Zeta functions of generalized permutations with application to their factorization formulas

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(Communicated by Shigefumi MORI, M.J.A., Sept. 12, 2012)

Abstract: We obtain a determinant expression of the zeta function of a generalized permutation over a finite set. As a corollary we prove the functional equation for the zeta function. In view of absolute mathematics, this is an extension from \(GL(n, \mathbb{F}_t)\) to \(GL(n, \mathbb{F}_{1^m})\), where \(\mathbb{F}_t\) and \(\mathbb{F}_{1^m}\) denote the imaginary objects “the field of one element” and “its extension of degree \(m\)”, respectively. As application we obtain a certain product formula for the zeta function, which is analogous to the factorization of the Dedekind zeta function into a product of Dirichlet \(L\)-functions for an abelian extention.

Key words: Zeta functions; the field with one element; absolute mathematics; generalized permutation groups.

1. Introduction. Let

\[
\zeta_\sigma(s) = \exp \left( \sum_{m=1}^{\infty} \frac{\text{Fix}(\sigma^m)}{m} e^{-ms} \right)
\]

be the zeta function of the \(\mathbb{Z}\)-dynamical system generated by a permutation \(\sigma \in S_n\), where \(S_n\) denotes the symmetric group over \(X_n = \{1, \ldots, n\}\). We see that \(\zeta_\sigma(s)\) is determined by the conjugacy class of \(\sigma\) in \(S_n\). By Proposition 1 below, it is also expressed by the Euler product over the set \(\text{Cyc}(\sigma)\) of primitive cycles of \(\sigma\):

\[
\zeta_\sigma(s) = \prod_{p \in \text{Cyc}(\sigma)} (1 - N(p)^{-s})^{-1},
\]

where \(N(p) = \ell(p)\) with \(\ell = \ell(p)\) being the length of a primitive cycle

\[
p : i \mapsto \sigma(i) \mapsto \sigma^2(i) \mapsto \cdots \mapsto \sigma^l(s) = i
\]

for some \(i \in \{1, \ldots, n\}\).

In our previous paper [3], we gave a proof of the determinant expression

\[
\zeta_\sigma(s) = \det(I - M(\sigma)e^{-s})^{-1},
\]

which enables us to obtain the functional equation of \(\zeta_\sigma(s)\).

Our first goal is to generalize such properties to the case of generalized permutations. Consequently we generalize \(\zeta_\sigma(s)\) to \(L_\sigma(s, \chi)\) with \(\chi\) a function over the set of cycles. As application we obtain a certain product formula for the zeta function, which is analogous to the factorization of the Dedekind zeta function into a product of Dirichlet \(L\)-functions in the case of an abelian extention.

We first briefly recall the definitions and settings on the generalized symmetric groups following the notation in [1].

Let \(\xi\) be a primitive \(m\)-th root of unity, and \(\mu_m\) be the multiplicative group of \(m\)-th roots of unity. The generalized permutation group \(W_m^n\) is the Wreath product of \(\mu_m\) by \(S_n\):

\[
1 \mapsto (\mu_m)^n \mapsto W_m^n \mapsto S_n \mapsto 1.
\]

It is also expressed as the group of permutations \(\tau\) of the set

\[
X_{n,m} := \{ \xi^k i \mid i = 1, \ldots, n, \ k = 0, 1, \ldots, m - 1 \}
\]

such that \(\tau(\xi^k i) = \xi^k \tau(i)\) for \(i = 1, \ldots, n\) and \(k = 0, 1, \ldots, m - 1\). The order of \(W_m^n\) is \(m^m n!\). The group \(W_m^n\) has the following presentation ([2]):

\[
W_m^n = \langle r_1, \ldots, r_{n-1}, w_1, \ldots, w_n : 
\]

\[
r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = c, \text{ if } |i - j| \geq 2,
\]

\[
w_i^n = c, \ w_i w_j = w_j w_i, \ r_i w_i = w_{i+1} r_i,
\]

\[
r_i w_j = w_j r_i, \text{ if } j \neq i, i+1 \rangle.
\]

We may identify \(r_i\) \(\text{for } i = 1, \ldots, n - 1\) with the transposition \((i, i+1)\) and therefore the symmetric group is
The elements $w_i$ may be identified with the mapping $X_{n,m} \rightarrow X_{n,m}$ defined by
\[ w_i(\xi^j) = \begin{cases} 
\xi^{j+1} & (j = i) \\
\xi^{j} & (j \neq i).
\end{cases} \]

