Area law and clustering of information in non-critical long-range interacting systems

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I. MOTIVATION AND BACKGROUND

Long-range interaction.— In recent years, fine-tuning of the interactions between particles has been realized in various experimental setups such as atomic, molecular, and optical systems [1–4]. These advances push the long-range interacting systems from the theoretical playground to the field relevant to practical applications. We here consider Hamiltonian H with power-law decaying interactions:

$$H = \sum_{i<j} h_{i,j} + \sum_{i=1}^{n} h_i, \quad \|h_{i,j}\| \le \frac{g}{r_{i,j}^{\alpha}}, \quad \alpha : \text{power exponent of the interaction decay} \tag{1}$$

where $\{h_{i,j}\}_{i<j}$ are the bi-partite interaction operators, $\{h_i\}_{i=1}^n$ are the local potentials, g is a constant of $\mathcal{O}(1)$ and $r_{i,j}$ is the distance between the spins i and j. One of the examples of controllable long-range interacting spin systems is the 1D long-range transverse Ising model, namely $h_{i,j} = J_{i,j}\sigma_i^x\sigma_j^x/r_{i,j}^{\alpha}$, where the exponent is tunable from $\alpha = 0$ to $\alpha = 3$ [3, 4]. These systems often exhibit novel quantum phases that do not appear in short-range interacting systems [5–7]. In both experimental and theoretical contexts, long-range-interacting systems play crucial roles in modern physics. So far, most existing analyses of short-range interacting systems require non-trivial modifications before they can be applied to systems with long-range interactions. In the present work, we focus on information theoretic properties in non-critical long-range interacting systems: i) the entanglement entropy in gapped ground state, ii) the conditional mutual information in Gibbs state above a temperature threshold.

Area-law conjecture.— We first consider the area-law of entanglement entropy for non-critical ground states in 1D long-range interacting systems. The area-law conjecture states that in every gapped ground states (i.e., in non-critical phases), the entanglement entropy scales at most as the boundary area of subregion. In 1D systems, for an arbitrary decomposition $\Lambda = L \cup R$ of the total system Λ, the area law is simply stated as follows:

$$S(\rho_L) \le \text{const.}, \quad \rho_L = \mathrm{tr}_R(|0\rangle\langle 0|), \tag{2}$$

where we denote the ground state as $|0\rangle$ and $S(\rho_L)$ is the von Neumann entropy, namely $S(\rho_L) = \mathrm{tr}(-\rho_L \log \rho_L)$ with $\log(\cdot)$ the natural logarithm. The area law plays a crucial role in the classification of quantum phases [8] and many of the numerical algorithms such as the density-matrix-renormalization-group algorithm [9]. Our question is whether the area law of the entanglement entropy (2) is still satisfied in the presence of long-range interactions. Typically, non-trivial quantum phases are induced by long-range interactions with power exponents smaller than three ($\alpha \le 3$). For $\alpha > 3$, the universality class is the same as that of short-range interacting systems [10] (i.e., $\alpha = \infty$). Hence, the long-range regimes (e.g, $\alpha \le 3$) are of particular interest in terms of novel quantum phases. Until now, the area law in long-range interacting systems has been highly controversial because the long-range interaction results in correlation patterns similar to the ones in critical phases. Although various numerical studies suggest that the area law holds at least for short-range regimes (i.e., $\alpha > 3$), the possibility of a sub-logarithmic correction to the standard area law (2) has been indicated for $\alpha \le 3$ [11]. In theoretical level, it has been pointed out by Hastings that there exist quantum states [12] having arbitrarily large entanglement entropy values while keeping the correlation length of order $\mathcal{O}[\log(n)]$ (i.e., corresponding to $\alpha = \infty$). Our aim is to settle the debate over the area law conjecture in 1D long-range interacting systems.

Conditional mutual information.— We second consider the decay rate of the conditional mutual information in quantum Gibbs states. When we consider the mutual information between the two subsystems A and C conditioned on the middle region B, the conditional mutual information $\mathcal{I}_\rho(A:C|B)$ is defined as follows:

$$\mathcal{I}_\rho(A:C|B) := S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^B), \tag{3}$$

where ρ^{AB} is the reduced density matrix in the subsystem ($AB = A \cup B$) and $S(\rho^{AB})$ is the von Neumann entropy, namely, $S(\rho^{AB}) := \mathrm{tr}(\rho^{AB} \log \rho^{AB})$. The conditional mutual information is a characterization of the tripartite correlation and is also deeply related to practical applications: for example, the accuracy of local recovery maps [13, 14], topological entanglement entropy [15, 16], and efficiency of the quantum Gibbs sampling [17], etc. In classical systems with short-range interaction, it has been shown that Gibbs states are equivalent to Markov networks, states that have vanishing conditional mutual information for certain partitions. For the quantum setting a similar theorem is known [18–20] but only gives rise to commuting Hamiltonians or restricted to 1D quantum systems with a not exactly but approximately vanishing condition [18]. Our aim is to prove the same (or stronger) result for any graph and both short-range and long-range interacting quantum systems.

