Symplectic invariants of parametric singularities

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Dedicated to Shing Tung Yau on the occasion of his 60-th birthday.

Abstract

We introduce the basic symplectic invariants of singular curves and surfaces: symplectic codimension, symplectic-isotropic codimension, symplectic defect and the number of isotropic double points. Their algebraic representations are constructed and relations between these invariants are derived. For isotropic multi-germs of maps from $\mathbf{C}^2$ to $\mathbf{C}^4$ the number of open umbrellas as a new invariant is introduced and its relation with Segre number of the image variety is found.

1 Introduction

We consider the classification problem for mappings to the symplectic space. The symplectomorphism classification problem is motivated naturally from Hamilton dynamics, the theory of differential equations, and differential geometry ([1]). For instance, the symplectic classification of Hamilton-Jacobi equations $V \subset T^*\mathbf{R}^n = \mathbf{R}^{2n}$ is of importance in the theory of first order PDE. Even for second order PDE, our classification problem appears in the study of singularities for generalized geometric solutions to symplectic Monge-Ampère equations: Given an n-form Ω on $T^*\mathbf{R}^n = \mathbf{R}^{2n}$, $f : (\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0)$ is called a generalized geometric solution to the Monge-Ampère equation associated to Ω if $f^*\omega = 0$, $f^*\Omega = 0$, where ω is the symplectic form. For example, for the MA-equation Hessian = constant, we take $\Omega = c \cdot dx_1 \wedge \cdots \wedge dx_n - dp_1 \wedge \cdots \wedge dp_n$. Moreover, the investigation

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of singularities of special Lagrangian varieties requires the basic symplectic singularity theory of parametrized Lagrangian varieties (cf. [6]). In fact $\Omega = \sum_{i=1}^{n} dx_1 \wedge \cdots \wedge dp_i \wedge \cdots \wedge dx_n - dp_1 \wedge \cdots \wedge dp_n$ gives the MA-equation Laplacian = Hessian of special Lagrangian varieties.

In general, in differential geometry, we represent a local solution surface to a differential system $\mathcal{I}$ on a manifold M in a parametric form $f : (\mathbf{R}^n, 0) \to M$ which satisfies $f^*\mathcal{I} = 0$. We identify solution surfaces up to parametrizations (right equivalence). The solution surface is called regular if it is represented by an immersion. Otherwise it is called *singular*. In the case of symplectic Monge-Ampère equations as above, the system $\mathcal{I}$ is generated by the symplectic form ω and Ω. Thus we are motivated to investigate the basic theory on singular parametric surfaces $f : (\mathbf{R}^m, 0) \to (T^*\mathbf{R}^n, 0)$ up to the natural equivalence relation, *symplectic equivalence*.

1.1 Symplectic equivalence.

Let $\omega = \sum_{i=1}^{n} dp_i \wedge dx_i$ be the standard symplectic form on $\mathbf{K}^{2n} = T^*\mathbf{K}^n$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Mappings are assumed to be real analytic or C^∞ for $\mathbf{K} = \mathbf{R}$ and complex analytic for $\mathbf{K} = \mathbf{C}$. Multi-germs $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ and $f' : (\mathbf{K}^m, S') \to (\mathbf{K}^{2n}, 0)$ to the symplectic space are called *symplectomorphic* (resp. *diffeomorphic*, *homeomorphic*) if the diagram

$$\begin{array}{ccc} (\mathbf{K}^m, S) & \xrightarrow{f} & (\mathbf{K}^{2n}, 0) \\ \sigma \downarrow & & \downarrow \tau \\ (\mathbf{K}^m, S') & \xrightarrow{f'} & (\mathbf{K}^{2n}, 0) \end{array}$$

is commutative for some diffeomorphism-germ σ and some symplectomorphism-germ τ, $\tau^*\omega = \omega$ (resp. for some diffeomorphism-germs σ, τ, for some homeomorphism-germs σ, τ). Here S, S' are finite sets.

For a map-germ $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$, the diffeomorphism class of the pull-back form $f^*\omega$ on $(\mathbf{K}^m, S)$ of the symplectic form ω is an obvious symplectic invariant of f: If f and f' are symplectomorphic, then $f^*\omega$ and $f'^*\omega$ are diffeomorphic, that is, for a diffeomorphism $\sigma : (\mathbf{K}^m, S) \to (\mathbf{K}^m, S')$, we have $\sigma^*(f'^*\omega) = f^*\omega$. We call $f^*\omega$ the *geometric restriction* of ω by f. In this connection, we mention a theorem which contains the classical Darboux theorem as the special case $m = 0$:

Theorem 1.1 (Darboux-Givental [4]) *Two immersion-mono-germs $f, f' : (\mathbf{K}^m, 0) \to (\mathbf{K}^{2n}, 0)$ are symplectomorphic if and only if the geometric restrictions $f^*\omega$ and $f'^*\omega'$ are diffeomorphic.*

Thus in the *non-singular case* (the case of immersion-mono-germs), the classification problem is reduced to that of the geometric restrictions of the symplectic form to the sources. Note that the pull-backs of symplectic forms are not arbitrary. To explain this, recall the standard notions: A submanifold M in the symplectic space $(\mathbf{K}^{2n}, \omega)$ is called coisotropic (resp. isotropic, symplectic) if the skew-orthogonal in $\mathbf{K}^{2n}$ to each tangent space T_pM, $p \in M$, to M contains T_pM (resp. the geometric restriction $\omega|_M$ is zero, $\omega|_M$ is symplectic). By the classical Darboux theorem, for a coisotropic submanifold, the local diffeomorphism class of the geometric restriction $\omega|_M$ is determined by just the dimension of M. Moreover, we know that a non-singular hypersurface is coisotropic. Then we have

Corollary 1.2 *All non-singular hypersurface-germs in $\mathbf{K}^{2n}$ are symplectomorphic. All coisotropic (resp. isotropic, symplectic) submanifold-germs of fixed dimension in $\mathbf{K}^{2n}$ are symplectomorphic.*

Note that all immersion-germs on a fixed dimensional source are diffeomorphic in our sense.

In the *singular case*, however, even if f and f' are diffeomorphic and $f^*\omega$ and $f'^*\omega$ are diffeomorphic, f and f' are not necessarily symplectomorphic. Therefore the symplectic classification is very different from the differential classification.

A mapping f is called *isotropic* if $f^*\omega = 0$, that is, if $\sum_{i=1}^n d(p_i \circ f) \wedge d(x_i \circ f) = 0$. If $m = 1$, then any germ $f : (\mathbf{K}, S) \to (\mathbf{K}^{2n}, 0)$ is isotropic. Moreover if $f : \mathbf{K}^n \to \mathbf{K}^{2n}, m = n$, then we often call isotropic f *Lagrangian*.

For the class of Lagrangian map-germs $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^{2n}, 0)$, the basic theory is established by Givental [14].

Let $N \subset (\mathbf{K}^{2n}, 0)$ be a germ of analytic variety. We assume the regular locus of N is dense in N. Consider de Rham complex (Ω^*_{2n}, d), the algebra of germs of differential forms on $(\mathbf{K}^{2n}, 0)$ and the exterior differential $d : \Omega^*_{2n} \to \Omega^*_{2n}$. Then de Rham complex $(\Omega^*(N), d)$ for N is defined as the quotient cochain complex of (Ω^*_{2n}, d) by the differential graded ideal $I^*(N)$ consisting of differential forms vanishing on the regular locus of N and the cohomology algebra $H^*(N) = H^*(\Omega^*_{2n}, d)$ from the cochain complex $(\Omega^*(N), d)$.

We call (N, ω) is *Lagrangian* if $\dim N = n$ and the restriction of a symplectic form ω to the regular locus of N vanishes. If (N, ω) is Lagrangian and $\omega = d\alpha$, then we have the well-defined cohomology class $[\alpha]$ in $H^1(N)$, which is called the *characteristic class* of (N, ω).

We call N *reduced* if it is not a product of an analytic set and a non-singular manifold of positive dimension. Then we have:

Theorem 1.3 ([14]) *Let (N, ω) be a reduced Lagrangian variety for a symplectic form $\omega = d\alpha$ on $(\mathbf{K}^{2n}, 0)$. Then any Lagrangian variety (N, ω') is symplectomorphic to (N, ω), provided the symplectic form $\omega' = d\alpha'$ is sufficiently near ω and $[\alpha'] = [\alpha] \in H^1(N)$.*

In general (N, ω) and (N', ω') are called *symplectomorphic* if there exists a diffeomorphism-germ $T : (\mathbf{K}^{2n}, 0) \to (\mathbf{K}^{2n}, 0)$ satisfying $T(N) = N'$ and $T^*\omega' = \omega$.