An element $\tau \in W_m^n$ is determined by the images from the base space $X_n$, which is embedded in $X_{nm}$ with $k = 0$ in (4). Namely, it can be written as
\[ \tau = \left( \begin{array}{ccc}
1 & 2 & \cdots & n \\
\xi^1 \sigma(1) & \xi^2 \sigma(2) & \cdots & \xi^n \sigma(n) \\
\end{array} \right) = \sigma \prod_{i=1}^{n} w_i^{s_i} \in W_m^n \]

with $\sigma \in S_n$ and $s_j \in \{0, 1, 2, \ldots, m-1\}$. Denote by $M$ the canonical representation $M : W_m^n \rightarrow GL_n(C)$ of $W_m^n$ defined by $M(\tau) = (\xi^j \delta_{\sigma(i),j})_{i,j=1,\ldots,n}$.

We define a function $\chi = \chi_\tau$ on the set of primitive cycles $p$ of $\sigma$ given by (2) as
\[ \chi : \text{Cyc}(\sigma) \rightarrow \mathbb{C}^* \]
\[ p \mapsto \xi^p, \]

with $\int_{p} \tau = \sum_{i \in p} s_i$. We also define the attached $L$-function as
\[ L_\sigma(s, \chi) = \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p)N(p)^{-s})^{-1}. \]

Our first main result is the determinant expression of $L_\sigma(s, \chi)$ described in Theorem 1 below. It is a natural extension of (3) from the viewpoint of absolute mathematics, because the symmetric group is interpreted as $S_n = GL(n, F_1)$, and the generalized permutation group is $W_m^n = GL(n, F_\infty)$. As corollaries of Theorem 1, we obtain the functional equation and the tensor structure of $L_\sigma(s, \chi)$.

Finally in the last section we reach a factorization formula which is an analog of the decomposition of the Dedekind zeta function of an abelian extension into Hecke $L$-functions.

2. Determinant expression. In our previous paper [3] we proved the following proposition.

Proposition 1. Let $X$ and $Y$ be finite sets. Put $|X| = n$. For $\sigma \in S_n$, the following properties hold.

(i) $\zeta_\sigma(s)$ has a determinant expression
\[ \zeta_\sigma(s) = \det(1 - M_0(\sigma)e^{-s})^{-1}, \]

where $M_0(\sigma) = (\delta_{\sigma(i),j})_{i,j=1,\ldots,n}$ is the matrix representation $M_0 : S_n \rightarrow GL_n(C)$.

(ii) $\zeta_\sigma(s)$ satisfies an analog of the Riemann hypothesis: $\zeta_\sigma(s) = \infty$ implies $\text{Re}(s) = 0$.

(iii) $\zeta_\sigma(s)$ satisfies the functional equation
\[ \zeta_\sigma(-s) = \zeta_\sigma(s)(-1)^{\#} \text{sgn}(\sigma)e^{-ns}. \]

(iv) $\zeta_\sigma(s)$ has the Euler product
\[ \zeta_\sigma(s) = \prod_{p \in \text{Cyc}(\sigma)} (1 - (N(p)^{-s})^{-1}. \]

(v) The singularities of $\zeta_\sigma(s)$ satisfy an additive structure under the tensor product. Namely, the sum of a pole of $\zeta_\sigma(s)$ for $\sigma \in \text{Aut}(X)$ and a pole of $\zeta_\tau(s)$ for $\tau \in \text{Aut}(Y)$ is a pole of $\zeta_{\sigma \tau}(s)$, and all poles of $\zeta_{\sigma \tau}(s)$ are given by this way. Here for $\sigma \in \text{Aut}(X)$ and $\tau \in \text{Aut}(Y)$, we denote their tensor product by $\sigma \otimes \tau \in \text{Aut}(X \times Y)$.

(vi) The Laurent expansion of $\zeta_\sigma(s)$ around $s = 0$ is given as follows:
\[ \zeta_\sigma(s) = s^{-m}c(s)^{-1} + O(s^{-m+1}), \]

where $m$ is the multiplicity of the eigenvalue 1 of $M_0(\sigma)$ and $c(s) = \prod_{p \in \text{Cyc}(\sigma)} l(p)$.

In this section we prove a generalization of this proposition to $L_\sigma(s, \chi)$.

