena were observed for high dimensional examples, see [21, 16, 26, 7, 27, 23]. On the other hand, the absence of wandering domains is the key of classification of one-dimensional unimodal as well as one-dimensional multimodal maps, in real analytic category, which were developed in [14, 24, 3, 13, 33], see the survey of van Strien [32]. For wandering domains of rational maps of the Riemann sphere, see [30, 25]. Moreover, Berry and Mestel [2] showed that any C^1 Lorenz map without gaps does not admit a wandering domain, but the corresponding assertion for the contracting case with gaps has not been shown yet, see [15].

Topics about the existence of non-trivial wandering domains in nonhyperbolic dynamics were first studied by Colli-Vargas [9] for some two-dimensional example

Date: November 13, 2018.

2010 Mathematics Subject Classification. Primary 37G25; 37C20; 37C29, Secondary 37C70.

Key words and phrases. heterodimensional cycle, wandering domain, Hopf bifurcation, Tatjer condition.

which is made up of an affine thick horseshoe with C^2 -persistent homoclinic tangencies. Moreover, their conjecture was recently proved to be true by the first and third authors [20, Theorem A]: any two-dimensional diffeomorphism in any Newhouse open set is contained in the C^r ($2 \leq r < \infty$) closure of diffeomorphisms having contracting non-trivial wandering domains for which the union of the ω -limit sets of points contains a basic set. This result moreover implies an affirmative answer in the C^r ($2 \leq r < \infty$) category to one of the open problems of van Strien [32] which is concerned with the existence of wandering domains for the Hénon family, see [20, Corollary B]. Compare with a negative answer in the C^ω category given in [28].

There is another well-studied nonhyperbolic phenomenon different from a homoclinic tangency, which is called a heterodimensional cycle. We say that a diffeomorphism has a *heterodimensional cycle* associated with saddle periodic points if there are saddle periodic points P and Q for the diffeomorphism such that

$$W^u(P) \cap W^s(Q) \neq \emptyset, W^u(Q) \cap W^s(P) \neq \emptyset, \text{u-ind}(P) \neq \text{u-ind}(Q),$$

where $\text{u-ind}(\cdot)$ is the dimension of the unstable bundle, called the *unstable index*, for a corresponding periodic point. Thus a natural question is the following:

Question. *Let f be a diffeomorphism having a heterodimensional cycle. Is the diffeomorphism f contained in the C^r closure of diffeomorphisms having contracting non-trivial wandering domains?*

The next theorem is one of the main results in the present paper, which gives an affirmative answer to the question in the C^1 -category.

Theorem 1.1. *Let f be a diffeomorphism on a three-dimensional manifold which has a heterodimensional cycle associated with two saddle periodic points at which both the derivatives for f have non-real eigenvalues. Then there exists a diffeomorphism g arbitrarily C^1 -close to f such that g has a contracting non-trivial wandering domain D and $\omega(D, g)$ is a nonhyperbolic transitive Cantor set without periodic points.*

Note that, in [9, 20], to construct non-trivial wandering domains near a homoclinic tangency of a 2-dimensional diffeomorphism F , they added a countable series of perturbations to F supported on some open sets which are respectively contained in mutually disjoint gaps in the complement of persistent tangencies. On the other hand in this paper, to show Theorem 1.1 we adopt a quite different procedure as follows:

- step 1:** C^1 -approximate the diffeomorphism f given in Theorem 1.1 by C^r , $r \geq 2$, diffeomorphisms with so called “non-transverse equidimensional cycles”;
- step 2:** C^r -approximate the diffeomorphisms with the non-transverse equidimensional cycles by other diffeomorphisms having so called “generalized homoclinic tangencies”;
- step 3:** Owing to one of Tatjer’s results [31], the generalized homoclinic tangencies lead to Bogdanov-Takens bifurcations, which create invariant circles;
- step 4:** Finally, by using a C^1 Denjoy-like construction along the invariant circle, one can obtain a diffeomorphism g satisfying the conditions as in Theorem 1.1.

In the next subsection, we give essential ingredients to carry out these steps.

1.2. Outline of the paper. Denote by \mathcal{A} the set of all diffeomorphisms which satisfy the assumption of Theorem 1.1. Furthermore denote by \mathcal{Z} the set of all diffeomorphisms which satisfy the conclusion of Theorem 1.1. Then, Theorem 1.1 can be rephrased as follows:

Corollary 1.2. *\mathcal{A} is contained in the C^1 closure of \mathcal{Z} .*

To show Theorem 1.1, we have to prepare auxiliary classes \mathcal{B} , \mathcal{C} , \mathcal{D} of diffeomorphisms on M satisfying the following inclusion relations:

$$\mathcal{A} \subset \overline{(\mathcal{B})}_{C^1}, \quad \mathcal{B} \subset \overline{(\mathcal{C})}_{C^r} \subset \overline{(\mathcal{D})}_{C^r}, \quad \mathcal{D} \subset \overline{(\mathcal{Z})}_{C^1},$$

where $\overline{(\cdot)}_{C^1}$ and $\overline{(\cdot)}_{C^r}$ respectively stand for the C^1 and C^r , $r \geq 2$, closures of the corresponding sets. We will give the definition of each class in the following sections. So, we here briefly explain what role each class plays in the proof of Theorem 1.1.

Section 2 contains step 1 where we show that any element of \mathcal{A} leads to a heterodimensional cycle containing an ‘‘intrinsic tangency’’. It is a non-transverse intersection between some invariant manifold and some leaf of an invariant foliation which is contained in some transverse heterodimensional intersections, see Lemma 2.4. Moreover, the intrinsic tangency yields the class \mathcal{B} of C^r diffeomorphisms for which each of elements has a ‘‘non-transverse equidimensional cycle’’, see Proposition 2.1.

Section 3 corresponds to steps 2 and 3. We here show that \mathcal{B} is contained in the C^r closure of the class \mathcal{C} of C^r diffeomorphisms where every element has a ‘‘generalized homoclinic tangency’’ presented by Tatjer [31]. See Proposition 3.1. In short, the generalized homoclinic tangency is a certain type of non-transverse codimension-two intersection which satisfies the Tatjer condition. Here the Tatjer condition consists of the geometric properties given in (C1)–(C3) of section 3. As a matter of fact, a couple of propositions 2.1 and 3.1 is the key of this paper which directly implies another main result:

Theorem 1.3. *Every element of \mathcal{A} can be arbitrarily C^1 -approximated by C^r diffeomorphisms having a generalized homoclinic tangency.*

Note that Tatjer presents several types of generalized homoclinic tangencies, which provide various phenomena according to the types, e.g., several types of limit return (Hénon-like) maps of renormalizations, birth of attracting or saddle type invariant circles via Bogdanov-Takens bifurcations, the existence of strange attractors and the Newhouse phenomenon. See [31, Theorem 1]. Indeed, by virtue of one of them, we can find the class \mathcal{D} of diffeomorphisms which have attracting invariant circles created by the Bogdanov-Takens bifurcation.