Moreover Givental ([14]) shows that, if N is quasi-homogeneous (for a positive weight), then de Rham complex $(\Omega^*(N), d)$ is acyclic (see also [10]). Therefore we have

Theorem 1.4 ([14]) *Suppose $N \subset (\mathbf{K}^{2n}, 0)$ is reduced and quasi-homogeneous. Then any Lagrangian (N, ω) and (N, ω') are symplectomorphic, provided $\mathbf{K} = \mathbf{C}$.*

Suppose, in the parametric form, two map-germs $f, f' : (\mathbf{K}^m, 0) \to (\mathbf{K}^{2n}, 0)$ are diffeomorphic for a fixed symplectic form ω by (σ, τ). If f, f' are symplectomorphic, then $(f(\mathbf{K}^n), \omega)$ and $(f(\mathbf{K}^n), \tau^*\omega)$ are symplectomorphic. Moreover, under the condition that f is a normalization of the image, if $(f(\mathbf{K}^n), \omega)$ and $(f(\mathbf{K}^n), \tau^*\omega)$ are symplectomorphic, then f and f' are symplectomorphic (cf. [23]).

Corollary 1.5 *Suppose two isotropic map-germs $f, f' : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$ are diffeomorphic. Assume that f is a normalization of the image, which is reduced, and f, f' are quasi-homogeneous for the same weight. Then f and f' are symplectomorphic.*

Remark 1.6 Corollary 1.5 does not hold in the case $\mathbf{K} = \mathbf{R}$. As a trivial example, $(t^3, t^5) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is diffeomorphic but not symplectomorphic to $(t^3, -t^5)$, since it is *chiral* (see [22]).

Example 1.7 ([19]) Let $f : (\mathbf{K}^2, 0) \to (\mathbf{K}^4, 0)$ be isotropic. Suppose f is diffeomorphic to

$$f_{\mathrm{ou}}(t, u) := \left(t^2, u, ut, \frac{2}{3}t^3 \right) = (x_1, x_2, p_1, p_2).$$

Then f is symplectomorphic to f_{ou} (Whitney's open umbrella). Moreover for any n there exists a class of open umbrellas characterised by the symplectic structural stability, and for them the Darboux-type theorem holds.

Note that the Darboux-type theorem follows from Givental's theory (Corollary 1.5) directly in the case $\mathbf{K} = \mathbf{C}$. The method is applied also to the case $\mathbf{K} = \mathbf{R}$.

Another generalization of Darboux-Givental theorem ([4]) to a singular case is given by the following result:

Theorem 1.8 ([11]) *For any $N, N' \subset \mathbf{K}^{2n}$ quasi-homogeneous, (N, ω) and (N', ω') are symplectomorphic if and only if the algebraic restrictions $[\omega]_N$ and $[\omega']_{N'}$ are diffeomorphic.*

The algebraic restriction $[\omega]_N$ is defined as the residue class of ω modulo the differential ideal $J^*(N) \subset \Omega^*_{2n}$ generated by functions vanishing on N. Note that $J^*(N) \subset I^*(N)$. We set $\Omega^*_{\text{alg}}(N) = \Omega^*_{2n}/J^*(N)$ and $H^*_{\text{alg}}(N) = H^*(\Omega^*_{\text{alg}}(N), d)$. Therefore $[\omega]_N \in H^2_{\text{alg}}(N)$. Note that there exists the canonical surjection $\pi : \Omega^*_{\text{alg}}(N) \to \Omega^*(N)$ of cochain complexes.

1.2 The problem of symplectic classification.

For the classification of curves in a symplectic space $\mathbf{K}^{2n}$ $(m = 1, n \geq 2)$, Arnold initiated the investigation on the difference between diffeomorphism and symplectic classifications ([2]). Then Kolgushkin [29] has completed the symplectic classification of simple multi-germs $(\mathbf{C}, S) \to (\mathbf{C}^{2n}, 0)$. Moreover Domitrz [9] has given several results on symplectic classification of multi-germs of curves by the method of algebraic restrictions.

Restricting ourselves to the case $m = n = 1$, namely to planar mono-curves $(\mathbf{K}, 0) \to (\mathbf{K}^2, 0)$, we have given both symplectic and differential exact classifications of differentially simple and uni-modal plane curve singularities, and clarified the difference between the differential and symplectic classifications ([22][24][26]). In our formulation, we do not fix diffeomorphism types but fix homeomorphism types of plane curve singularities. Actually we fix Puiseux characteristics and then we have symplectic classification results in a unified manner (§2.1).

Roughly distinguishing the classification problems in the presence of various geometric structures, we observe that there are, at least, two types:
(V) Classification of mappings and varieties, and
(D) Classification of differential forms and dynamical systems.

For classifications of type (V), we have finite lists of simplest objects and finite dimensional moduli for complicated objects. Moreover the finite determinacy holds, except for an infinite codimensional set of objects.

On the other hand, for classifications of type (D), we have finite lists of simplest objects, but functional moduli for complicated objects. The finite determinacy does not hold for objects of finite codimension.

Therefore, we ask whether our symplectic classification problem falls into type (V) or (D). Actually, this depends on the class of mappings we treat: The classification of isotropic (or Lagrangian) varieties (or mappings) under symplectomorphisms falls into type (V), and, in fact, several finiteness theorems are proved for them [20][22][23][24]. These results clarify the difference between geometric and algebraic restrictions.

On the other hand the classification of coisotropic varieties under symplectomorphisms falls into type (D). A map-germ $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ to the symplectic space $(\mathbf{K}^{2n}, \omega)$ is called *coisotropic* if $m \geq n$ and f lifts to an isotropic mapping $\tilde{f} : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n} \times \mathbf{K}^{2k}, 0) = (\mathbf{K}^{2m}, 0)$, with $k = m - n$. Here we regard $\mathbf{K}^{2m}$ as a symplectic space with the symplectic form $\omega \ominus \eta = \pi_1^*\omega - \pi_2^*\eta$ for the canonical symplectic form η of $\mathbf{K}^{2k} = T^*\mathbf{K}^k$ and projections $\pi_1 : \mathbf{K}^{2n} \times \mathbf{K}^{2k} \to \mathbf{K}^{2n}$, $\pi_2 : \mathbf{K}^{2n} \times \mathbf{K}^{2k} \to \mathbf{K}^{2k}$. Then the classification of coisotropic map-germs $(\mathbf{K}^3, 0) \to (\mathbf{K}, 0)$ has functional moduli ([25]).

1.3 Basic invariants for classification.

For the exact classification problem of singularities, the notion of *codimension* is the most basic one to measure the complexity or degeneracy of singularities. For instance, the classification of a class of singularities of mappings proceeds from small codimension to large. In general, for a map-germ $f : (\mathbf{K}^n, S) \to (\mathbf{K}^p, 0)$, the $\mathcal{A}_e$-codimension of f is defined by

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} V_f/[f_*(V_S) + (V_p) \circ f],$$

the dimension of the quotient of the infinitesimal deformations of f by those induced from right-left equivalences, $f_* = tf : V_S \to V_f$, $wf(\eta) = \eta \circ f$, $wf : V_p \to V_f$, [33][39]. We often write $\text{cod}(f) = \mathcal{A}_e\text{-cod}(f)$ briefly. The codimension $\mathcal{A}_e\text{-cod}(f)$ is finite if and only if f is finitely $\mathcal{A}$-determined. Moreover the codimension is estimated by other geometric invariants such as 0-stable invariants in terms of “disentanglement” ([27][35][36]). For instance, the $\mathcal{A}_e$-codimension of an $\mathcal{A}$-finite germ $f : (\mathbf{C}, S) \to (\mathbf{C}^2, 0)$ is estimated as

$$\mathcal{A}_e\text{-cod}(f) \leq \delta(f) - r + 1, \quad \cdots\cdots (*)$$

where $r = \#S$ and $\delta(f) = \dim_{\mathbf{K}} \mathcal{O}_S/f^*\mathcal{O}_2$, the number of double points of f. Here $\mathcal{O}_S$ (resp. $\mathcal{O}_n$) denotes the $\mathbf{K}$-algebra of C^∞ or holomorphic function-germs on $(\mathbf{K}, S)$ (resp. $(\mathbf{K}^n, 0)$). Moreover the equality holds if and only if f is quasi-homogeneous ([37]). See also [8][16].