Theorem 1. Let $X$ be a finite set with $|X| = n$, and $\xi \in C$ be a primitive $m$-th root of unity. For a generalized permutation $\tau \in W_m^n$ with a decomposition given by (5), the $L$-function $L_\sigma(s, \chi)$ satisfies the determinant expression
\[ L_\sigma(s, \chi) = \det(1 - M(\tau)e^{-s})^{-1}. \]

Note that the matrix $M(\tau)$ is not uniquely determined for each given $\chi$. In other words, more than one $\tau$‘s (or $s_i$‘s) may possibly correspond to the same $\chi$. The determinant in (8), however, is well-defined for each $\chi$ not depending on the choice of $\tau$ or $s_i$‘s.

Proof of Theorem 1. We put the decomposition of a permutation $\sigma$ into cyclic permutations as
\[ \sigma = \sigma_1 \cdots \sigma_r = (i_1, \ldots, i_{l(1)}) (i_{l(1)+1}, \ldots, i_{l(1)+l(2)}) \cdots (i_{l(1)+\ldots+l(r-1)+1}, \ldots, i_n). \]

Let $\pi \in S_n$ be the permutation such that $\pi(k) = i_k$ for $k = 1, 2, 3, \ldots, n$. Then
\[ \pi^{-1} \sigma \pi = (1 \cdots l(1)) (l(1) + 1 \cdots (l(1) + l(2)) \cdots (l(1) + \cdots + l(r-1) + 1 \cdots n). \]
Hence
\[ M(\pi)^{-1}M(\tau)M(\pi) = \text{diag}(C_{l(1)}, C_{l(2)}, \ldots, C_{l(l)}) \]
with
\[
C_{l(k)} = \begin{pmatrix}
0 & \mu_m \\
\mu_m & \ddots \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
\xi_{l-1} & \cdots & \cdots & 0
\end{pmatrix}
\]
being a generalized cyclic permutation matrix of size \( l(k) \). We define integers \( t_1, \ldots, t_n \in \{0, 1, 2, \ldots, m - 1\} \) by
\[
C_{l(k)} = \begin{pmatrix}
0 & \xi_{l(k-1)+1} \\
\xi_{l(k-1)+1} & \ddots \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
\xi_{l-1} & \cdots & \cdots & 0
\end{pmatrix}
\]
with \( l(0) = 0 \) by convention. Note that \( \{t_j\} \) is a reordered sequence of \( \{s_j\} \). Since a cyclic permutation is corresponding to a cycle, we may write \( \chi(C_{l(k)}) = \prod_{j=1}^{l(k)} \xi^{t_{l(j)+1}} \) by taking the definition (6) into consideration. Then
\[
\det(1 - M(\tau)e^{-s}) = \prod_{j=1}^{r} \det(I_{l(j)} - C_{l(j)}e^{-s})
\]
where the last identity is deduced by the following lemma. It holds that
\[
\det(1 - M(\tau)e^{-s}) = \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p)N(p)^{-s}).
\]

**Lemma 1.** Let
\[
C_I = \begin{pmatrix}
0 & \xi^1 \\
\xi^1 & \ddots \\
& \ddots & \ddots \\
& & \xi^{l-1} \\
\xi^{l-1} & 0
\end{pmatrix}
\]
be a generalized permutation matrix. Put
\[
\chi(C_I) = \prod_{j=1}^{l} \xi^{t_j}.
\]
The following identity hold:
\[
\det(I_I - C_Iu) = 1 - \chi(C_I)u^l.
\]

**Proof.**
\[
\det(I_I - C_Iu) = \det\begin{pmatrix}
1 & -\xi^1u \\
& \ddots & \ddots \\
& & \ddots & -\xi^{l-1}u \\
& & & 1
\end{pmatrix}
\]
\[
= \prod_{j=1}^{r} (1 - \chi(C_{l(j)})u^{-s}),
\]
where \( \chi(\overline{C_I}) = \chi(C_I) \).

**Corollary 1 (Functional equation).** For a generalized permutation \( \tau \in W_n^m \) with a decomposition given by (5), the L-function \( L_\sigma(s, \chi) \) satisfies the functional equation
\[
L_\sigma(-s, \chi) = (-1)^n \det M(\tau)^{-1} e^{-ns} L_\sigma(s, \overline{\chi})
\]
where \( \overline{\chi} \) is the complex conjugation of \( \chi \) which is given by replacing \( \xi \) with \( \overline{\xi} \).