Section 4 contains step 4 where we finally perform a Denjoy-like construction for a normal tubular neighborhood of the attracting invariant circle of any diffeomorphism in \mathcal{D} to detect non-trivial wandering domains.

In closing, we note that all approximations in this paper can be done in C^r -category with any integer $r \geq 2$, except for Lemma 2.2 in step 1 and Proposition 4.2 in step 4.

2. INTRINSIC TANGENCIES OF CYCLES

Let M be a three-dimensional Riemannian manifold and N_1 and N_2 be submanifolds of M . We say that a point $\mathbf{x} \in N_1 \cap N_2$ is a *transverse intersection* if it satisfies $T_{\mathbf{x}}M = T_{\mathbf{x}}N_1 + T_{\mathbf{x}}N_2$. Denote by $N_1 \pitchfork N_2$ the set of all transverse intersections of N_1 and N_2 . On the other hand, a point $\mathbf{y} \in N_1 \cap N_2$ satisfying $T_{\mathbf{y}}M \neq T_{\mathbf{y}}N_1 + T_{\mathbf{y}}N_2$ is called a *tangency*. In this section we consider the set \mathcal{B} of all C^r , $r \geq 2$, diffeomorphisms on M for which any element has a *non-transverse equidimensional cycle*, that is, for any $f \in \mathcal{B}$, there are saddle periodic points P and P' for f satisfying the following conditions:

- (B1) $\text{u-ind}(P) = \text{u-ind}(P') = 2$, and the unstable eigenvalues of P are non-real, while the unstable eigenvalues of P' are real;
- (B2) P and P' are *homoclinically related* to each other, i.e., $W^s(P') \pitchfork W^u(P) \neq \emptyset$ and $W^u(P') \pitchfork W^s(P) \neq \emptyset$;
- (B3) there is a quadratic tangency between $W^s(P')$ and $W^u(P)$.

Here the tangency $\mathbf{y} \in W^s(P') \cap W^u(P)$ is said to be *quadratic* (or a *contact of order 1*) if there exist an arc $\ell \subset W^s(P')$, a regular surface $S \subset W^u(P)$, and some C^2 change of coordinates on an open neighborhood $U(\mathbf{y})$ of \mathbf{y} such that (i) $\mathbf{y} = (0, 0, 0)$, $S = \{(x, y, z) \in U(\mathbf{y}); z = 0\}$; (ii) ℓ has a regular parametrization $\ell(t) = (x(t), y(t), z(t))$ with $\ell(0) = (0, 0, 0)$; (iii) $z'(0) = 0$ and $z''(0) \neq 0$.

The main result of this section is the following proposition.

Proposition 2.1. \mathcal{A} is contained in the C^1 closure of \mathcal{B} .

Since our setting is in dimension three, the following lemma can be immediately obtained from Bonatti-Díaz' result [5, Theorem 2.1].

Lemma 2.2. *Let f be any element of \mathcal{A} which has a heterodimensional cycle associated with P and Q at which both the derivatives have non-real eigenvalues. Then every C^1 neighborhood of f contains a diffeomorphism f_1 with a heterodimensional cycle having real central eigenvalues. Moreover, this cycle for f_1 can be taken associated with saddles P'_{f_1} and Q'_{f_1} which are homoclinically related to the continuations P_{f_1} of P and Q_{f_1} of Q , respectively. \square*

Here the heterodimensional cycle for f_1 is said to have *real central eigenvalues* if there are an expanding real eigenvalue of $Df_1^{\text{Per}(f_1)}(P_{f_1})$ and a contracting real eigenvalue of $Df_1^{\text{Per}(f_1)}(Q_{f_1})$ whose multiplicities are equal to 1. Note that this lemma implies that $\text{u-ind}(P) = \text{u-ind}(P_{f_1})$ and $\text{u-ind}(Q) = \text{u-ind}(Q_{f_1})$. See Figure 2.2 for the configuration of each ingredient in Lemma 2.2.

Remark 2.3. The heterodimensional cycle for f_1 in Lemma 2.2 is *simple* in the sense that the local dynamics in small neighborhoods of P_{f_1} and of Q_{f_1} are linear while the transitions between the neighborhoods are affine and preserve a partially hyperbolic splitting. See [5, Definition 3.4] for the precise description. Moreover the cycle contains a transverse intersection associated with the saddles P'_{f_1} and Q'_{f_1} , see [5, §5].

We here introduce a concept of intrinsic tangencies. Let f be a diffeomorphism on a three-dimensional manifold M having saddle periodic points P and Q with $\text{u-ind}(P) = \text{u-ind}(Q) + 1$. Suppose that all eigenvalues of the derivative $Df^{\text{Per}(Q)}(Q)$ are real. We say that $W^u(P)$ and $W^s(Q)$ have an *intrinsic tangency* if there is a leaf ℓ^{ss} of the C^1 strong stable foliation $\mathcal{F}^{ss}(Q)$ in $W^s(Q)$ such that ℓ^{ss} and $W^u(P)$ have

a tangency. See Figure 2.1-(a). Note that the intrinsic tangency is not necessarily contained in a heterodimensional tangency between $W^u(P)$ and $W^s(Q)$ as shown in Figure 2.1-(b). See [11, 19] for its precise definition. Indeed, it is not difficult to give examples where intrinsic tangencies are contained in $W^u(P) \pitchfork W^s(Q)$, *e.g.*, circular transverse heterodimensional intersections in [11, 19] contain at least two such intrinsic tangencies.

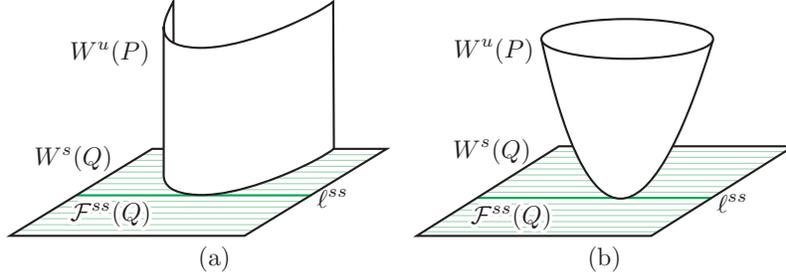


FIGURE 2.1. Intrinsic tangencies in transverse intersection and heterodimensional tangency