Let $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$ be a multi-germ of isotropic mapping (or Lagrangian immersion with singularities). Then we set

$$\text{sp-cod}(f) = \dim_{\mathbf{K}} VI_f/[f_*(V_S) + (VH_{2n}) \circ f],$$

and call it the *symplectic codimension* (or the *symplectic-isotropic codimension*) of $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$. Here VI_f is the space of infinitesimal *isotropic* deformations of f:

$$VI_f = \{v : (\mathbf{K}^m, S) \to T\mathbf{K}^{2n} \mid v^*\dot{\omega} = 0, \pi \circ v = f\},$$

for the natural symplectic lifting $\dot{\omega}$ of ω on $T\mathbf{K}^{2n}$, $\dot{\omega} = \sum_{i=1}^{n} d\varphi_i \wedge dx_i + dp_i \wedge d\xi_i$ for the coordinates $(x, p; \xi, \varphi)$ of $T\mathbf{K}^{2n}$, and $\pi : T\mathbf{K}^{2n} \to \mathbf{K}^{2n}$ is the bundle projection. Moreover we denote by VH_{2n} the space of holomorphic Hamiltonian vector fields over $(\mathbf{K}^{2n}, 0)$, and by V_S the space of holomorphic vector fields over $(\mathbf{K}^n, S)$. The symplectic codimension sp-cod(f) is regarded as the minimal number of parameters for "the symplectically versal isotropic unfolding"of f, if f is of corank one.

1.4 Symplectic classification of plane curves.

In the case $n = 1$, any planar curve $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$ is isotropic and the notion of the *symplectic codimension* of f is given by

$$\text{sp-cod}(f) = \dim_{\mathbf{K}} V_f/[f_*(V_S) + (VH_2) \circ f].$$

It is introduced in [22] and shown to be equal to $\delta(f) = \dim_{\mathbf{K}} \mathcal{O}_1/f^*\mathcal{O}_2$, the number of double points, in the case of mono-germs, $r = 1$. The result is easily generalized to multi-germs, for general r, and in fact we have

$$\text{sp-cod}(f) = \delta(f) - r + 1,$$

for a multi-germ $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$. Therefore Mond's formula (*) is rewritten as

$$\mathcal{A}_e\text{-cod}(f) \leq \text{sp-cod}(f),$$

and the difference $sd(f) = \text{sp-cod}(f) - \mathcal{A}_e\text{-cod}(f)$ was called the *symplectic defect*, which measures the difference of symplectomorphism and diffeomorphism classifications, i.e. the dimension of the symplectic moduli space. It is also called the *symplectic multiplicity* in [11].

A curve-germ f is called *p-modal*, for a non-negative integer p, if a finite number of s-parameter families ($0 \leq s \leq p$) of diffeomorphism classes form a neighborhood of f in the space of parametric curve-germs. A 0-modal (resp. 1-modal, 2-modal) singularity is called a *simple* (resp. *uni-modal, bi-modal*) singularity. Note that the notion of modality on the holomorphic map-germs $(\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ differs from that defined for function-germs on their images (the modality on the space of equations $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$). Let $F : (\mathbf{C} \times \mathbf{C}^\delta, 0) \to (\mathbf{C}^2 \times \mathbf{C}^\delta, 0)$ be a symplectically mini-versal unfolding. Set $F(t, u) = (f_u(t), u)(u \in \mathbf{C}^\delta)$. Then the symplectic moduli space of f is obtained as a quotient space of a component $\{u \in \mathbf{C}^\delta \mid \delta(f_u) = \delta(f)\}$ in $\mathbf{C}^\delta$ of the δ-constant locus in $\mathbf{C}^\delta$.

In [7], Bruce and Gaffney classified the simple singularities of parametric plane curve-germs (under the diffeomorphism equivalence) into the classes $A_{2\ell}$, $E_{6\ell}$, $E_{6\ell+2}$, W_{12}, W_{18} and $W^\#_{1,2\ell-1}$ ($\ell = 1, 2, 3, \dots$). In [22][24][26], following O. Zariski's Puiseux expansion technique ([41]), we obtain the symplectic classification of simple and uni-modal singularities of parametric plane-curve germs, i.e. we construct the orbit structure of simple and uni-modal diffeomorphism singularities under the symplectomorphic equivalence. Some of results are reviewed in §2 in this paper.

The diffeomorphism classifications for simple multi-germs of curves are developed in [3][30][31]. The symplectic classification of them has been given in [29]. Moreover the notions of *full-simplicity* and *full-modality* have been introduced by Zhitomirskii [42], which is natural via global problem of mappings. Then it would be natural to ask the symplectic classification of full-simple and full-uni-modal singularities of multi-germs of curves $(\mathbf{C}, S) \to (\mathbf{C}^{2n}, 0)$.

1.5 New symplectic invariants in higher dimensions.

For $n \geq 2$, there is no such simple relation between the $\mathcal{A}_e$-codimension and the symplectic codimension, because the symplectic-isotropic codimension indicates the codimension in a subspace of map-germs of an orbit of a subgroup of $\mathcal{A}$. To measure the difference between symplectomorphism equivalence and diffeomorphism equivalence for isotropic map-germs we introduce another symplectic invariant diff-cod(f) = diff-cod$_I(f)$, the differential-isotropic codimension instead of the symplectic-isotropic codimension sp-cod(f) = sp-cod$_I(f)$ of f. Then we set

$$\mathrm{sd}(f) = \text{sp-cod}(f) - \text{diff-cod}(f).$$

We give an algebraic description of $\mathrm{sd}(f)$ and show that both sp-cod(f) and diff-cod(f) are $\mathcal{A}$-invariants, hence so is $\mathrm{sd}(f)$.

In this paper, we consider new geometric symplectic invariants of isotropic mappings for $\mathbf{K} = \mathbf{C}$. If a multi-germ of isotropic mapping $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank ≤ 1, and sp-cod$(f) < \infty$, then f can be perturbed to a symplectically stable isotropic mapping $\tilde{f}$ whose singularities consist of open umbrellas and transverse self-intersection points (double points). See §4. The number of transverse self-intersection points of the perturbation $\tilde{f}$ does not depend on the perturbation. It is called *the number of isotropic double points* of f and denoted by $\delta_I = \delta_I(f)$. Note that, for $n = 1$, $\delta_I(f) = \delta(f)$.

We give a relation between the two symplectic invariants sp-cod$(f) =$ sp-cod$_I(f)$ and $\delta_I(f)$ for isotropic map-germs $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$. Moreover, we introduce another invariant $u_I(f)$, the number of open umbrellas, for isotropic map-germs $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ and provide a relation of $\delta_I(f)$ and $u_I(f)$ with the Segre number of the image variety of f using Gaffney's result [13].

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2 Singularities of planar curves

First we recall symplectic classification of planar curves obtained in the series of papers [22][24][26] and related results to provide insights for the study of the higher dimensional case. Let $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$ be a multi-germ of planar curve. We assume that the base point set S consists of r points.

Theorem 2.1 *Let $f : (\mathbf{K}, S) \to (\mathbf{K}^2, 0)$ be an $\mathcal{A}$-finite plane curve with r components. Then* sp-cod(f) *and $\delta(f)$ are both finite and we have*

$$\text{sp-cod}(f) = \delta(f) - r + 1,$$

where $r = \#S$ and $\delta(f) = \dim_{\mathbf{C}} \mathcal{O}_S / f^* \mathcal{O}_2$, the number of double points of a stable perturbation of f.

