A NUMERICAL SCHEME CROSS-2D WITH POSITIVITY CONDITION CONSIDERING CROSS-DERIVATIVES FOR 2-D FLOW FIELDS

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Abstract
We develop a numerical scheme taking into consideration the cross-derivatives between \( z \)- and \( y \)-coordinates in a two-dimensional flow fields. The present scheme possesses positive coefficients under an allowance condition among the computational parameters for two-dimensional advection-diffusion equations. Numerical experiments of 2-D initial shock propagation show numerical solutions of good quality with the present scheme.

Key Words: Numerical scheme, Numerical stability, Advection-diffusion equation, Positive coefficients condition, Cross-derivatives

1 Introduction
In CFD (Computational Fluid Dynamics), it is much concern to construct stable numerical schemes with higher-order accuracy because of a trend of trade-off relationships between numerical accuracy and stability. So far we have constructed stable schemes with high accuracy based on an analytical solution of advection-diffusion equations. Namely, by involving the properties of the exact solutions of liner and nonliner advection-diffusion equations into a numerical scheme, we constructed the numerical schemes Ano [1] and Cole [2] with monotonically preserving properties for unsteady linear and nonlinear equations, respectively.

According to Godunov's theorem, there exist no polynomial expansion schemes with positive difference coefficients except for the first-order accuracy for advection equations. In this connection, the conventional high-order polynomial schemes such as QUICK [3], KAWAMURA [4] and UTOPIA schemes tend to bring forth unstable solutions even for advection-diffusion equations especially in case of large gradient fields. However we showed [5] that there exists a stable polynomial expansion scheme with third-order accuracy in case of advection-diffusion equations by adjusting a parameter associated with the difference coefficients so that the stability condition is to be fulfilled. Thus we constructed [5] a third-order polynomial scheme optimized with respect to stability and truncation errors under an allowance condition among the Courant numbers and the diffusion numbers for advection-diffusion equations. This scheme was extended to be applicable to two-dimensional and three-dimensional equations.

In case of multidimensional flow analyses, usually one-dimensional schemes are applied independently to each coordinate axis, where the partial derivative \( f_{x,y} = \frac{\partial^2 f}{\partial x \partial y} \) of a quantity \( f(t,x,y) \) are not taken into consideration and the numerical accuracy of one-dimensional schemes would never be maintained. Hereafter we call \( f_{x,y} \) cross-derivative. In those conventional calculations, false numerical diffusion errors are dominant in a region where a computational grid line and the flow streamlines are not closely aligned. To cope this problem, skew upwind differencing provides a useful alternative to other upstream schemes. In this paper, we develop a numerical scheme to rigorously take into consideration the cross-derivatives between \( x \) - and \( y \)-coordinates in a two-dimensional flow fields. This cross-derivatives approach is expected to bring forth more accurate calculations for multidimensional flow fields as well as in a region where a computational grid line and the flow streamlines are not closely aligned.

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2. Numerical Stability

Regarding the convergence of numerical solution \( f(n\Delta t, i\Delta x) = f_i^n \) at the time step number \( n \) and spatial mesh number \( i \) to the exact solution \( f(t, x) \) of differential equations, there exist the Lax’s equivalence theorem and the TVD (Total Variation Diminishing) equivalence theorem for linear and nonlinear partial differential equations, respectively. According to those theorems, the stability condition of solutions with the consistency condition of numerical schemes are the necessary and sufficient conditions for convergence.

Regarding the numerical stability, there exist mainly boundedness of numerical solutions and numerical oscillations (unphysical oscillations). The former is due to round-off errors and can be controlled by the von-Neumann stability condition, while the latter is due to the behaviour of the exact solution itself of finite difference equations and a change of the finite difference equation is necessary. Usually the numerical scheme is changed locally to a low-order scheme in a region where a steep gradient exists.

According to Lax’s equivalence theory regarding convergence, the boundedness and consistency assure the convergence of numerical solution to the exact solution of partial differential equations. Hence the numerical oscillation, if any, vanishes when \( \Delta t \) and \( \Delta x \) go to 0. However, we solve the partial differential equations by using finite difference values \((\Delta t, \Delta x)\), and a numerical solution free from numerical oscillations is to be desired, which brings forth improvement of quality of numerical solutions.