Let f be any element of \mathcal{A} which has a heterodimensional cycle associated with saddle periodic points P and Q at which both the derivatives have non-real eigenvalues. One may suppose that $\text{u-ind}(P) > \text{u-ind}(Q)$, $W^u(P) \pitchfork W^s(Q) \neq \emptyset$ and $W^u(Q) \cap W^s(P)$ contains a quasi-transverse intersection. By Lemma 2.2 and Remark 2.3, one obtains a C^1 diffeomorphism f_1 arbitrarily C^1 -close to f which satisfies the following properties:

- (A1) f_1 has saddle periodic points P'_{f_1} and Q'_{f_1} with the following conditions:
 - (i) the eigenvalues of the derivatives of f_1 at P'_{f_1} and Q'_{f_1} are distinct real numbers;
 - (ii) P'_{f_1} is homoclinically related to the continuation P_{f_1} of P , while Q'_{f_1} is homoclinically related to the continuation Q_{f_1} of Q ;
 - (iii) $W^u(P_{f_1}) \cap W^s(Q_{f_1})$ contains a transverse intersection;
- (A2) f_1 has a heterodimensional cycle associated with P'_{f_1} and Q'_{f_1} , *i.e.*,
 - (i) $W^u(P'_{f_1}) \cap W^s(Q'_{f_1})$ contains a transverse intersection;
 - (ii) $W^s(P'_{f_1}) \cap W^u(Q'_{f_1})$ contains a quasi-transverse intersection X' .

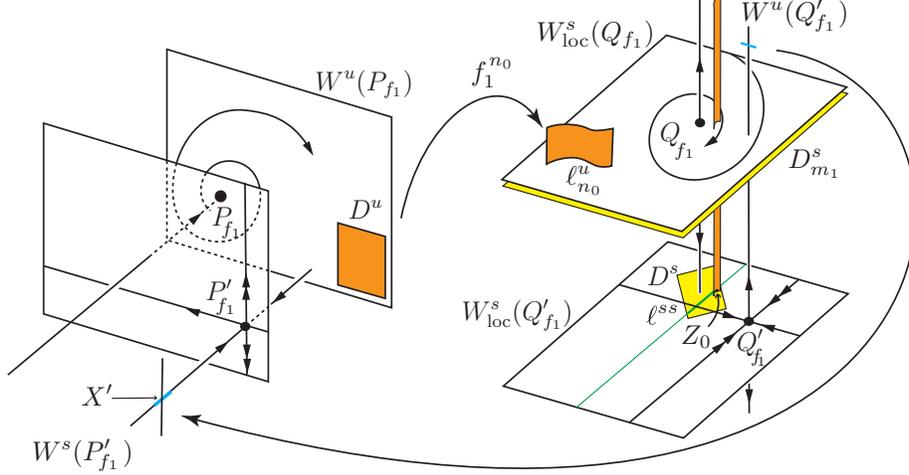
Here X' is called a *quasi-transverse intersection* if it satisfies $T_{X'}W^s(P'_{f_1}) + T_{X'}W^u(Q'_{f_1}) = T_{X'}W^s(P'_{f_1}) \oplus T_{X'}W^u(Q'_{f_1})$. See Figure 2.2.

Lemma 2.4. *Arbitrarily C^1 -close to f_1 satisfying (A1) and (A2), there is a C^r diffeomorphism f_2 satisfying (A1) and (A2) such that $W^u(P_{f_2}) \pitchfork W^s_{\text{loc}}(Q'_{f_2})$ contains an intrinsic tangency, where P_{f_2} and Q'_{f_2} are the continuations of P_{f_1} and Q'_{f_1} , respectively.*

Together with Lemma 2.2 and Remark 2.3, it immediately implies the next result.

Corollary 2.5. *The intrinsic tangency is obtained by an arbitrarily small C^1 -perturbation of any $f \in \mathcal{A}$. \square*

Proof of Lemma 2.4. We here recall that the above $f \in \mathcal{A}$ has the saddle periodic point Q such that $Df^{\text{Per}(Q)}(Q)$ has a pair of non-real contracting eigenvalues.

FIGURE 2.2. Heterodimensional and equidimensional cycles for f_1

Hence, there exist a small neighborhood V of Q_{f_1} , a local chart (x, y, z) in V and real constants $a_s, a_u, \vartheta \in \mathbb{R}$ with $0 < |a_s| < 1 < |a_u|$ such that

- $\overline{V} \cap W^u_{\text{loc}}(P_{f_1}) = \emptyset$ and $\overline{V} \cap \mathcal{O}_{f_1}(X') = \emptyset$, where \overline{V} is the closure of V and $\mathcal{O}_{f_1}(\cdot)$ is the orbit of the corresponding point for f_1 ;
- $Q_{f_1} = (0, 0, 0)$ and

$$(2.1) \quad f_1^{\text{Per}(Q_{f_1})}(x, y, z) = ((x, y) {}^t A_s, a_u z), \quad A_s = a_s \begin{pmatrix} \cos 2\pi\vartheta & -\sin 2\pi\vartheta \\ \sin 2\pi\vartheta & \cos 2\pi\vartheta \end{pmatrix}$$

for any $(x, y, z) \in V$. Moreover, after a small local perturbation if necessary, we may assume that the above ϑ is irrational.

On the one hand, by (A1)-(iii), there exist an unstable disk

$$D^u \subset W^u_{\text{loc}}(P_{f_1}) \cap V^c$$

and a positive integer n_0 such that the set of transverse intersections $f_1^{n_0}(D^u) \pitchfork W^s_{\text{loc}}(Q_{f_1})$ contains an arc $\ell^u_{n_0}$. See Figure 2.2. On the other hand, by (A1)-(ii) and the Inclination Lemma, it follows that $W^s(Q'_{f_1})$ contains two-dimensional disks for which the backward images converge to $W^s_{\text{loc}}(Q_{f_1})$ in C^1 topology. Hence, for any $\epsilon > 0$, there exist an integer $m_0 \geq 0$ and a stable disk $D^s \subset W^s_{\text{loc}}(Q'_{f_1})$ containing a point of $W^u(Q_{f_1}) \pitchfork W^s_{\text{loc}}(Q'_{f_1})$ and such that, for any integer $m \geq 0$,

$$d_{C^1}(D^s_m, W^s_{\text{loc}}(Q_{f_1})) < \epsilon$$

where $d_{C^1}(\cdot, \cdot)$ is the C^1 distance between corresponding submanifolds, and D^s_m is a component of $f_1^{-m\text{Per}(Q_{f_1})-m_0}(D^s) \cap V$.