Proof: We denote by $J_S \subset \mathcal{O}_S$ the ideal of $\mathcal{O}_S$ consisting of the functions which vanish on S, the Jacobson radical of $\mathcal{O}_S$. If $S = \{x_0\}$, $r = 1$, then $J_S = m_{x_0} \subset \mathcal{O}_{x_0}$, the unique maximal ideal. For each $v = v_1 \left(\frac{\partial}{\partial x} \circ f\right) +$

$v_2\left(\frac{\partial}{\partial p} \circ f\right) \in V_f$, we take the unique function $h \in J_S$ ("generating function") such that

$$dh = v_2 d(x \circ f) - v_1 d(p \circ f) \ (= v^*\dot{\theta}),$$

the pull-back of the Louville 1-form on $T\mathbf{K}^2$ by $v : (\mathbf{K}, S) \to T\mathbf{K}^2 \cong T^*\mathbf{K}^2$. Then the generating function h belongs to

$$\mathcal{R}_f = \{h \in \mathcal{O}_S \mid dh \in \langle d(x \circ f), d(p \circ f)\rangle_{\mathcal{O}_S}\}.$$

Thus we have a linear mapping $e : V_f \to \mathcal{R}_f \cap J_S$. Clearly the mapping e is surjective. Moreover we have $e|_{f_*(V_S)} = 0$ and $e(X_H \circ f) = (H - H(0)) \circ f$, for the Hamiltonian vector field X_H with the Hamiltonian $H \in \mathcal{O}_2$. Then we have an exact sequence of vector spaces

$$0 \to \frac{V'_f}{f_*(V_S)} \to \frac{V_f}{f_*(V_S) + (VH_2) \circ f} \to \frac{\mathcal{R}_f \cap J_S}{f^* m_2} \to 0,$$

where V'_f is the space of vector fields along f having zero generating functions.

Let $S = \{s_1, \dots, s_r\}$. Denote by f_i the germ of f at s_i. Assume that the order of f_i at s_i is equal to k_i. Then we have $V'_f/f_*(V_S) \cong \oplus_{i=1}^r V'_{f_i}/f_{i*}(V_{s_i})$ and it has dimension $\sum_{i=1}^r (k_i - 1)$ over $\mathbf{K}$. On the other hand $\mathcal{O}_S/(\mathcal{R}_f \cap J_S) \cong \oplus_{i=1}^r \mathcal{O}_{s_i}/m_{s_i}^{k_i}$ and it has dimension $\sum_{i=1}^r k_i$ over $\mathbf{K}$. Thus we have

$$\begin{aligned}
\text{sp-cod}(f) &= \dim_{\mathbf{K}} \frac{V_f}{f_*(V_S) + (VH_2) \circ f} \\
&= \dim_{\mathbf{K}} \frac{V'_f}{f_*(V_S)} + \dim_{\mathbf{K}} \frac{\mathcal{R}_f \cap J_S}{f^* m_2} \\
&= \dim_{\mathbf{K}} \frac{\mathcal{O}_S}{\mathcal{R}_f \cap J_S} - r + \dim_{\mathbf{K}} \frac{\mathcal{R}_f \cap J_S}{f^* m_2} \\
&= \dim_{\mathbf{K}} \frac{\mathcal{O}_S}{f^* m_2} - r \ = \ \dim_{\mathbf{K}} \frac{\mathcal{O}_S}{f^* \mathcal{O}_2} - r + 1 \ = \ \delta(f) - r + 1.
\end{aligned}$$

□

Remark 2.2 If we set

$$\mathcal{G}_f = \{h \in \mathcal{O}_S \mid dh \in \langle d(x \circ f), d(p \circ f)\rangle_{f^*\mathcal{O}_2}\},$$

then we have

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} \frac{\mathcal{O}_S}{\mathcal{G}_f}.$$

Moreover

$$\mathrm{sd}(f) = \dim_{\mathbf{K}} \frac{\mathcal{G}_f}{f^*\mathcal{O}_2} - r + 1.$$

Note that $\mathcal{O}_S, \mathcal{R}_f$ and $\mathcal{G}_f$ are defined via the exterior derivative and any locally constant functions belong to them, which is not the case for $f^*\mathcal{O}_2$.

In general, for each homeomorphism class of planar curves, the symplectic moduli space is mapped canonically onto the differential moduli space. The dimension of the fiber over a diffeomorphism class $[f]$ equals $\mathrm{sd}(f)$. It is known that $\mathrm{sd}(f) = \mu(f) - \tau(f)$, where $\mu(f) = 2\delta(f)$ is the Milnor number of f and $\tau(f)$ is the Tyurina number of f ([38][32][10]). Let $s(f)$ be the symplectic modality, that is, the number of parameters in the symplectic normal form of f. Moreover let $c(f)$ be the codimension of the locus in the parameter space corresponding to germs diffeomorphic to f. Then $s(f) - c(f) = \mathrm{sd}(f)$. Thus we have the formula, even for multi-germs, for the Tyurina number (by means of Varchenko-Lando's formula):

$$\tau(f) = 2\delta(f) + c(f) - s(f).$$

See [24][26] for details.

2.1 Puiseux characteristics

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$, $f(t) = (x(t), y(t))$, be a germ of holomorphic parametric plane curve. Let m be the minimum of the order of $x(t)$ and that of $y(t)$ at $t = 0$. Then, using a re-parametrization and the symplectomorphism $(x, y) \mapsto (y, -x)$ if necessary, we see that f is symplectomorphic to $(t^m, \sum_{k=m}^{\infty} a_k t^k)$. Suppose $m \geq 2$, that is, f is not an immersion.

Set $\beta_1 = \min\{k \mid a_k \neq 0, m \nmid k\}$ and let e_1 be the greatest common divisor of m and β_1. Inductively set $\beta_j = \min\{k \mid a_k \neq 0, e_{j-1} \nmid k\}$, and let e_j be the greatest common divisor of β_j and e_{j-1}, $j \geq 2$. Then $e_{q-1} > 1$, $e_q = 1$ for a sufficiently large q, and we call $(m = \beta_0, \beta_1, \beta_2, \dots, \beta_q)$ the *Puiseux characteristic* of f. The Puiseux characteristic is a basic diffeomorphism invariant, and it determines exactly the homeomorphism class of f ([41]). For example, setting $e_0 = m$, we have the number of double points $\delta(f)$ described by $\delta(f) = \frac{1}{2}\sum_{j=1}^{q}(\beta_j - 1)(e_{j-1} - e_j)$ ([34][40]).

Then f is symplectomorphic to a germ of the form

$$(t^m, t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k).$$

In [26], we characterize simple and uni-modal singularities by means of their Puiseux characteristics using an infinitesimal method :

Lemma 2.3 *Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ be a curve-germ with Puiseux characteristic $(m, \beta_1, \dots)$. If $m = 4$ and $\beta_1 \geq 13$, or $m = 5$ and $\beta_1 \geq 9$, or $m \geq 6$, then the modality of f is at least 2.*

2.2 Symplectic normal forms

Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ be a holomorphic map-germ.

We briefly recall the theory developed in [22], which is applied to the complex analytic case: The *symplectic codimension* of f is defined by

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} \frac{V_f}{tf(V_1) + wf(VH_2)}$$

as an infinitesimal symplectic invariant of Mather's type. Here V_f is the space of germs of holomorphic vector fields $v : (\mathbf{C}, 0) \to T\mathbf{C}^2$ along f, which is the space of infinitesimal deformations of f, V_1 the space of germs of holomorphic vector fields over $(\mathbf{C}, 0)$ and VH_2 the space of germs of holomorphic Hamiltonian vector fields over $(\mathbf{C}^2, 0)$. The homomorphisms $tf : V_1 \to V_f$ and $wf : VH_2 \to V_f$ are defined by $tf(\xi) := f_*(\xi),\ \xi \in V_1$ and $wf(\eta) := \eta \circ f$ respectively as in §2.1. Here we do not assume $\xi(0) = 0$ nor $\eta(0,0) = 0$.

An unfolding $F : (\mathbf{C} \times \mathbf{C}^r, (0,0)) \to (\mathbf{C}^2 \times \mathbf{C}^r, (0,0))$ of f, $F(t,u) = (f_u(t), u)$, is *symplectically versal* if $\frac{\partial f_u}{\partial u_1}(t,0), \dots, \frac{\partial f_u}{\partial u_r}(t,0)$ generate V_f, over $\mathbf{C}$, up to the space $tf(V_1) + wf(VH_2)$ of deformations which are covered by symplectomorphisms ([22], Proposition 7.1).

For example, a symplectically versal unfolding of the germ $f(t) = (t^5, t^6)$ of type N_{20}, $\delta(f) = 10$, is given by

$$\begin{cases} x &= t^5 + \mu_1 t^3 + \mu_2 t^2 + \mu_3 t, \\ y &= t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14} + \nu_1 t^4 + \nu_2 t^3 + \nu_3 t^2 + \nu_4 t, \end{cases}$$

with 10-parameters $\mu_1, \mu_2, \mu_3, \lambda_1, \lambda_2, \lambda_3, \nu_1, \nu_2, \nu_3, \nu_4$.