Regarding the stability condition, there is the positive coefficient condition that all the coefficients of difference equations should be positive, when \( f_i^{n+1} \) is expressed by a linear combination in terms of \( f_i^n \) as \( f_i^{n+1} = \sum a_i f_i^n \) with the consistency condition \( \sum a_i = 1 \). This positive coefficient condition assures the following conditions:

1) Monotonicity of the numerical scheme, namely \( \frac{\partial f_i^{n+1}}{\partial f_i^n} \geq 0 \).

2) Monotonicity preserving condition: If \( f(t, x) \) decreases (or increases) monotonously all over the space at time \( t \), \( f(t + \Delta t, x) \) decreases (or increases) monotonously all over the space at the time \( t + \Delta t \), namely

\[
f_i^n \leq f_i^{n+1} \quad (\text{for all } i) \quad \Rightarrow \quad f_i^{n+1} \leq f_i^{n+1} \quad (\text{for all } i).
\]

3) Maximum principle: \( f(t + \Delta t, x) \) at new time \( t + \Delta t \) exists between the minimum and maximum values of \( f(t, x) \) at time \( t \), namely

\[
\text{Min} \ [f(t, x)] \leq f(t + \Delta t, x) \leq \text{Max} \ [f(t, x)].
\]

which is so called the maximum principle inequality over a specified domain and is a property of analytical solutions of linear and nonlinear advection-diffusion equations without source terms.

4) TVD (Total Variation Diminishing) condition, namely

\[
TV(f^{n+1}) = \sum |f_i^{n+1} - f_i^n| \leq TV(f^n)
\]

5) Boundedness condition, namely when we expand \( f(t, x_i) \) in terms of the Fourier series as \( f(t, x_i) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} F_k^n \exp[2\pi ki/N] \), the amplification factor \( A_k \) is less than 1: \( A_k = |F_k^{n+1}/F_k^n| \leq 1 \).

Thus the positive coefficients condition with the consistency condition is sufficient condition for both the numerical oscillations and the boundedness of the numerical solutions.

3. Mathematical Formulation

3.1 Model Equation

Here we consider two-dimensional advection-diffusion equations without source terms as follows:
\[ \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} = \nu \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right), \]  

where \( t \) denotes the time, \( x \) and \( y \) the spatial coordinates, \( u \) and \( v \) the components of velocity in the \( x \)- and \( y \)-axis, respectively and \( \nu \) the diffusion parameter.

3.2 Computational Grid

We use a computational domain with the uniform mesh sizes \( \Delta x \) and \( \Delta y \) as shown in Fig.1. We express the solution \( f(t, x, y) = f(t^n, i\Delta x, j\Delta y) \) or \( f_{i,j}^n \) on this computational grid, where the super subscript \( n \) denotes the time step number and \( t = n\Delta t \) with the temporal mesh size \( \Delta t \).

![Computational Grid Diagram](image)

Fig. 1 A computational grid and stencils associated with the mesh point \((i, j)\) under consideration.

4. The Taylor series expansion

4.1 Cross-Derivatives Formulation Using 9 Stencils

We expand \( f_{i\pm1,j\pm1} \) at 8 mesh points surrounding the mesh point \((i, j)\) in Fig.1 into the Taylor series by making use of the Taylor expansion formula as follows:

\[
\begin{align*}
 f(x_i + \Delta x, y_j) &= f(x_i, y_j) + \sum_{k=1}^{km} \frac{1}{k!} \frac{\partial^k f(x_i, y_j)}{\partial x^k} (\Delta x)^k + o((\Delta x)^{km+1}) \\
 f(x_i, y_j + \Delta y) &= f(x_i, y_j) + \sum_{k=1}^{km} \frac{1}{k!} \frac{\partial^k f(x_i, y_j)}{\partial y^k} (\Delta y)^k + o((\Delta y)^{km+1}) \\
 f(x_i + \Delta x, y_j + \Delta y) &= f(x_i, y_j) + \sum_{k=1}^{km} \frac{1}{k!} \left( \frac{\Delta x}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^k f(x_i, y_j) + o((\Delta x^2 + \Delta y^2)^{(km+1)/2})
\end{align*}
\]