Note that, by Remark 2.3 together with (A1)-(i), one has the strong stable foliation $\mathcal{F}^{ss}(Q'_{f_1})$ of Q'_{f_1} whose leaves are of codimension one in $W^s(Q'_{f_1})$. Moreover, observe that, by the irrational rotation of (2.1) with $\vartheta \notin \mathbb{Q}$, the images of $\ell^u_{n_0}$ by the forward iterations of $f_1^{\text{Per}(Q_{f_1})}$ rotate and converge to Q_{f_1} in $W^s_{\text{loc}}(Q_{f_1})$, see Figure

2.3-(a). Hence, there are an integer $m_1 \geq 0$ and an arc $\ell^{ss} \subset D^s \cap \mathcal{F}_{\text{loc}}^{ss}(Q'_{f_1})$ such that

- $\ell_{n_0}^u \cap \ell_{m_1}^{ss} \neq \emptyset$ where $\ell_{m_1}^{ss} := \pi^s \left(f_1^{-m_1 \text{Per}(Q_{f_1}) - m_0}(\ell^{ss}) \right)$ for the canonical projection $\pi^s : V \rightarrow W_{\text{loc}}^s(Q_{f_1})$ with $\pi^s(x, y, z) = (x, y, 0)$;
- there is a point $\mathbf{p}_0 \in \ell_{n_0}^u \cap \ell_{m_1}^{ss}$ such that

$$\angle(T_{\mathbf{p}_0} \ell_{n_0}^u, T_{\mathbf{p}_0} \ell_{m_1}^{ss}) < \epsilon,$$

where $\angle(\cdot, \cdot)$ stands for the angle between the corresponding subspaces in $T_{\mathbf{p}_0} W_{\text{loc}}^s(Q_f)$.

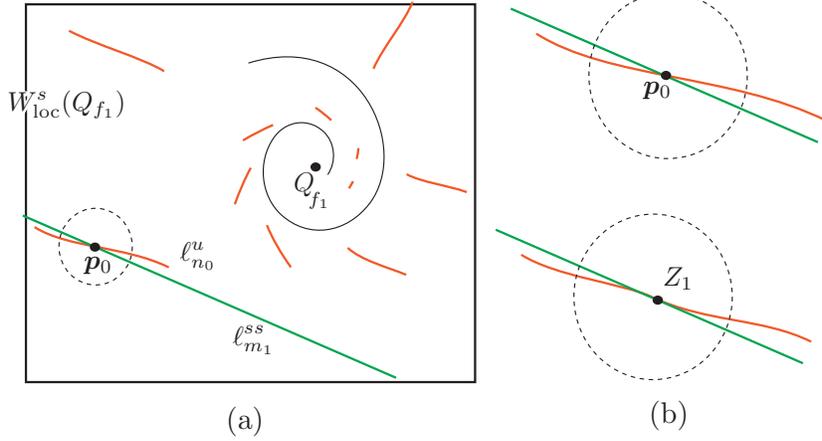


FIGURE 2.3. Images of $\ell_{n_0}^u$ and $\ell_{m_1}^{ss}$

Therefore, one can obtain a diffeomorphism f_2 with a tangency Z_1 near \mathbf{p}_0 between $\ell_{n_0}^u$ and $\ell_{m_1}^{ss}$, if necessary perturbing $f_1^{n_0}$ slightly in a small neighborhood of $f_1^{-n_0}(\mathbf{p}_0)$ in D^u . It follows from the above conditions that such a C^r perturbation deforming $\ell_{n_0}^u$ as shown in Figure 2.3-(b) can be defined by the composition of appropriate bump function and isometry. Thus we have the intrinsic tangency

$$Z_0 := f_2^{m_1 \text{Per}(Q_{f_2}) + m_0}(Z_1)$$

between $W^u(P_{f_2})$ and $W_{\text{loc}}^s(Q'_{f_2})$. This concludes the proof of Lemma 2.4. \square

Proof of Proposition 2.1. Arbitrarily C^1 close to $f \in \mathcal{A}$, from Lemmas 2.2, one has a diffeomorphism f_1 with a heterodimensional cycle associated with saddles P'_{f_1} and Q'_{f_1} with distinct real eigenvalues which are homoclinically related to the continuations P_{f_1} and Q_{f_1} , respectively. Moreover, it follows by Lemma 2.4 and Corollary 2.5 that, arbitrarily C^1 near f_1 , one obtains a C^r diffeomorphism f_2 satisfying (A1) and (A2) which has an intrinsic tangency Z_0 between $W^u(P_{f_2})$ and a leaf ℓ^{ss} of the strong stable foliation $\mathcal{F}^{ss}(Q'_{f_2})$. Note that ℓ^{ss} is almost parallel to $W_{\text{loc}}^{ss}(Q'_{f_2})$ in the linearizing coordinates in a neighborhood of Q'_{f_2} . See Figure 2.4-(a).

Recall that, by (A2), $W_{\text{loc}}^s(P'_{f_2}) \cap W^u(Q'_{f_2})$ contains the quasi-transverse intersection X' as shown in Figure 2.2. Hence one has a positive integer k_0 such that the point $X'_0 = f_2^{k_0}(X')$ is contained in $W_{\text{loc}}^u(Q'_{f_2})$. Note that since $\bar{V} \cap \mathcal{O}_{f_2}(X') = \emptyset$

where V is the neighborhood V of Q_{f_1} , $X'_0 \notin \bar{V}$. We here consider a small segment $L^s \subset W^s(P'_{f_2})$ with $X'_0 \in L^s$ and $L^s \cap \bar{V} = \emptyset$. Observe that if necessary perturbing f_2 slightly in a small neighborhood of L^s , it follows from Inclination Lemma that, for any large integer $m > 0$, the backward image $f_2^{-m\text{Per}(Q'_{f_2})}(L^s)$ contains a segment which is sufficiently C^1 -close to $W_{\text{loc}}^{ss}(Q'_{f_2})$. Denote $f_2^{-m\text{Per}(Q'_{f_2})}(X'_0)$ by X'_m . See Figure 2.4-(b).