**Proof.** By Theorem 1, it follows that
\[
L(-s, \chi) = \det(1 - M(\tau)e^{-s})^{-1}
\]
\[
= \det((1 - M(\tau)e^{-s})(1 - M(\tau)^{-1}e^{-s}))^{-1}
\]
\[
= (-1)^n \det M(\tau)^{-1} e^{-ns} \det(1 - M(\tau)^{-1}e^{-s})^{-1}.
\]
Let $\xi_k$ be a primitive $m_k$-th root of unity for $k = 1, 2$. For generalized permutations $\tau_1 \in W_{m_1}$ over $X_{n_1} = \{1, \ldots, n_1\}$ and $\tau_2 \in W_{m_2}$ over $X_{n_2} = \{1, \ldots, n_2\}$ with their decomposition given by

$$
\tau_k = \left( \begin{array}{ccc}
1 & 2 & \cdots & n_k \\
e^{\xi_k^1} & 
e^{\xi_k^2} & \cdots & 
e^{\xi_k^{n_k}} \\
\end{array} \right) = \sigma_k \prod w_{k,i} \in W_m,
$$

we define their tensor product $\tau_1 \otimes \tau_2 \in W^{m_1 m_2}$ as follows. As we saw in the notation (5), any element in $W^{m_1 m_2}$ is determined if we give the image of every element in the base space $X_{m_1} \times X_{m_2}$, which is given by

\[
\tau_1 \otimes \tau_2 : \quad X_{n_1} \times X_{n_2} \rightarrow X_{n_1 m_1} \times X_{n_2 m_2} \\
(i, j) \mapsto (\xi_1^i \sigma_1(i), \xi_2^j \sigma_2(j))
\]

with $\xi$ a primitive $m_1 m_2$-th root of unity. In other words, if we identify $\tau_1 \in W_{m_1}$ as the linear map $\tau_2 : \mathbb{C}^{m_1} \rightarrow \mathbb{C}^{n_1}$ introduced by the representation $M$, the tensor product

$$
\tau_1 \otimes \tau_2 : \quad \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \rightarrow \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}
$$

is defined by the usual tensor product of linear maps with the representation matrix given by the Kronecker tensor product $M(\tau_1) \otimes M(\tau_2)$ of matrices.

In the following corollary, we define $\chi_1 = \chi_1$, $\chi_2 = \chi_2$, and $\chi_1 \otimes \chi_2 := \chi_{1 \otimes 2}$.

**Corollary 2** (Tensor structure). The singularities of $L_\sigma(s, \chi)$ satisfy an additive structure under the tensor product. Namely, the sum of a pole of $L_1(s) := L_\sigma(s, \chi_1)$ and a pole of $L_2(s) := L_\sigma(s, \chi_2)$ is a pole of $L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2)$, and all poles of $L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2)$ are given in this way.

*Proof.* By Theorem 1,

$$
L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2) = \det(1 - M(\tau_1) \otimes M(\tau_2))^{-1}.
$$

We put the eigenvalues of $M(\tau_1)$ and $M(\tau_2)$ as $\alpha_j$ ($j = 1, \ldots, n_1$) and $\beta_k$ ($k = 1, 2, \ldots, n_2$), respectively. We see from Theorem 1 that the poles of $L_{\sigma_1}(s, \chi_1)$ and $L_{\sigma_2}(s, \chi_2)$ are given by $s \equiv \log \alpha_j$ (mod $2\pi i \mathbb{Z}$) and $s \equiv \log \beta_k$ (mod $2\pi i \mathbb{Z}$). Thus the set of poles of $L_{\sigma_1 \otimes \sigma_2}(s, \chi_1 \otimes \chi_2)$ is given by

$$
\{ \log \alpha_j \beta_k \mod 2\pi i \mathbb{Z} \mid 1 \leq j \leq n_1, 1 \leq k \leq n_2 \}.
$$

The result follows from

$$
\log \alpha_j \beta_k \equiv \log \alpha_j + \log \beta_k \mod 2\pi i \mathbb{Z}.
$$

Theorem 1 also describes the order of the $L$-function at $s = 0$ as follows.