Here \( km \) is the maximum number of the series and we take 3 for \( km \). The Taylor series expansions of typical mesh points \((x_{i+1}, y_{j+1})\) and \((x_{i+1}, y_{j-1})\) are as follows:
\[ f(x_{i+1, j+1}) = f(x_i, y_j) + \left( \Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right) f(x_i, y_j) + \frac{1}{2} \left( \Delta x^2 \frac{\partial^2}{\partial x^2} + 2\Delta x \Delta y \frac{\partial^2}{\partial x \partial y} + \Delta y^2 \frac{\partial^2}{\partial y^2} \right) f(x_i, y_j) \]
\[ + \frac{1}{6} \left( \Delta x^3 \frac{\partial^3}{\partial x^3} + 3\Delta x^2 \Delta y \frac{\partial^3}{\partial x^2 \partial y} + 3\Delta x \Delta y^2 \frac{\partial^3}{\partial x \partial y^2} + \Delta y^3 \frac{\partial^3}{\partial y^3} \right) f(x_i, y_j) + o((\Delta x^2 + \Delta y^2)^2) \]  
(4a)

\[ f(x_{i-1, j+1}) = f(x_i, y_j) + \left( \Delta x \frac{\partial}{\partial x} - \Delta y \frac{\partial}{\partial y} \right) f(x_i, y_j) + \frac{1}{2} \left( \Delta x^2 \frac{\partial^2}{\partial x^2} - 2\Delta x \Delta y \frac{\partial^2}{\partial x \partial y} + \Delta y^2 \frac{\partial^2}{\partial y^2} \right) f(x_i, y_j) \]
\[ + \frac{1}{6} \left( \Delta x^3 \frac{\partial^3}{\partial x^3} - 3\Delta x^2 \Delta y \frac{\partial^3}{\partial x^2 \partial y} + 3\Delta x \Delta y^2 \frac{\partial^3}{\partial x \partial y^2} - \Delta y^3 \frac{\partial^3}{\partial y^3} \right) f(x_i, y_j) + o((\Delta x^2 + \Delta y^2)^2) \]  
(4b)

Taking a linear combination of the above 8 Taylor expansion series equations with the multiplication of eight parameters \((\alpha, \beta, \gamma, \delta, k, l, m, n)\) yields

\[ \alpha f_{i+1, j+1} + \beta f_{i+1, j-1} + \gamma f_{i-1, j+1} + \delta f_{i-1, j-1} + k f_{i+1, j} + l f_{i-1, j} + m f_{i, j+1} + n f_{i, j-1} \]
\[ = (\alpha + \beta + \gamma + \delta + k + l + m + n) f_{i, j} \]
\[ + (\alpha + \beta - \gamma - \delta + k - l) \Delta x \frac{\partial f_{i, j}}{\partial x} + (\alpha + \beta - \gamma - \delta + m - n) \Delta y \frac{\partial f_{i, j}}{\partial y} \]
\[ + \frac{1}{2} (\alpha + \beta + \gamma + \delta + k + l) \Delta x^2 \frac{\partial^2 f_{i, j}}{\partial x^2} + (\alpha + \beta - \gamma - \delta) \Delta x \Delta y \frac{\partial^2 f_{i, j}}{\partial x \partial y} \]
\[ + \frac{1}{2} (\alpha + \beta + \gamma - \delta + m + n) \Delta y^2 \frac{\partial^2 f_{i, j}}{\partial y^2} + (\alpha + \beta - \gamma - \delta + k - l) \Delta x^3 \frac{\partial^3 f_{i, j}}{\partial x^3} \]
\[ + \frac{1}{2} (\alpha + \beta + \gamma - \delta) \Delta x^2 \Delta y \frac{\partial^3 f_{i, j}}{\partial x^2 \partial y} + \frac{1}{2} (\alpha + \beta - \gamma - \delta) \Delta x \Delta y^2 \frac{\partial^3 f_{i, j}}{\partial x \partial y^2} \]
\[ + \frac{1}{6} (\alpha + \beta + \gamma - \delta + m - n) \Delta y^3 \frac{\partial^3 f_{i, j}}{\partial y^3} + o((\Delta x^2 + \Delta y^2)^2) \]  
(5)