The symplectically mini-versal unfolding is unique up to symplectomorphism of unfoldings.

In general, some parameters of the symplectically versal unfolding correspond to deformations into less singular germs, and the remaining parameters provide the symplectic normal form within a given *equi-singular class* up to discrete symplectomorphic equivalence. For instance, in the above example, setting $\mu_1 = \mu_2 = \mu_3 = 0, \nu_1 = \nu_2 = \nu_3 = \nu_4 = 0$, we have the symplectic normal form for N_{20}.

Let f be of Puiseux characteristic $(m, \beta_1, \ldots, \beta_q)$. A monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$ can be calculated by considering the *order semigroup*

$$S(f) = \{\mathrm{ord}(k) \mid k \in f^*\mathcal{O}_2\} \subseteq \mathbf{N}.$$

In fact $\{t^r \mid r \in \mathbf{N} \setminus S(f),\ r > 0\}$ forms a monomial basis of $\mathcal{O}_1/f^*\mathcal{O}_2$.

Then we have the following general result ([26]) on symplectic classification via the order semigroup:

Theorem 2.4 *Let $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$, $f(t) = (t^m,\ t^{\beta_1} + \sum_{k=\beta_1+1}^{\infty} b_k t^k)$, be a germ of Puiseux characteristic $(m, \beta_1, \ldots, \beta_q)$. Let $r_1+m, \ldots, r_s+m (r_1 < \cdots < r_s)$ be all elements of $\mathbf{N} \setminus S(f)$ with $r_j > \beta_1 (1 \le j \le s)$. Then f is symplectomorphic to*

$$f_\lambda(t) = (t^m,\ t^{\beta_1} + \lambda_1 t^{r_1} + \lambda_2 t^{r_2} + \cdots + \lambda_s t^{r_s})$$

for some $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbf{C}^s$.

A family $f_\lambda(t) (\lambda \in \mathbf{C}^s)$, is called a *symplectic normal form* for the Puiseux characteristic $(m, \beta_1, \ldots, \beta_q)$ if any plane curve-germ of Puiseux characteristic $(m, \beta_1, \ldots, \beta_q)$ is symplectomorphic to $f_\lambda(t)$ for some $\lambda \in \mathbf{C}^s$. And those $\lambda \in \mathbf{C}^s$ for which f_λ is symplectomorphic to a given plane branch form a discrete subset of $\mathbf{C}^s$.

If there exists a symplectic normal form, then we have a surjective mapping of $\mathbf{C}^s$ into the space of symplectic moduli with discrete fibers.

Then we have the following results on symplectic normal forms

Proposition 2.5 *Under the same notation as in Theorem 2.4, we have the following:*

(1) *If the Puiseux characteristic is (m, β_1), then the family*

$$f_\lambda(t) = (t^m,\ t^{\beta_1} + \lambda_1 t^{r_1} + \cdots + \lambda_s t^{r_s}),$$

$\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbf{C}^s$, is a symplectomorphic normal form.

(2) *If the Puiseux characteristic is $(4, 6, 2\ell + 5)$, then $s = \ell + 1$ and $r_1 = 7, r_2 = 9, \ldots, r_{\ell-1} = 2\ell + 3, r_\ell = 2\ell + 5, r_{\ell+1} = 2\ell + 7$. Within the family*

$$f_c(t) = (t^4, t^6 + c_1 t^7 + c_2 t^9 + \cdots + c_{\ell-1} t^{2\ell+3} + c_\ell t^{2\ell+5} + c_{\ell+1} t^{2\ell+7}),$$

the subfamily

$$f_\lambda(t) = (t^4, t^6 + \lambda_1 t^{2\ell+5} + \lambda_2 t^{2\ell+7}),$$

$\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2, \lambda_1 \neq 0$*, is a symplectic normal form.*

(3) *If the Puiseux characteristic is* $(4, 10, 2\ell + 9)$*, then* $s = \ell + 4$ *and* $r_1 = 11, r_2 = 13, r_3 = 15, \ldots, r_{\ell-1} = 2\ell+7, r_\ell = 2\ell+9, r_{\ell+1} = 2\ell+11, r_{\ell+2} = 2\ell + 13, r_{\ell+3} = 2\ell + 17, r_{\ell+4} = 2\ell + 21$*. Within the family*

$$\begin{aligned} f_c(t) &= (t^4, t^{10} + c_1 t^{11} + c_2 t^{13} + c_3 t^{15} + \cdots + c_{\ell-1} t^{2\ell+7} \\ &\quad + c_\ell t^{2\ell+9} + c_{\ell+1} t^{2\ell+11} + c_{\ell+2} t^{2\ell+13} + c_{\ell+3} t^{2\ell+17} + c_{\ell+4} t^{2\ell+21}), \end{aligned}$$

the subfamily

$$f_\lambda(t) = (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}),$$

$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \mathbf{C}^5, \lambda_1 \neq 0$*, is a symplectic normal form.*

The above Proposition 2.5 implies the following exact list of normal forms under symplectomorphic equivalence:

Theorem 2.6 *A simple or uni-modal singularity* $f : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ *is symplectomorphic to a germ which belongs to one of the following families (called "symplectic normal forms"):*

$$\begin{array}{ll}
A_{2\ell}: & (t^2, t^{2\ell+1}), \\
E_{6\ell}: & (t^3, t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2}: & (t^3, t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12}: & (t^4, t^5 + \lambda t^7), \\
W_{18}: & (t^4, t^7 + \lambda t^9 + \mu t^{13}), \\
W^{\#}_{1,2\ell-1}: & (t^4, t^6 + \lambda t^{2\ell+5} + \mu t^{2\ell+9}), \lambda \neq 0 (\ell = 1, 2, \ldots), \\
N_{20}: & (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}), \\
N_{24}: & (t^5, t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}), \\
N_{28}: & (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}), \\
W_{24}: & (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}), \\
W_{30}: & (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}), \\
W^{\#}_{2,2\ell-1}: & (t^4, t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \\
 & \lambda_1 \neq 0 \ (\ell = 1, 2, \ldots).
\end{array}$$

Considering the symplectomorphism equivalence, we have given the classification of uni-modal planar curve-germs and we observe that there exists the difference (or "quotient") between differential and symplectic classifications:

Theorem 2.7 ([24]) *For planar curves $f : (\mathbf{K}, 0) \to (\mathbf{K}^2, 0)$, symplectic moduli appear from $\mathcal{A}_e$-codim $= 5$ on (E_{12}); while differential moduli appear from $\mathcal{A}_e$-codim $= 8$ on (N_{20}).*

We can say that symplectic moduli appear *earlier* than differential moduli.

For a detailed symplectic classification of planar-mono-germs see [24][26].

3 Symplectic-isotropic codimension

Let κ be a germ of 2-form on $(\mathbf{K}^m, S)$, S being finite. Then we denote by $\mathcal{O}^{\kappa}_{m,2n}$ the set of map-germs $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ with the geometric restriction $f^*\omega = \kappa$.

A deformation f_t of $f_0 = f \in \mathcal{O}^{\kappa}_{m,2n}$ is called *isotropic* if $f_t \in \mathcal{O}^{\kappa}_{m,2n}$, i.e. $f_t^*\omega = f^*\omega$ $(= \kappa)$. Then we set

$$\text{sp-cod}(f) = \dim_{\mathbf{C}} VI_f/[f_*(V_{S,\kappa}) + (VH_{2n}) \circ f],$$

and call it the *symplectic codimension* (or the *symplectic-isotropic codimension*) of $f : (\mathbf{C}^m, S) \to (\mathbf{C}^{2n}, 0)$. Here we set

$$V_{S,\kappa} = \{\xi \in V_S \mid L_\xi \kappa = 0\},$$

the space of vector fields which leave κ invariant. Note that $V_{S,\kappa} = V_S$ if $\kappa = 0$.

Example 3.1 A germ f is coisotropic if and only if $f^*\omega = g^*\eta$ for some $g : (\mathbf{K}^m, S) \to (\mathbf{K}^{2k}, 0)$. A coisotropic map-germ $f : (\mathbf{K}^m, S) \to (\mathbf{K}^{2n}, 0)$ is a *coisotropic map-germ with regular reduction* if g can be taken to be a submersion. Then $\kappa = g^*\eta$ is of constant rank and the *coisotropic deformation* of f is investigated by studying the space $\mathcal{O}^{\kappa}_{m,2n}$. The *characteristic foliation* $\mathcal{F}_f$ is generated by the kernel field defined by $f^*\omega = g^*\eta$. Then any vector field in $V_{S,\kappa}$ preserves $\mathcal{F}_f$.