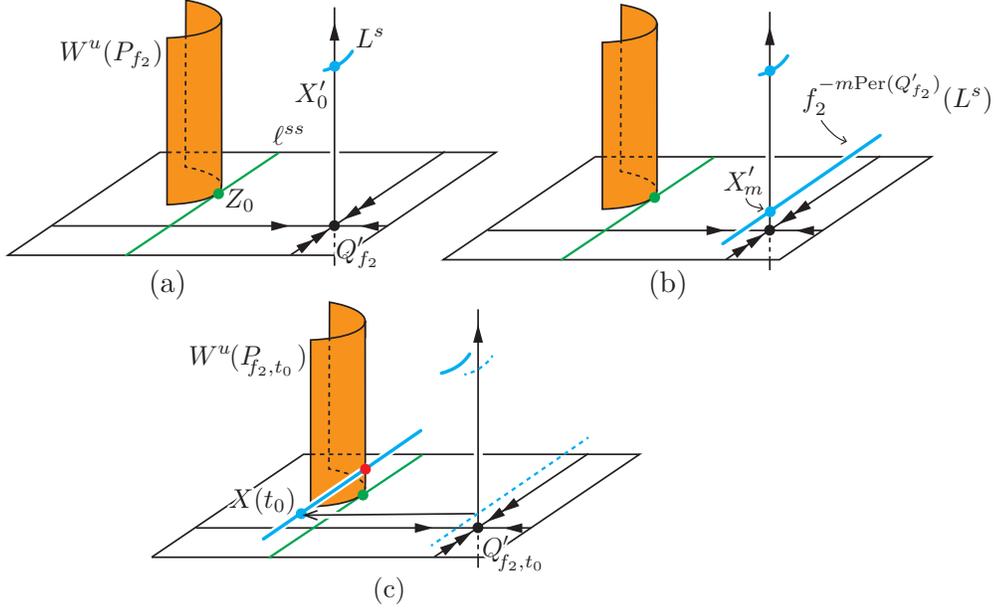


FIGURE 2.4. The appearance of an intrinsic tangency

We here consider a one-parameter family of C^r diffeomorphisms, $\{f_{2,t}\}_{t \in \mathbb{R}}$ with $f_{2,0} = f_2$, which *unfolds generically* the quasi-transverse intersection at X'_m used in [11, 18], that is, there are a C^1 -curve $\{X(t)\}_{t \in \mathbb{R}}$ and a C^1 map $\rho : \mathbb{R} \rightarrow \mathbb{R}^+$ with $X(t) \in f_{2,t}^{k_0+m}(W_{\text{loc}}^u(P_{f_2,t}))$ for every t and $\rho(0) > 0$ such that:

- $X(0) = X'_m$ and $\text{dist}\left(X(t), W_{\text{loc}}^u(Q'_{f_2,t})\right) = |t|\rho(t)$;
- $T_{X'_m} W^s(P'_{f_2,t}) \oplus T_{X'_m} W^u(Q'_{f_2,t}) \oplus N = T_{X'_m} M$, where N is the one-dimensional space spanned by $\frac{dX}{dt}(0)$.

Observe that, there is a real number t_0 arbitrarily near 0 such that $f_{2,t_0}^{-m\text{Per}(Q'_{f_2,t_0})}(L^s)$ and $W^u(P_{f_2,t_0})$ have a quadratic tangency. See Figure 2.4-(c). Let us denote f_{2,t_0} by g . It follows that g satisfies the condition (B3).

Note that it follows from (A1)-(ii) that (B2) holds for g . In addition, since the amount of all perturbations can be taken arbitrarily small, (B1) holds for g . In conclusion, g is contained in \mathcal{B} . This ends the proof of Proposition 2.1. \square

The above proof contains the following remark:

Remark 2.6. *Every C^r diffeomorphism satisfying (A1) and (A2) can be C^r approximated by diffeomorphisms with a non-transverse equidimensional cycle satisfying the conditions (B1), (B2) and (B3).*

3. HOMOCLINIC TANGENCIES WITH THE TATJER CONDITION

We first recall a diffeomorphism with a generalized homoclinic tangency satisfying some conditions presented in [31], which plays an important role in the proof of Theorem 1.1. The purpose of this section is to show Proposition 3.1, where we claim that diffeomorphisms satisfying the Tatjer condition are *not* so special in a neighborhood of \mathcal{B} .

Let f be a diffeomorphism on a three-dimensional Riemannian manifold M which has a homoclinic tangency of a saddle periodic point P' . We here note that one of the requirement in Tajter's conditions is that the central-stable bundle $E^{cs} = E^s \oplus E^c$ at P' must be extended along the stable manifold $W^s(P')$ of P' . See the explanation just below (C2). To guarantee it as well as the quadratic condition in (C1), we have to assume that the regularity of f is at least C^2 . Suppose the derivative for f at P' has real eigenvalues λ_s, λ_{cu} and λ_u satisfying $|\lambda_s| < 1 < |\lambda_{cu}| < |\lambda_u|$. Assume that there are C^1 linearizing coordinates (x, y, z) for f on a neighborhood U' of P' such that

$$P' = (0, 0, 0), \quad f(x, y, z) = (\lambda_{cu}x, \lambda_u y, \lambda_s z)$$

for any $(x, y, z) \in U'$. In U' , the local unstable and stable manifolds of P' are given respectively as

$$W_{\text{loc}}^u(P') = \{(x, y, 0) ; |x|, |y| < \delta\}, \quad W_{\text{loc}}^s(P') = \{(0, 0, z) ; |z| < \delta\}$$

for some $\delta > 0$. Moreover one has the local strong unstable C^1 foliation $\mathcal{F}_{\text{loc}}^{uu}(P')$ in $W_{\text{loc}}^u(P')$ such that, for any point $\bar{x} = (\bar{x}, \bar{y}, 0) \in W_{\text{loc}}^u(P')$, the leaf $\ell^{uu}(\bar{x})$ of $\mathcal{F}_{\text{loc}}^{uu}(P')$ containing \bar{x} is given as

$$\ell^{uu}(\bar{x}) = \{(\bar{x}, y, 0) ; |y| < \delta\}.$$

We say that a homoclinic tangency satisfies the *Tatjer condition* (which corresponds to the type I of case B in [31, Theorem 1]) if the following (C1)–(C3) hold.

- (C1) $W^u(P')$ and $W^s(P')$ have a quadratic tangency at \mathbf{x}_0 which does not belong to the strong unstable manifold $W_{\text{loc}}^{uu}(P')$ of P' ;
- (C2) $W^s(P')$ is tangent to the leaf $\ell^{uu}(\mathbf{x}_0)$ of $\mathcal{F}_{\text{loc}}^{uu}(P')$ at \mathbf{x}_0 .

For the tangency \mathbf{x}_0 , we here consider the forward image $\bar{\mathbf{x}}_0 = f^{n_0}(\mathbf{x}_0)$ for a large $n_0 \geq 0$ satisfying $\bar{\mathbf{x}}_0 \in W_{\text{loc}}^s(P')$. In addition, we consider a plane $S(\bar{\mathbf{x}}_0)$ containing $\bar{\mathbf{x}}_0$ such that $T_{\bar{\mathbf{x}}_0}S(\bar{\mathbf{x}}_0)$ is generated by $(\frac{\partial}{\partial x})_{\bar{\mathbf{x}}_0}, (\frac{\partial}{\partial z})_{\bar{\mathbf{x}}_0} \in T_{\bar{\mathbf{x}}_0}M$. Note that by the chosen linearizing coordinates on a neighborhood of P' , the plane $S(\bar{\mathbf{x}}_0)$ in $T_{\bar{\mathbf{x}}_0}M$ corresponds to the central-stable bundle at this tangent point. See Figure 3.1. The last condition is the following:

- (C3) $S(\bar{\mathbf{x}}_0)$ and $W^u(P')$ are transverse at $\bar{\mathbf{x}}_0$.