**Corollary 3.** The Laurent expansion of $L_\sigma(s, \chi)$ around $s = 0$ is given as follows:

$$
L_\sigma(s, \chi) = s^{-K} c(\tau) + O(s^{-K+1}),
$$

where $K$ is the multiplicity of the eigenvalue $1$ of $M(\tau)$ and

$$
c(\tau) = \prod_{p \in \text{Cyc}(\sigma)} (l(p))^{-1} \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p))^{-1}.
$$

Moreover, $K$ is equal to the number of primitive cycles $p$ of $\sigma$ such that $\chi(p) = 1$.

*Proof.* By Theorem 1, we have

$$
L_\sigma(s, \chi) = \det(1 - M(\tau_1) e^{-s})^{-1} = \left(1 - e^{-s}\right)^{-K} \prod_{\alpha \neq 1} (1 - \alpha e^{-s})^{-1},
$$

where in the last product $\alpha$ runs through the eigenvalues of $M(\tau)$ such that $\alpha \neq 1$. Hence $L_\sigma(s, \chi)$ has a pole of order $K$ at $s = 0$. The leading coefficient is calculated from (iv):

$$
\prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p) N(p)^{-s})^{-1} = \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p) + \chi(p) l(p)s + O(s^2))^{-1} = s^{-K} \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p))^{-1}
$$

$$
\times \prod_{p \in \text{Cyc}(\sigma)} (1 - \chi(p) l(p)s + O(s^{K+1})).
$$

3. Factorization formulas. It is classical that for any finite abelian extension $K/k$ of algebraic number fields of finite degree, the Dedekind zeta function $\zeta_K(s)$ is decomposed into the product of Dirichlet $L$-functions over Dirichlet characters:

$$
\zeta_K(s) = \prod_{\chi} L_k(s, \chi).
$$
In this section we obtain an analog of this phenomenon by restricting ourselves to the case when the function $\chi$ has the form

$$\chi(p) = \theta^p \quad (\forall p \in \text{Cyc}(\sigma))$$

for some fixed $\theta \in \mu_m$. Namely,

$$L_\sigma(s, \chi) = \prod_{p \in \text{Cyc}(\sigma)} (1 - \theta^p e^{-l(p)s})^{-1} = \zeta_\sigma(s - \log \theta).$$

For $\theta = \exp(\frac{2\pi i}{m})$ ($m \in \mathbb{N}$), we denote $\chi = \chi_m$. The following factorization formula is analogous to (10).

**Theorem 2.** Let $\sigma \in S_n$, and $\tau = \sigma \prod_{i=1}^s w_i \in W_n$.

Put $\tilde{\sigma}$ to be the permutation $\tau$ regarded as an element in $S_{nm}$. Then it holds for any $m \in \mathbb{N}$ that

$$\zeta_\sigma(s) = \prod_{b=0}^{m-1} L_\sigma(s, \chi_m^b).$$

Before proving this theorem, we set up some analogous notions on lifting and splitting by following the theory of extensions of number fields. Consider the following commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\sim} & \tilde{X} \\
\downarrow f & & \downarrow f \\
X & \xrightarrow{\sigma} & X,
\end{array}
$$

where $f : \tilde{X} \to X$ is a surjective map of finite sets with $\sigma \in \text{Aut}(X)$ and $\tilde{\sigma} \in \text{Aut}(\tilde{X})$. Then a primitive cycle $p \in \text{Cyc}(\tilde{\sigma})$ is called a lift of $p \in \text{Cyc}(\sigma)$ if and only if $f(p) = p$. The inverse image $f^{-1}(p)$ of $p \in \text{Cyc}(\sigma)$ is a (not necessarily primitive) cycle of $\tilde{\sigma}$, and it can be decomposed into the form $f^{-1}(p) = \sum_{i=1}^s p_i$, with each $p_i$ a lift of $p$. In this setting we say that $p$ remains primitive if $g = 1$, and that $p$ splits if $g \geq 2$. Moreover, when $|f^{-1}(x)| = m$ for all $x \in X$, it holds that $g \leq m$, and we say that $p$ splits completely if $g = m$.