Now, for an isotropic $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$, we define

$$\text{diff-cod}(f) = \dim_{\mathbf{K}} \frac{VI_f}{f_*(V_S) + (V_{2n} \circ f) \cap VI_f},$$

while

$$\text{sp-cod}(f) = \dim_{\mathbf{K}} \frac{VI_f}{f_*(V_S) + VH_{2n} \circ f},$$

and

$$\mathcal{A}_e\text{-cod}(f) = \dim_{\mathbf{K}} \frac{V_f}{f_*(V_S) + V_{2n} \circ f}.$$

Moreover we set

$$\text{sd}(f) = \text{sp-cod}(f) - \text{diff-cod}(f) \quad (\geq 0),$$

the *symplectic defect* or *symplectic multiplicity* of f.

Note that, for $n = 1$, we have $VI_f = V_f$: any infinitesimal deformation is isotropic.

We define subspaces $\mathcal{O}_S \supseteq \mathcal{R}_f \supseteq \mathcal{G}_f \supseteq f^*\mathcal{O}_{2n}$ by

$$\begin{aligned}
\mathcal{R}_f &= \{e \in \mathcal{O}_S \mid de \in \mathcal{O}_S \cdot f^*(\Omega^1_{2n})\}, \\
\mathcal{G}_f &= \{e \in \mathcal{O}_S \mid de \in f^*(\Omega^1_{2n})\},
\end{aligned}$$

where de is the exterior differential of the function e, Ω^1_{2n} is the space of 1-forms on $(\mathbf{K}^{2n}, 0)$. Then we have algebraic formulae for symplectic invariants.

Theorem 3.2 *Let $n \geq 2$. Let $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$ be isotropic. If f is a normalization of its image and the codimension of non-immersive locus* $\text{cod}_{\mathbf{C}}\Sigma(f) \geq 2$, *then*

$$\begin{aligned}
\text{sp-cod}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{R}_f}{f^*\mathcal{O}_{2n}} - r + 1, \\
\text{diff-cod}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{R}_f}{\mathcal{G}_f}, \\
\text{sd}(f) &= \dim_{\mathbf{K}} \frac{\mathcal{G}_f}{f^*\mathcal{O}_{2n}} - r + 1,
\end{aligned}$$

where $r = \#S$.

Remark 3.3 The mono-germ case of Theorem 3.2 is proved in [23].

Proof of Theorem 3.2: We refer to the proof of Theorem 2.1 in the case $n = 1$. Each isotropic vector field $v \in VI_f$ has the unique generating function $h \in \mathcal{R}_f \cap J_S$ such that $dh = v^*\dot{\theta}$. If we denote by VI'_f is the space of isotropic vector fields with zero generating function, we have the exact sequence

$$0 \to VI'_f \to VI_f \to \mathcal{R}_f \cap J_S \to 0,$$

which induces the exact sequence

$$0 \to \frac{VI'_f}{f_*(V_S)} \to \frac{VI_f}{f_*(V_S) + (VH_{2n}) \circ f} \to \frac{\mathcal{R}_f \cap J_S}{f^* m_{2n}} \to 0.$$

Since the singular locus of f is of codimension ≥ 2, we have $VI'_f = f_*(V_S)$. Thus we see that

$$\frac{VI_f}{f_*(V_S) + (VH_{2n}) \circ f} \cong \frac{\mathcal{R}_f \cap J_S}{f^* m_{2n}}.$$

We claim that

$$\dim_{\mathbf{K}} \frac{\mathcal{R}_f \cap J_S}{f^* m_{2n}} = \dim_{\mathbf{K}} \frac{\mathcal{R}_f}{f^* \mathcal{O}_{2n}} - r + 1.$$

In fact, we consider the linear map $\mathcal{R}_f \to \mathbf{K}^r = \{S \to \mathbf{K}\}$ defined by $h \mapsto h|_S$, and the induced exact sequence

$$0 \to \mathcal{R}_f \cap J_S \to \mathcal{R}_f \to \mathbf{K}^r \to 0.$$

The last sequence induces the exact sequence

$$0 \to \frac{\mathcal{R}_f \cap J_S}{f^* m_{2n}} \to \frac{\mathcal{R}_f}{f^* \mathcal{O}_{2n}} \to \mathbf{K}^r/\mathbf{K} \cong \mathbf{K}^{r-1} \to 0,$$

where $\mathbf{K}^r/\mathbf{K}$ is the quotient by the diagonal translations.

An isotropic vector field v along f belongs to $(V_{2n} \circ f) \cap VI_f$ if and only if its generating function belongs to $\mathcal{G}_f \cap J_S$. Furthermore any element of $\mathcal{G}_f \cap J_S$ is a generating function of $(V_{2n} \circ f) \cap VI_f$. Therefore we have

$$\frac{VI_f}{f_*(V_S) + (V_{2n} \circ f) \cap VI_f} \cong \frac{\mathcal{R}_f \cap J_S}{\mathcal{G}_f \cap J_S}.$$

Moreover we see that the inclusion $\mathcal{R}_f \cap J_S \to \mathcal{R}_f$ induces an isomorphism

$$\frac{\mathcal{R}_f \cap J_S}{\mathcal{G}_f \cap J_S} \cong \frac{\mathcal{R}_f}{\mathcal{G}_f}.$$

Thus we have the remaining equalities. □

Since $\mathcal{R}_f, \mathcal{G}_f$ are defined independently of the symplectic structure, we have:

Corollary 3.4 *For isotropic map-germs* $f : (\mathbf{K}^n, S) \to (\mathbf{K}^{2n}, 0)$*,* sp-cod$(f)$ *and* diff-cod(f) *are differential invariants. Namely, if* f, f' *are diffeomorphic, then* sp-cod$(f) =$ sp-cod(f') *and* diff-cod$(f) =$ diff-cod(f').

4 Symplectic codimension and double points

In what follows we suppose $\mathbf{K} = \mathbf{C}$.

We recall the Artin-Nagata formula (Mumford's formula) [5]: For an $\mathcal{A}$-finite map-germ $f : X = (\mathbf{C}^n, S) \to Y = (\mathbf{C}^{2n}, 0)$, *the number of double points* is given by $\delta(f) = \frac{1}{2}\dim_{\mathbf{C}} \epsilon$, where $\epsilon = \mathrm{Ker}(\mathcal{O}_{X\times_Y X} \to \mathcal{O}_X)$ is the kernel of the induced morphism from the diagonal map $X \to X \times_Y X$ to the fiber product of f. For a map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$, we have as in [13]:

$$\delta(f) = \dim_{\mathbf{C}} \frac{\langle x_1 - \tilde{x}_1, \ldots, x_n - \tilde{x}_n\rangle_{\mathcal{O}_{2n}}}{\langle f_1(x) - f_1(\tilde{x}), \ldots, f_{2n}(x) - f_{2n}(\tilde{x})\rangle_{\mathcal{O}_{2n}}}.$$

Also we have $\delta(f) = \frac{1}{2}\dim_{\mathbf{C}}\ \mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} (\mathcal{O}_X/f^*\mathcal{O}_Y)$. See also [28].

For $n \geq 2$, the inequality $\mathcal{A}_e$-cod$(f) \leq \delta(f) - r + 1$ does not hold in general.

Example 4.1 ([5]): Let $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ be an immersion whose image consists of three planes intersecting transversely to each other at $0 \in \mathbf{C}^4$. Then $\mathcal{A}_e$-cod$(f) = 2$, $\delta(f) = 3$, $\#S = r = 3$,

This example (Mumford example) was constructed so that $\delta(f) \neq \dim_{\mathbf{C}} \mathcal{O}_n/f^*(\mathcal{O}_{2n})$. In fact, $\dim_{\mathbf{C}} \mathcal{O}_n/f^*(\mathcal{O}_{2n}) = 4$ for that example.