Denote by \mathcal{C} the set of all C^r diffeomorphisms on M which have homoclinic tangencies satisfying the Tatjer condition. This set \mathcal{C} is contained in the class of diffeomorphisms having “generalized homoclinic tangencies” defined in [31].

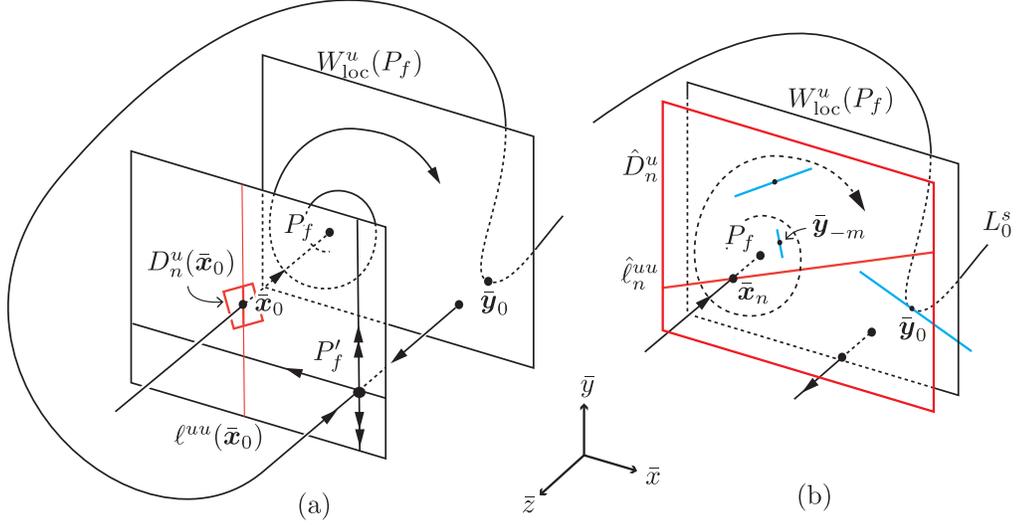


FIGURE 3.2. Non-transverse equidimensional cycle with rotation

n , there is a two-dimensional disk $D_n^u(\bar{x}_0) \subset W^u(P_f')$ containing \bar{x}_0 such that $f^{n\text{Per}(P_f)}(D_n^u(\bar{x}_0))$ converges to $W_{\text{loc}}^u(P_f)$ in the C^r topology as $n \rightarrow \infty$. Write

$$\hat{D}_n^u := f^{n\text{Per}(P_f)}(D_n^u(\bar{x}_0))$$

for each $n > 0$. Note that there is a segment $\ell_n^{uu}(\bar{x}_0)$ contained in the leaf through \bar{x}_0 of the strong unstable foliation $\mathcal{F}^{uu}(P_f')$, which is carried to \hat{D}_n^u by $f^{n\text{Per}(P_f)}$. For each $n > 0$, define

$$\hat{\ell}_n^{uu} := f^{n\text{Per}(P_f)}(\ell_n^{uu}(\bar{x}_0)).$$

See Figure 3.2-(b). Let us take the following steps:

Step 1: Let \mathbf{v}_n^{uu} be a unit vector tangent to $\hat{\ell}_n^{uu}$ at \bar{x}_n . Since the sequence $\{\bar{x}_n\}$ converges to P_f in M as $n \rightarrow \infty$, we have a subsequence $\{\mathbf{v}_{n_i}^{uu}\}$ of $\{\mathbf{v}_n^{uu}\}$ converging to a unit vector $\mathbf{v}_\infty \in T_{P_f}W_{\text{loc}}^u(P_f)$ as $n_i \rightarrow \infty$ in the tangent vector space TM . This implies that, for any $\varepsilon > 0$, there is an integer $n_0 := n_{i_0} > 0$ such that

$$(3.2) \quad |P_f - \bar{x}_{n_0}| < \varepsilon/2, \quad \text{dist}_{TM}(\mathbf{v}_\infty, \mathbf{v}_{n_0}^{uu}) < \varepsilon/2.$$

This finishes the first step. ■

Next, we consider a quadratic tangency $\bar{\mathbf{y}}_0$ between $W^s(P_f')$ and $W_{\text{loc}}^u(P_f)$ which is given by the condition (B3). For every integer $m \geq 0$, we write

$$\bar{\mathbf{y}}_{-m} := f^{-m\text{Per}(P_f)}(\bar{\mathbf{y}}_0).$$

See Figure 3.2-(b).

Step 2: Let L_{-m}^s be a subarc of $W^s(P_f')$ passing through $\bar{\mathbf{y}}_{-m}$ for any integer $m \geq 0$. Consider a unit vector \mathbf{w}_{-m}^s tangent to L_{-m}^s at $\bar{\mathbf{y}}_{-m}$. Since $|b_u| > 1$ in (3.1), $\{\bar{\mathbf{y}}_{-m}\}$ converges to P_f in M as $m \rightarrow \infty$. Since moreover ϑ is irrational, there exists a subsequence $\{m_i\}$ of $\{m\}$ such that $\{\mathbf{w}_{-m_i}^s\}$ converges to \mathbf{v}_∞ in TM . This implies that, for any $\varepsilon > 0$, there is an integer $m_0 = m_{i_0} > 0$ such that

$$(3.3) \quad |P_f - \bar{\mathbf{y}}_{-m_0}| < \varepsilon/2, \quad \text{dist}_{TM}(\mathbf{v}_\infty, \mathbf{w}_{-m_0}^s) < \varepsilon/2.$$

This ends the second step. ■

In the next final step, we combine the above two steps.

Step 3: From (3.2) and (3.3), we have

$$(3.4) \quad |\bar{x}_{n_0} - \bar{y}_{-m_0}| < \varepsilon, \quad \text{dist}_{TM}(\mathbf{v}_{n_0}^{uu}, \mathbf{w}_{-m_0}^s) < \varepsilon.$$