**Proof of Theorem 2.** We appeal to the cyclotomic equation

$$\prod_{b=0}^{m-1} (1 - \zeta_k^b X) = 1 - X^k$$

with $\zeta_k$ a primitive $k$-th root of unity. By putting $X = e^{-l(p)s}$ and $k = \frac{m}{(m,l,p)}$, we have

$$\prod_{b=0}^{m-1} L_\sigma(s, \chi_m^b) = \prod_{b=0}^{m-1} \prod_{p \in \text{Cyc}(\sigma)} (1 - \zeta_m^b e^{-l(p)s})^{-1} = \prod_{p \in \text{Cyc}(\sigma)} \prod_{b=0}^{m-1} (1 - \zeta_m^b e^{-l(p)s})^{-1} = \prod_{p \in \text{Cyc}(\sigma)} \prod_{b=0}^{m-1} \left(1 - e^{\frac{m(l,p)}{(m,l,p)} s}\right)^{-1}.$$

It remains to prove that the lifts of $p \in \text{Cyc}(\sigma)$ are $(m,l(p))$ primitive cycles of $\tilde{\sigma}$ which are of length $\frac{m(l,p)}{(m,l,p)}$.

To see this, we use the expression (4). Let $\xi_k i \in X_{n,m}$ be a fixed point of $\tilde{\sigma}$. Then,

$$\tilde{\sigma}^i(\xi_k i) = \xi_k i \iff \sigma^i(i) = i \quad \text{and} \quad \theta^i \xi_k = \xi_k,$$

where $p$ is the primitive cycle to which $i \in X_n$ belongs. Thus the length of the orbit of $\xi_k i$ is equal to the least common multiple of $l(p)$ and $m$, which is $\frac{m(l,p)}{(m,l(p))}$.

The number of elements belonging to $f^{-1}(p)$ in $\tilde{X}$ is $m l(p)$. Thus the number of lifts of $p$ is $(m,l(p))$ with their length $\frac{m(l,p)}{(m,l(p))}$.

From the proof of Theorem 2, we have the following facts immediately.

**Corollary 4.** Let $\sigma$ be a permutation of $X_n$, and $p$ be a primitive cycle which belongs to $\text{Cyc}(\sigma)$ with $l = l(p)$ defined as in (2).

In the lifted permutation

$$\tilde{\sigma} : X_{n,m} \to X_{n,m}$$

of $\sigma : X_n \to X_n$, it holds that

$$\begin{cases} p \text{ remains primitive} & \text{if } (l,m) = 1, \\ p \text{ splits} & \text{if } (l,m) > 1. \end{cases}$$

In the extreme case, $p$ splits completely, if and only if $m | l$.

This is analogous to the decomposition law of prime ideals for finite extensions of number fields.
Example 1. $n = 5$, $\sigma = (1\ 2)(3\ 4\ 5)$.

Cyc($\sigma$) consists of two primitive cycles $p_1$ and $p_2$, where $l(p_1) = 2$ and $l(p_2) = 3$. Consider the covering with $m = 2$, that is, $\xi = -1$. The cycle $p_1$ splits completely, since there exist two cycles above $p_1$, which are $(1 \mapsto -2 \mapsto 1)$ and $(2 \mapsto -1 \mapsto 2)$. Thus we find that $p_1$ splits completely in the extension $X_{5,2}$ of $X_5$. This is the case with $(m, l) = (2, 2) = 2$, which satisfies $m|l$.

On the other hand, the cycle $p_2$ remains primitive, because $p_2 = (3 \mapsto 4 \mapsto 5 \mapsto 3)$ is lifted to only one cycle $(3 \mapsto -4 \mapsto 5 \mapsto -3 \mapsto 4 \mapsto -5 \mapsto 3)$ of length 6. This is the case with $(l, m) = (3, 2) = 1$.

Example 2. $n = 8$, $\sigma = (1\ 2)(3\ 4\ 5\ 6\ 7\ 8)$.

Cyc($\sigma$) consists of two primitive cycles $p_1$ and $p_2$, where $l(p_1) = 2$ and $l(p_2) = 6$. Consider the covering with $m = 4$, that is, $\xi = \sqrt{-1} = i$. Above the cycle $p_1$ there exist two cycles of length 4, which are $(1 \mapsto 2i \mapsto -1 \mapsto -2i \mapsto 1)$ and $(2 \mapsto i \mapsto -2 \mapsto -i \mapsto 2)$. We find that $p_1$ splits in the extension $X_{8,4}$ of $X_8$.

This is the case with $(l, m) = (2, 4) = 2 > 1$. The other cycle $p_2$ also splits, because there exists two cycles of length 12 above $p_2$. This is the case with $(l, m) = (6, 4) = 2 > 1$.

Acknowledgement. The authors would like to express their gratitude to the referee for many valuable comments.

References