On the other hand, Gaffney [13] showed the following: For an $\mathcal{A}$-finite map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n}, 0)$,

$$\begin{aligned}\delta(f) \quad = \quad & \frac{1}{2}\left[\ \mathrm{Segre}_{2n}\langle f_1(x) - f_1(\tilde{x}), \ldots, f_{2n}(x) - f_{2n}(\tilde{x})\rangle_{\mathcal{O}_{2n}}\right.\\ & \qquad \left. - \mathrm{Whitney}(\pi \circ f : (\mathbf{C}^n, 0) \to (\mathbf{C}^{2n-1}, 0))\ \right]\end{aligned}$$

is half of [*the Segre number of the ideal defining the double points in* $\mathcal{O}_{2n} = \mathcal{O}_{\mathbf{C}^n\times\mathbf{C}^n}$ *minus the number of Whitney umbrellas of a generic projection* $\pi : \mathbf{C}^n \to \mathbf{C}^{2n-1}$ *composed with* f].

Now we consider symplectic-isotropic singularities: If an isotropic map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank one and is stable among isotropic perturbations under symplectomorphisms, then f is symplectomorphic to an *open umbrella*, which can be explicitly represented as a polynomial normal form, and projects to the Whitney umbrella (Theorem 1.7, [19]). Note that, though the result was stated in the real C^∞ case, even in the holomorphic and local case, similar results follow.

If an isotropic map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ is of corank ≤ 1 and sp-cod$(f) < \infty$, then f can be perturbed to a symplectically stable isotropic mapping $\tilde{f}$ whose singularities consist of "open umbrellas"(singularities of codimension 2) and transverse self-intersection points (double points). The number of transverse self-intersection points of the perturbation $\tilde{f}$ does not depend on symplectically stable perturbations. It is called *the number of isotropic double points* of f and denoted by $\delta_I = \delta_I(f)$.

We set

$$B_\varepsilon = \{x \in \mathbf{C}^{2n} \mid |x| < \varepsilon\}.$$

Then we have

Proposition 4.2 *Let $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ be a multi-germ of an isotropic mapping of corank ≤ 1 and* sp-cod$(f) < \infty$*. Then a representative $f : f^{-1}(B_\varepsilon) \to \mathbf{C}^{2n}$ can be perturbed to a symplectically stable isotropic mapping $\tilde{f} : \tilde{f}^{-1}(B_\varepsilon) \to \mathbf{C}^{2n}$ whose singularities consist of open umbrellas and transverse double points. The number of double points is independent of the perturbation, provided $\varepsilon > 0$ is sufficiently small.*

We need to show the following to get an algebraic formula for the number of double points after isotropic stable perturbations.

Lemma 4.3 *Let $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ be a multi-germ of an isotropic mapping. If f is of corank ≤ 1 and the isotropic codimension* sp-cod$(f) <$ *∞, then f is a finite mapping, and the sheaf $f_*\mathcal{R}_f/\mathcal{O}_{2n}$ is a coherent $\mathcal{O}_{2n}$-module.*

Proof: Suppose f is isotropic and sp-cod$(f) = \dim_{\mathbf{C}} VI_f/[f_*(V_S) + VH_{2n} \circ f] < \infty$. Then

$$\dim_{\mathbf{C}} \mathcal{R}_f/f^*\mathcal{O}_{2n} = \dim_{\mathbf{C}}(\mathcal{R}_f \cap J_S)/(f^*m_{2n}) - r + 1$$

is finite dimensional over $\mathbf{C}$. Note that the above equality was used in the proof of Theorem 3.2, but it holds under the assumption that f is isotropic. Thus we deduce that $\mathcal{R}_f$ is a finite $\mathcal{O}_{2n}$-module. Moreover suppose that f is of corank ≤ 1. Then we see that f is a finite mapping (see the proof of

Proposition 2.3 of [19] and Remark 2.3 of [17]). Now consider the de Rham complex (Ω, d) of holomorphic differential forms on $(\mathbf{C}^n, S)$ defined by the exterior differential d, and the differential ideal $\mathcal{I}$ generated by the exterior differentials of components of f. Then the induced complex $(\Omega/\mathcal{I}, d)$ is a coherent $\mathcal{O}_n$-module. Then, by the finite coherence theorem (see for instance [15]), $(f_*(\Omega/\mathcal{I}), d)$ is a coherent $\mathcal{O}_{2n}$-module. Thus the 0-th cohomology $f_*\mathcal{R}_f$ is also a coherent $\mathcal{O}_{2n}$-module. (see Proposition 1.1 of [18]). Therefore $f_*\mathcal{R}_f/\mathcal{O}_{2n}$ is a coherent $\mathcal{O}_{2n}$-module as required. □

Example 4.4 Let $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$, S be a set of transverse double points, $\#S = r = 2$. Then $\dim_{\mathbf{C}} \mathcal{R}_f/f^*\mathcal{O}_{2n} = 1$.

Example 4.5 For an open umbrella $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$ (Theorem 1.7), f is of corank one and its singular locus is of codimension 2. Moreover we have $\mathcal{A}_e$-cod$(f) = 1$, $\delta(f) = 1$. The open umbrella is symplectically stable under isotropic deformations. Therefore we have diff-cod(f) = sp-cod$(f) = 0$ and $\delta_I(f) = 0$.

Theorem 4.6 *For an isotropic map-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ of corank one and with* sp-cod$(f) < \infty$*, we have*

$$\dim_{\mathbf{C}} \frac{\mathcal{R}_f}{f^*\mathcal{O}_{2n}} \ \geq \ \delta_I(f).$$

Therefore we have

$$\text{diff-cod}(f) \ \leq \ \text{sp-cod}(f) \ \geq \ \delta_I(f) - r + 1.$$

Proof: For a stable isotropic perturbation $\tilde{f}$ of f, the support of the sheaf $\tilde{f}_*\mathcal{R}_{\tilde{f}}/\mathcal{O}_{2n}$ is the set of double points of $\tilde{f}(\tilde{f}^{-1}B_\varepsilon) = \tilde{V}$ ([12]). Therefore $\delta_I(f)$ is obtained as the sum of the dimensions of $\tilde{f}_*\mathcal{R}_{\tilde{f}}/\mathcal{O}_{2n}$ at the double points. Let $F : (\mathbf{C}^n \times \mathbf{C}, (S, 0)) \to (\mathbf{C}^{2n} \times \mathbf{C}, (0, 0))$, $F(x, t) = (f_t(x), t)$, $f_0 = f$, be an isotropic unfolding of f which induces a stable isotropic perturbation. We denote by D_F the closure of the locus of double points of F. Denote by $\pi : D_F \to \mathbf{C}$ the projection to the parameter space $\mathbf{C}$. Then π is a finite mapping. Moreover the stalk $F_*\mathcal{R}_F/\mathcal{O}_{2n+1}$ at a point $(y, t) \in \mathbf{C}^{2n+1}$ is $\mathbf{C}$-isomorphic to $(f_t)_*\mathcal{R}_{f_t}/\mathcal{O}_{2n}$ at $y \in \mathbf{C}^{2n}$. Therefore $\delta_I(f)$ is obtained as the sum of the dimensions of $F_*\mathcal{R}_F/\mathcal{O}_{2n+1}$ on $\pi^{-1}(t) \subset D_F$

for $t \neq 0$. Thus we have

$$\begin{array}{rcl}\dim_{\mathbf{C}} \mathcal{R}_f/f^*\mathcal{O}_{2n} & = & \dim_{\mathbf{C}} \pi_* \left(F_*\mathcal{R}_F/\mathcal{O}_{2n+1}\right)_0 \\ & \geq & \dim_{\mathbf{C}} \pi_* \left(F_*\mathcal{R}_F/\mathcal{O}_{2n+1}\right)_t \\ & = & \sum_{y\in\pi^{-1}(t)} \dim_{\mathbf{C}} \left(F_*\mathcal{R}_F/\mathcal{O}_{2n+1}\right) = \delta_I(f).\end{array}$$

□

Again we remark that, in the inequality sp-cod$(f) \geq \delta_I(f) - r + 1$, equality holds in the case $n = 1$, but not in general for $n \geq 2$. Therefore, setting

$$i(f) = \text{sp-cod}(f) - (\delta_I(f) - r + 1),$$

it is natural to ask for the interpretation of $i(f)$ in symplectic terms. We remark that the numbers $\delta(f) - r + 1$ and $\delta_I(f) - r + 1$ have a clear topological meaning.