See Figure 3.3-(a). This concludes the third step. ■

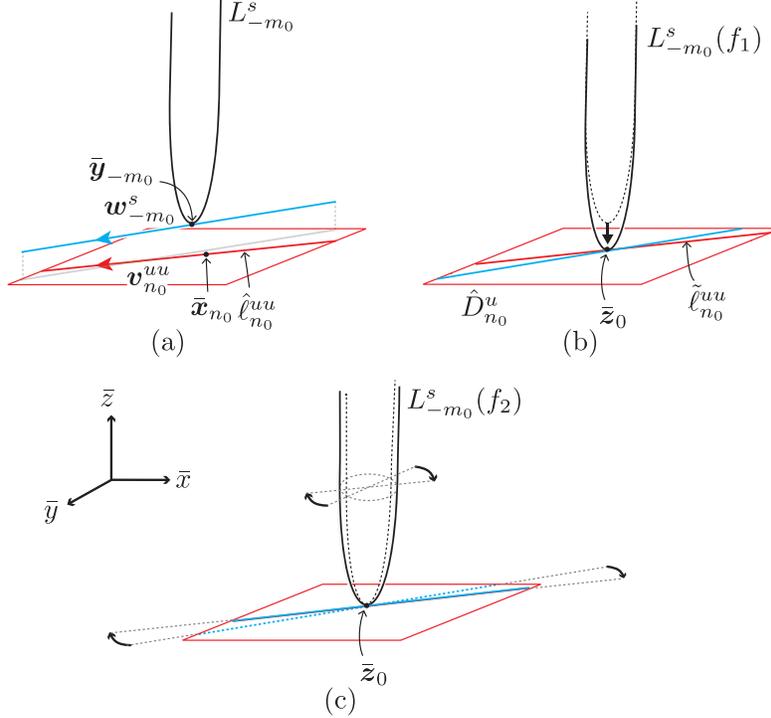


FIGURE 3.3. The advent of tangency of codimension two

Now we add a small perturbation to f on a neighborhood of \bar{y}_0 to obtain a C^r diffeomorphism f_1 which is C^r -close to f and such that the continuation $L^s_{-m_0}(f_1)$ is obtained from $L^s_{-m_0}$ by a “shifting down” operation along the \bar{z} -axis and has a quadratic tangency \bar{z}_0 with $\hat{D}_{n_0}^u$. See Figure 3.3-(b). Let $\tilde{\ell}_{n_0}^{uu}$ be a curve in $\hat{D}_{n_0}^u$ passing through \bar{z}_0 and such that $f_1^{-n_0 \text{Per}(P_{f_1})}(\tilde{\ell}_{n_0}^{uu})$ is contained in one of leaves of $\mathcal{F}^{uu}(P'_{f_1})$. In general, $T_{\bar{z}_0} L^s_{-m_0}(f_1)$ does not coincide with $T_{\bar{z}_0} \tilde{\ell}_{n_0}^{uu}$. However, the condition (3.4) implies that these spaces are sufficiently close to each other. Hence, by adding a perturbation again to f_1 on the neighborhood of \bar{y}_0 , we obtain a C^r diffeomorphism f_2 which is C^r -close to f_1 and such that the continuation $L^s_{-m_0}(f_2)$ of $L^s_{-m_0}(f_1)$ has a quadratic tangency with $\hat{D}_{n_0}^u$ at \bar{z}_0 and satisfies

$$T_{\bar{z}_0} L^s_{-m_0}(f_2) = T_{\bar{z}_0} \tilde{\ell}_{n_0}^{uu},$$

see Figure 3.3-(c). More precisely, $L^s_{-m_0}(f_2)$ is obtained from $L^s_{-m_0}(f_1)$ by a small rotation around the axis meeting $\hat{D}_{n_0}^u$ orthogonally at \bar{z}_0 .

At last we have obtained the C^r diffeomorphism $g := f_2$ which is C^r -near f and such that

- $W^s(P'_g)$ and $\hat{D}_{n_{i_0}}^u$ have a quadratic tangency at \bar{z}_0 ;
- moreover, at the point \bar{z}_0 , $W^s(P'_g)$ and $\tilde{\ell}_{n_0}^{uu}$ have a tangency with

$$c_{\bar{z}_0}(W^s(P'_g), \tilde{\ell}_{n_0}^{uu}) = 2.$$

This implies the condition (C2). Note that, since $\bar{x}_0 \notin W^{uu}(P'_f)$, the quadratic tangency \bar{z}_0 does not belong to $W^{uu}(P'_f)$. It implies that (C1) holds for g . Moreover, since (C3) is the condition about the transversality, it still holds after arbitrarily small perturbations of f . Consequently, g belongs to \mathcal{C} . This completes the proof of Proposition 3.1. \square

In the end of this section, we present the next lemma which is indispensable for our discussion in the final section. Since this is just an extract from the main result of [31], we here skip the proof.

Lemma 3.2 (see the items 1–3 in [31, Theorem 1]). *Let f be a C^r ($r \geq 2$) diffeomorphism on a three-dimensional manifold M which has a homoclinic tangency of a saddle periodic point P' with real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying $|\lambda_1| < 1 < |\lambda_2| < |\lambda_3|$. In addition, suppose that the homoclinic tangency satisfies the Tatjer condition. Then there are a two-parameter family $\{f_{a,b}\}_{a,b \in \mathbb{R}}$ with $f_{0,0} = f$ and a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ of the parameter values with $(a_n, b_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$ such that, for any sufficiently large n , f_{a_n, b_n} has an n -periodic smooth attracting invariant circle.* \square

We remark that the attracting invariant circles in Lemma 3.2 are really generated by the Hopf (also known as the Neimark-Sacker) bifurcation for three-dimensional diffeomorphisms, which is a part of the codimension-two bifurcation called Bogdanov-Takens, see Broer et al [8], and also [31, §2.4].

Addendum (about strange attractors/ininitely many sinks). By using other results in [31, Theorem 1], one can show that any element of \mathcal{C} can be approximated by diffeomorphisms having strange attractors or infinitely many sinks. However, by [19, Theorem C] together with [18, Corollary B], both the phenomena can be directly derived from the existence of intrinsic tangencies in the proof of Proposition 2.1 without detouring to the constructing of generalized homoclinic tangencies. That is, any element of \mathcal{A} can be approximated by diffeomorphisms having these phenomena. \square

4. CONSTRUCTIONS OF WANDERING DOMAINS

Let \mathcal{D} be the set of all C^r diffeomorphisms on a three-dimensional manifold M having smooth attracting invariant circles which are given by the Hopf bifurcation as in the conclusion of Lemma 3.2. Here the regularity r should be at least 5 so that it can be expressed by the following normal form (4.1) of the Hopf bifurcation, see [29, §7.5]. More specifically, for any $f \in \mathcal{D}$, there exists a one-parameter family $\{f_\mu\}_{\mu \in \mathbb{R}}$ of C^r diffeomorphisms such that f_0 is arbitrarily C^5 -close to f and f_μ undergoes the generic Hopf bifurcation at an n -periodic point \mathbf{p} for some integer $n \geq 0$ for $\mu = 0$ which creates an attracting invariant circle. In fact, one has polar

coordinates (r, θ) and a real coordinate t in a small neighborhood of \mathbf{p} such that $\mathbf{p} = (0, 0, 0)$ and

$$(4.1) \quad f_\mu^n(r, \theta, t) = ((1 + \mu)r - a_\mu r^3 + O_\mu(r^4), \theta + \beta_\mu + O_\mu(r^2), \gamma t)$$

where $O_\mu(r^k)$ is a smooth function of order r^k with $k \in \mathbb{N}$ near $(r, \mu) = (0, 0)$ which depends on μ smoothly, a_μ, β_μ are real constants depending on μ smoothly with $a_0 > 0$, and γ is a real constant with $0 < |\gamma| < 1$. Notice that f restricted to the $r\theta$ -space has the same form as the normal form of the original two-dimensional Hopf bifurcation, see [29, §7.5] or [22, §§4.6–4.7] for more details. Observe that, as $\mu > 0$, this form has a saddle periodic point at $\mathbf{p} = (0, 0, 0)$ and has an attracting invariant circle surrounding \mathbf{p} of radius $(\mu a_\mu^{-1})^{1/2} + O(\mu)$.