4.2 Consistency with Advection-Diffusion Equation

Here we require the consistency of Eq.(5) with the advection-diffusion equation.

\[ \alpha + \beta - \gamma - \delta + k - l = \frac{u}{\Delta x} \]  
(6a)

\[ \alpha + \beta - \gamma - \delta + m - n = \frac{\nu}{\Delta y} \]  
(6b)

\[ \alpha + \beta + \gamma + \delta + k + l = -\frac{2\nu}{\Delta x^2} \]  
(6c)

\[ \alpha + \beta + \gamma + \delta + m + n = -\frac{2\nu}{\Delta y^2} \]  
(6d)

4.3 Requirements for Higher Numerical Accuracy

Further we require the numerical accuracy as high as possible for Eq.(5).

\[ \alpha - \beta - \gamma + \delta = 0 \]  
(7a)

\[ \alpha - \beta + \gamma - \delta = 0 \]  
(7b)

\[ \alpha + \beta - \gamma - \delta = 0 \]  
(7c)

We can solve Eqs.(6a), (6b), \cdots (7b), (7c) by using \(m\) as a parameter as follows:
\[ \alpha = \beta = \gamma = \delta = \frac{m}{2} + \frac{\nu}{4\Delta y} - \frac{\nu}{2\Delta y^2} \] (8a)

\[ k = \frac{m}{2\Delta x} - \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} \] (8b)

\[ l = \frac{m}{2\Delta x} - \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} \] (8c)

\[ n = m - \frac{\nu}{\Delta y} \] (8d)

### 4.4 Finite Difference Equation

Substituting Eqs.(8a)-(8d) for Eq.(5) yields an approximate finite difference equation for the steady advection-diffusion equation as follows:

\[ u \frac{\partial f_{i,j}}{\partial x} + v \frac{\partial f_{i,j}}{\partial y} - \nu \left( \frac{\partial^2 f_{i,j}}{\partial x^2} + \frac{\partial^2 f_{i,j}}{\partial y^2} \right) = \alpha f_{i+1,j+1} + \beta f_{i+1,j-1} + \gamma f_{i-1,j+1} + \delta f_{i-1,j-1} + k f_{i+1,j} + l f_{i-1,j} + m f_{i,j+1} + n f_{i,j-1} \]

\[ + \frac{\alpha + \beta + \gamma + \delta + k + l + m + n}{\nu} f_{i,j} + O ((\Delta x^2 + \Delta y^2)^{1/2}) \] (9)

Regarding numerical stability, the positive coefficients condition means that the finite difference coefficients associated with transported quantities \( f \) in a finite difference equation with a single step and explicit scheme with respect to time are positive. Hence we adopt the Euler explicit scheme with a single time step for the temporal term of Eq.(2), and apply Eq.(9) for advection-diffusion terms. Thus we obtain the finite difference equation for Eq.(2) as follows:

\[ f_{i,j}^{n+1} = \left[1 + \Delta t \left( \alpha + \beta + \gamma + \delta + k + l + m + n \right) \right] f_{i,j}^n \]

\[ - \Delta t \left( \alpha f_{i+1,j+1} + \beta f_{i+1,j-1} + \gamma f_{i-1,j+1} + \delta f_{i-1,j-1} + k f_{i+1,j} + l f_{i-1,j} + m f_{i,j+1} + n f_{i,j-1} \right) \]

\[ + O ((\Delta x^2 + \Delta y^2)^{1/2}) + O (\Delta t) \] (10)

### 4.5 Stability Condition

According to the positive coefficients conditions for the stability, we have the following stability conditions for Eq.(10):

\[ \alpha < 0, \quad \beta < 0, \quad \gamma < 0, \quad \delta < 0, \quad k < 0, \quad l < 0, \quad m < 0, \quad n < 0 \] (11a)

\[ 1 + \Delta t \left( \alpha + \beta + \gamma + \delta + k + l + m + n \right) > 0 \] (11b)