Proposition 4.7 *For $\mathcal{A}$-finite $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$, the disentanglement (the image of a stable perturbation) is homotopically equivalent to the bouquet of $\delta(f) - r + 1$ circles. For an isotropic $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ of corank ≤ 1 with* sp-cod$(f) < \infty$ *the isotropic disentanglement (the image of an isotropically stable perturbation) is homotopically equivalent to the bouquet of $\delta_I(f) - r + 1$ circles.*

Proof: The image of each $2n$-ball of $\tilde{f}^{-1}B_\varepsilon$ has, as a deformation retract, a finite tree with vertices which are double points of $\tilde{f}$. Thus the perturbed image is homotopically equivalent to a compact 1-dimensional complex. Therefore $\tilde{f}(\tilde{f}^{-1}B_\varepsilon)$ is homotopically equivalent to $\bigvee^m S^1$ for some m. Moreover we have

$$\chi(\tilde{f}(\tilde{f}^{-1}B_\varepsilon)) = r\chi(D^2n) - \delta = r - \delta.$$

Hence $\chi = 1 - m$. Thus we have $m = \delta - r + 1$. □

Remark 4.8 Any open umbrella $V \subset (\mathbf{C}^{2n}, 0)$ has local trivial topology: $(\mathbf{C}^{2n}, V, 0)$ is homeomorphic to $(\mathbf{C}^{2n}, \mathbf{C}^n, 0)$.

5 Symplectic invariants of surfaces

First we observe

Lemma 5.1 *For an isotropic map-germ $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ of corank ≤ 1,* sp-cod$(f) < \infty$ *if and only if* $\mathcal{A}_e$-cod$(f) < \infty$.

Remark 5.2 The similar result to Lemma 5.1 for $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ with $n \geq 3$ never hold. In fact the three dimensional open umbrella $f : (\mathbf{C}^3, 0) \to (\mathbf{C}^6, 0)$ has 1-dimensional singular locus, therefore $\mathcal{A}_e$-cod$(f) = \infty$, while sp-cod$(f) = 0$ ([19]). Note that map-germs $(\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, 0)$ with $\mathcal{A}_e$-cod$(f) < \infty$ must be immersive off S.

Proof of Lemma 5.1 : The condition $\mathcal{A}_e$-cod$(f) < \infty$ is characterized by Gaffney's criterion: For any $y \in \mathbf{C}^{2n}$ near 0 and for any finite $S \subset f^{-1}(y)$, the multi-germ $f : (\mathbf{C}^n, S) \to (\mathbf{C}^{2n}, y)$ is $\mathcal{A}$-stable. (For the finite determinacy of map-germs, see the seminal paper [39]). In our case a multi-germ is $\mathcal{A}$-stable if and only if it is an embedding with at most one transversal self-intersection. Actually we have that f satisfies $\mathcal{A}_e$-cod$(f) < \infty$ if and only if f is an embedding off S. For an isotropic map-germs of corank ≤ 1, the symplectic stability is described by the transversality in isotropic jet space to the symplectic orbit as in the ordinary case ([20]). Then, f is perturbed to an isotropic map-germ which is multi-transversal off S to symplectic orbits by an isotropic perturbation of arbitrary higher order. A multi-germ of isotropic mapping is symplectically stable if and only if it is an embedding with at most one transversal self-intersection or at most one open Whitney umbrella (Example 1.7). Actually we have that f satisfies the condition sp-cod$(f) < \infty$ if and only if f is an embedding off S, in the case $n \leq 2$. □

For an isotropic $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ of corank ≤ 1, we can define "the number of open umbrellas" $u_I = u_I(f)$, in addition to $\delta_I = \delta_I(f)$. Then the sum of the number of open umbrellas $u_I(f)$ and the number of isotropic double points $\delta_I(f)$ is equal to the number of double points $\delta(f)$:

$$\delta_I(f) + u_I(f) = \delta(f),$$

because $\delta = 1$ for each open umbrella. Moreover, by the isotropic nature of f, we have

Lemma 5.3 *Let $f : (\mathbf{C}^2, S) \to (\mathbf{C}^4, 0)$ be an isotropic map-germ of corank ≤ 1. Here* $\mathrm{corank}(f) = \max_{s \in S} \mathrm{corank}_s(f)$*. Then,*

$$u_I(f) = \mathrm{Whitney}(\pi \circ f),$$

the number of Whitney umbrellas of a generic projection $\pi : \mathbf{C}^4 \to \mathbf{C}^3$ composed with f.

Proof: Suppose f is of corank 1 at $s_1, \ldots, s_{r'}$ and is immersive at $s_{r'+1}, \ldots, s_r$. Let $\ell_i = f_*(T_{s_i}\mathbf{C}^2) \subset T_0\mathbf{C}^4, 1 \leq i \leq r'$. Take the skew-orthogonal $\ell_i^{\perp} = \{v \in$

$T_0\mathbf{C}^4 \mid \omega(v, \ell_i) = 0\}$ to ℓ_i, which is of dimension 3. Then take any line $\ell \subset T_0\mathbf{C}^4$ such that

$$(\ell_1^\perp \cup \cdots \cup \ell_{r'}^\perp \cup \Pi_{r'+1} \cup \cdots \cup \Pi_r) \cap \ell = \{0\},$$

where $\Pi_j = f_*(T_{s_j}\mathbf{C}^2)$, and take the projection along ℓ as a generic projection. Then, for any isotropic perturbation $\tilde{f}$, the tangent space $\tilde{f}_*(T_p\mathbf{C}^2), p \in (\mathbf{C}^2, S)$. does not contain ℓ. In fact, $\tilde{f}_*(T_p\mathbf{C}^2)$ contains a line ℓ' ($\neq \ell$) near $\ell_1, \ldots, \ell_{r'}, \Pi_{r'+1}, \ldots, \Pi_r$. Moreover $\ell \not\subset (\ell')^\perp$. If $\ell \subset \tilde{f}_*(T_p\mathbf{C}^2)$, then $\omega(\ell, \ell') \neq 0$. This leads to a contradiction, since $\tilde{f}$ is isotropic. Therefore any singular point of $\pi \circ \tilde{f}$ comes from a singular point of $\tilde{f}$. Thus the number of Whitney umbrellas of $\pi \circ \tilde{f}$ is equal to the number of open umbrellas of $\tilde{f}$. □

Therefore we have, by Gaffney's formula,

Proposition 5.4 *For an isotropic map-germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$ with* sp-cod$(f) < \infty$*, we have*

$$\text{Segre}_4 = 2\delta_I + 3u_I.$$

Proof: By Gaffney's formula $2\delta = \text{Segre}_4 - \text{Whitney}(\pi \circ f)$. We have shown that $\delta = \delta_I + u_I$ and $u_I = \text{Whitney}(\pi \circ f)$. Therefore we have

$$\text{Segre}_4 = 2\delta + \text{Whitney}(\pi \circ f) = 2(\delta_I + u_I) + u_I = 2\delta_I + 3u_I.$$

□

Example 5.5 *Consider again the isotropic map-germ*

$$f_{\text{ou}} := (x_1, x_2, p_1, p_2) = \left(t^2,\ u,\ ut,\ \frac{2}{3}t^3\right) : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$$

as in Theorem 1.7. Then we have $\mathcal{R}_f = \mathcal{G}_f = f^*\mathcal{O}_4$. Moreover we have sp-cod$(f_{\text{ou}}) = 0$*,* sd$(f_{\text{ou}}) = 0$*, $\delta_I = 0$, $u_I = 1$, $\delta = 1$,* $\text{Segre}_4 = 3$*.*

Example 5.6 (multiple open umbrella): The isotropic map-germ $f_{\text{mou}}^{\pm}(t, u) := (t^2,\ u,\ t^3 \pm u^2 t,\ \frac{4}{3}ut^3) : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$, has isolated singularity at 0 and is quasi-homogeneous for the weights $w(t) = 1$ and $w(u) = 2$. By Corollary 1.5, sd$(f_{\text{mou}}^{\pm}) = 0$. In fact, we have $\mathcal{R}_f \supsetneq \mathcal{G}_f = f^*\mathcal{O}_{2n}$ and sp-cod$(f_{\text{mou}}^{\pm}) = 1$. Moreover $f_{\text{mou}}^{\pm}$ is isotropically perturbed into two open umbrellas and one double point, and therefore $\delta_I = 1$, $u_I = 2$, $\delta = 3$, $\text{Segre}_4 = 8$.

Remark 5.7 An algebraic formula for the number u_I of open umbrellas is known ([21]): For $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^4, 0)$, we have

$$u_I = \dim_{\mathbf{C}} \frac{\mathcal{O}_2}{J_f},$$

where J_f is the ideal generated by the 2-minors of the Jacobi matrix of f.

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