Using the notations \mathcal{C} and \mathcal{D} , one can rewrite Lemma 3.2 as follows:

Corollary 4.1. *\mathcal{C} is contained in the C^1 closure of \mathcal{D} .*

The next result is the final step for the proof of Theorem 1.1.

Proposition 4.2. *Let f be a diffeomorphism contained in \mathcal{D} . There is a diffeomorphism g arbitrarily C^1 -close to f such that g has a contracting non-trivial wandering domain D for which $\omega(D, g)$ is a transitive nonhyperbolic Cantor set without periodic points.*

Proof. For $f \in \mathcal{D}$, one has a C^r diffeomorphism f_μ arbitrarily C^1 -close to f such that f_μ^n given by (4.1) has an attracting invariant circle S with $f_\mu^i(S) \cap S = \emptyset$ for any $i = 0, \dots, n-1$. By perturbing f_μ slightly if necessary, we may assume that $\beta_\mu/(2\pi) \notin \mathbb{Q}$. Moreover, since μ is close to 0, we have a diffeomorphism \tilde{f} C^1 -near f_μ such that

$$(4.2) \quad \tilde{f}^n(r, \theta, t) = ((1 + \mu)r - a_\mu r^3, \theta + \beta_\mu, \gamma t)$$

in the neighborhood of \mathbf{p} of radius $2(\mu a_\mu^{-1})^{1/2}$. By this it follows that there is an attracting invariant circle \tilde{S} and the restriction of \tilde{f}^n to \tilde{S} is an irrational rigid rotation. Hence, by the same construction as that of Denjoy's counter-example, see [10] and [17], we have a C^1 diffeomorphism g arbitrarily C^1 -close to \tilde{f} , a sequence $\{\ell_i\}_{i \geq 0}$ of open arcs which are contained in a circle S_g sufficiently C^1 -close to \tilde{S} and satisfy the following conditions:

- S_g is an attracting invariant circle for g^n , and $g^n|_{S_g}$ is a rigid irrational rotation;
- for any $i, j \geq 0$ with $i \neq j$,

$$(4.3) \quad g^n(\ell_i) = \ell_{i+1}, \quad \ell_i \cap \ell_j = \emptyset;$$

- $\omega(\ell_0, g^n)$ is a topologically transitive Cantor set on S_g without periodic points, where g^n has zero Lyapunov exponent.

We here consider a normal tubular neighborhood of each arc ℓ_i , see Figure 4.1, which is defined as

$$D_i := \bigcup_{x \in \ell_i} \Delta_i(x),$$

where $\Delta_i(x)$ is the open disk of radius δ centered at $x \in \ell_i$ which lies in a plane normal to ℓ_i for each $i \geq 0$, where $\delta > 0$ is a given small number independent of x and i . By the form of (4.1), the restrictions of $f_\mu^n(r, \theta, t)$ to the first and third

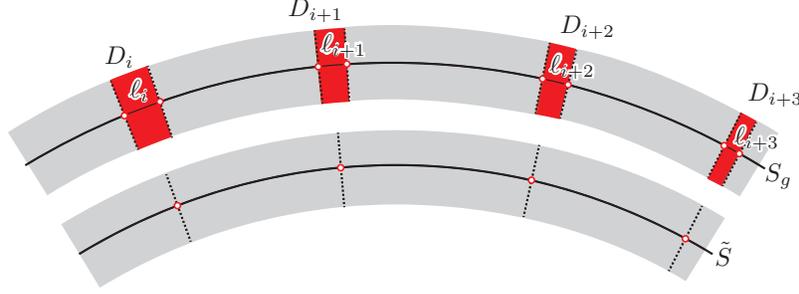


FIGURE 4.1. Denjoy-like construction

entries are contracting maps. It follows from this fact together with the wandering condition (4.3) that

$$g^n(D_i) \subset D_{i+1}, \quad D_i \cap D_j = \emptyset$$

for every $i, j \geq 0$ with $i \neq j$. In consequence, the open set D_0 is a contracting non-trivial wandering domain for g^n . Since $g^i(S_g) \cap S_g = \emptyset$ for $i = 1, \dots, n-1$, D_0 is a contracting non-trivial wandering domain also for g . \square

We are at last ready to show the main result of this paper.

Proof of Theorem 1.1. By Propositions 2.1, 3.1, and Corollary 4.1,

$$\mathcal{A} \subset \overline{(\mathcal{B})}_{C^1}, \quad \mathcal{B} \subset \overline{(\mathcal{C})}_{C^r} \subset \overline{(\mathcal{D})}_{C^r}.$$

Hence, for any $f \in \mathcal{A}$, there is an $\hat{f} \in \mathcal{D}$ arbitrarily C^1 -close to f . By Proposition 4.2, we obtain a diffeomorphism g arbitrarily C^1 -close to \hat{f} which has a contracting non-trivial wandering domain. It implies that g is an element of \mathcal{L} . This completes the proof. \square

ACKNOWLEDGEMENTS.

This paper was partially supported by JSPS KAKENHI Grant Numbers 25400112 and 26400093. The authors thank hospitality of the Kyoto Dynamical Seminar. The authors also thank helpful comments and suggestions by Alexandre Rodrigues and the anonymous referees.

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DEPARTMENT OF MATHEMATICS, TOKAI UNIVERSITY, 4-1-1 KITAKANAME, HIRATUKA, KANAGAWA, 259-1292, JAPAN

E-mail address: kiriki@tokai-u.jp

KITAMI INSTITUTE OF TECHNOLOGY, 165 KOEN-CHO, KITAMI, HOKKAIDO 090-8507, JAPAN

E-mail address: nakano@mail.kitami-it.ac.jp

DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES, TOKYO METROPOLITAN UNIVERSITY, MINAMI-OHSAWA 1-1, HACHIOJI, TOKYO 192-0397, JAPAN

E-mail address: tsoma@tmu.ac.jp