By solving Inequalities (11a) and (11b) with substitution of Eqs(8a), (8b), (8c) and (8d) into Eqs.(11a) and (11b), we obtain the following results:

Under the condition of

\[ 0 < \Delta x < \frac{2\nu}{|u|}, \quad 0 < \Delta y < \frac{2\nu}{|u|}, \]

and

...
We have
\[ \lambda_4 < m < \min[\lambda_1, \lambda_2, \lambda_3], \] (13)
where
\[ \lambda_1 = 0, \] (14a)
\[ \lambda_2 = -\frac{u}{2\Delta x} + \frac{\nu}{2\Delta y} + \frac{\nu}{\Delta x^2} - \frac{\nu}{\Delta y^2}, \] (14b)
\[ \lambda_3 = \frac{u}{2\Delta x} + \frac{\nu}{2\Delta y} + \frac{\nu}{\Delta x^2} - \frac{\nu}{\Delta y^2}, \] (14c)
\[ \lambda_4 = -\frac{1}{2\Delta t} + \frac{\nu}{2\Delta y} + \frac{\nu}{\Delta x^2}. \] (14d)

4.6 Determination of Optimum Value of \( m \) and Other Parameters

Next we determine the optimum value of \( m \) so that the truncation errors of Eq.(9) may be minimum. The truncation errors with the fourth order are as follows:
\[ TE = \frac{1}{24} \left( \alpha + \beta + \gamma + \delta + k + m \right) \frac{\partial^4 f}{\partial x^4} \Delta x^4 + \frac{1}{6} \left( \alpha - \beta + \gamma - \delta \right) \frac{\partial^4 f}{\partial x^3 \partial y} \Delta x^3 \Delta y \]
\[ + \frac{1}{4} \left( \alpha + \beta + \gamma + \delta \right) \frac{\partial^4 f}{\partial x^2 \partial y^2} \Delta x^2 \Delta y^2 + \frac{1}{6} \left( \alpha - \beta + \gamma - \delta \right) \frac{\partial^4 f}{\partial x \partial y^3} \Delta x \Delta y^3 \]
\[ + \frac{1}{24} \left( \alpha + \beta + \gamma + \delta + l + n \right) \frac{\partial^4 f}{\partial y^4} \Delta y^4 \] (15)

Substituting Eq.(8) into the coefficients of right hand side equation of \( TE \) yields
\[ \frac{1}{24} \left( \alpha + \beta + \gamma + \delta + k + m \right) = \frac{1}{24} \left( \frac{u}{2\Delta x} + \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} - \frac{\nu}{\Delta y^2} \right), \] (16a)
\[ \frac{1}{6} (\alpha - \beta + \gamma - \delta) = 0, \] (16b)
\[ \frac{1}{4} (\alpha + \beta + \gamma + \delta) = \frac{1}{4} \left( -2m + \frac{\nu}{\Delta y} - \frac{2\nu}{\Delta y^2} \right), \] (16c)
\[ \frac{1}{6} (\alpha - \beta + \gamma - \delta) = 0 \] (16d)
and
\[ \frac{1}{24} (\alpha + \beta + \gamma + \delta + l + n) = \frac{1}{24} \left( \frac{u}{2\Delta x} + \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} - \frac{\nu}{\Delta y^2} \right). \] (16e)

Therefore the first and fifth coefficients in \( TE \) can not always be zero. From the requirement that the third term of \( TE \) should be as small as possible, the optimum value of \( m \) is \( \lambda_4 \) under the condition given by Eq.(13). Finally we obtain the optimum values for all parameters as follows:
\[ m = -\frac{1}{2\Delta t} + \frac{\nu}{2\Delta y} + \frac{\nu}{\Delta x^2} \]  
\[ \alpha = \beta = \gamma = \delta = -\frac{m}{2} + \frac{\nu}{4\Delta y} - \frac{\nu}{2\Delta y^2} = -\frac{1}{2\Delta t} + \frac{\nu}{4\Delta y} - \frac{1}{2}\left( \frac{\nu}{\Delta x^2} - \frac{\nu}{\Delta y^2} \right) \]  
\[ k = m + \frac{u}{2\Delta x} - \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} = -\frac{1}{2\Delta t} + \frac{u}{2\Delta x} + \frac{\nu}{\Delta y^2} \]  
\[ l = m - \frac{u}{2\Delta x} - \frac{\nu}{2\Delta y} - \frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2} = -\frac{1}{2\Delta t} - \frac{u}{2\Delta x} + \frac{\nu}{\Delta y^2} \]  
\[ n = m - \frac{\nu}{\Delta y} = -\frac{1}{2\Delta t} - \frac{\nu}{2\Delta y} + \frac{\nu}{\Delta x^2} \]  

It may seems strange that parameters (\( \alpha, \beta, \gamma, \delta, k, l, n \)) include \( \Delta t \), not withstanding Eq. (9) is a steady equation. This comes from that we applied the stability condition for unsteady equations. If we construct a numerical scheme for steady equations, we apply the stability condition for steady equations, resulting in excluding \( \Delta t \).

### 4.7 Computational Procedure

Thus a computational procedure is outlined below.

1. **Determination of \( \Delta x \) and \( \Delta y \):**
   \[ \Delta x = \xi_x \left( \frac{2\nu}{|u_{\text{max}}|} \right) \quad \text{and} \quad \Delta y = \xi_y \left( \frac{2\nu}{|v_{\text{max}}|} \right), \quad \xi_x = \xi_y = 0.9. \]  

2. **Determination of \( \Delta t \):**
   \[ \Delta t = \xi_t \left( \frac{1}{2\text{Max}\left( \frac{\nu}{\Delta x^2} + \frac{\nu}{\Delta y^2}, \frac{\nu}{\Delta y^2} + \frac{u}{2\Delta x} \right)} \right), \quad \xi_t = 0.9, \]  

3. **Determination of the optimum value of \( m \):**
   \[ m = \lambda_4, \]
   in which a slightly large value than \( \lambda_4 \) is adopted for \( m \) in order that Inequality (13) may be satisfied.

4. **Determination of other parameters \( (\alpha, \beta, \gamma, \delta, k, l, n) \).**

Further, we are performing the mathematical formulations for 13 stencils including 4 mesh points \((i+2, j), (i-2, j), (i, j+2)\) and \((i, j-2)\).
Fig. 2: Computational domain and boundary conditions.

Fig. 3: 2-D solution with the present optimized scheme \( n \): time step number.
5 Numerical Experiments

We solve an advection-diffusion equation Eq.(2). Figure 3 shows the computational geometry and the boundary conditions. Namely, \( f = 0 \) on the bottom and right boundaries, \( f = 100 \) on the left and top boundaries and \( f = 50 \) at the two corners \((i = j = 1)\) and \((i = j = 51)\). We employed a computational grid \( 51 \times 51 \) (uniform meshes) with uniform velocity fields \( u = v \). Outside the boundaries, the same values on the boundaries are set except for the line \( i = j \), where \( f = 50 \) is assigned. Thus at every points outside the boundaries, the value of \( f \) is assigned and any special treatment near the boundaries is not necessary resulting in keeping the same accuracy all over the computational domain. Initially, the values of \( f \) are set 0 all over the inner domain. Thus, the solution of this problem approaches to a stationary solution. We employed the second-order Crank-Nicolson scheme for time discretization.

Both the KAWAMURA scheme and the UTOPIA scheme could not continue to calculate because of numerical instability. The present CROSS-2D scheme gave a stable solution, which is shown at the mesh point \((i,j)\) along a diagonal line with \( j = 51 - i \) \((i = 1, 2, \ldots, 51)\) at the time step number \( n = 10, 20, 30, 40, 50, 60 \) and 100 in Fig.3. The solution at \( n = 100 \) almost reached to the stationary state.

6 Conclusion

We investigated a numerical method to rigorously take into consideration the cross-derivatives between \( x \)- and \( y \)-coordinates in a two-dimensional flow fields, and developed a numerical scheme CROSS-2D. The present scheme possesses positive coefficients under an allowance condition among the computational parameters such as \( \Delta t, \Delta x \) and \( \Delta y \) for two-dimensional advection-diffusion equations. We performed numerical experiments of initial shock propagation and confirmed the quality of solutions with the present CROSS-2D scheme.

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