II. SUMMARY OF MAIN RESULTS AND APPICATIONS

We first summarize our main results on the area law and decay rate of the conditional mutual information (see attached paper [21] and [22] for more rigorous statements). We second show several implications.

1. We resolve the area-law conjecture for non-critical ground states in one-dimensional long-range interacting systems in the following sense. For arbitrary one-dimensional Hamiltonians in the form of (1) with $\alpha > 2$, the entanglement entropy $S(\rho_L)$ is bounded from above by

$$S(\rho_L) \le c \log^2(d) \left(\frac{\log(d)}{\Delta}\right)^{1+2/(\alpha-2)+\eta} \tag{4}$$

where d is a dimension of local Hilbert space, Δ is the spectral gap, η is an arbitrary positive constant and c is a constant of $\mathcal{O}(1)$.

2. We show the decay rate of the conditional mutual information in generic quantum Gibbs state above a temperature threshold. Let A, B, and C be arbitrary subsystems in Λ ($A, B, C \subset \Lambda$). For arbitrary D-dimensional Hamiltonians in the form of (1) with $\alpha > D$, we consider the Gibbs state $\rho := e^{-\beta H}/Z$ (Z: partition function). Then, above a temperature threshold β_c which does not depend on the system size n, the Gibbs state satisfies

$$\mathcal{I}_\rho(A:C|B) \le \beta \min(|A|, |C|) \frac{C_\beta}{d_{A,C}^{\alpha-D}} \tag{5}$$

with $d_{A,C}$ the distance between the subsystems A and C, where $|A|$ is the cardinality of A and C_β is a constant of $\mathcal{O}(1)$ which depends on β and β_c. We give an explicit value of β_c in the attached paper [22]. In short-range interacting systems (i.e., $\alpha = \infty$), we can prove that the conditional mutual information decays exponentially with respect to the distance $d_{A,C}$ [22]. If we select B as an empty set (i.e., $B = \emptyset$), the conditional mutual information reduces to bipartite mutual information: $\mathcal{I}_\rho(A:C|\emptyset) = \mathcal{I}_\rho(A:C)$, where $\mathcal{I}_\rho(A:C) := S(\rho^A) + S(\rho^C) - S(\rho^{AC})$.

***1**a. (Remark on tightness of our area-law bound)— In the short-range limit (i.e., $\alpha \to \infty$), our area-law bound reduces to $\log^3(d)/\Delta$ and reproduces the state-of-the-art bound in short-range interacting systems by Arad et al. [23, 24]. Also, our condition $\alpha > 2$ covers important classes of long-range interactions such as van der Waals interactions ($\alpha = 6$) and dipole-dipole interactions ($\alpha = 3$). More importantly, it includes the long-range regimes of $\alpha \le 3$ where novel quantum phases can appear owing to their long-range nature. We believe that the condition $\alpha > 2$ is the best general condition. If the exponent α is smaller than two, the norm of the boundary interaction along a cut diverges in the thermodynamics limit ($n \to \infty$). Then, for $\alpha \le 2$ the system energy possesses a high-dimensional character, and hence its 1D character should be lost.

*1b. (Remark on MPS representation of gapped ground state)— On the MPS representation of the ground state $|0\rangle$, we prove the following statement: If $\alpha > 2$ and the spectral gap is nonvanishing, there exists an MPS $|\psi_D\rangle$ with bond dimensions $D = \exp[c' \log^{5/2}(1/\delta)]$ (c': constant) such that

$$\|\mathrm{tr}_{X^c}(|\psi_D\rangle\langle\psi_D|) - \mathrm{tr}_{X^c}(|0\rangle\langle 0|)\|_1 \le \delta |X| \tag{6}$$

for an arbitrary concatenated subregion $X \subset \Lambda$, where $\|\cdot\|_1$ is the trace norm. From the approximation (6), to achieve an approximation error of $\delta = 1/\mathrm{poly}(n)$, we need quasi-polynomial bond dimensions, namely $D = \exp[\log^{5/2}(n)]$. Our result justifies the MPS ansatz with small bond dimensions, obtained at a moderate computational cost.

*2a. (Remark on Quantum Gibbs sampling) Based on the Fawzi–Renner theorem [13, 14] and the results in Ref. [17], we can discuss the efficiency of quantum Gibbs sampling. For the simplicity, we here consider the $\alpha \to \infty$ limit (i.e., short-range interacting systems). Then, under the assumption of Ineq (5), there exists a $(D+1)$-depth circuit of quantum channels $\mathbb{F} = \mathbb{F}_{D+1} \cdots \mathbb{F}_2 \mathbb{F}_1$ such that

$$\|\mathbb{F}(\psi) - \rho\|_1 = 1/\mathrm{poly}(n), \tag{7}$$

where D is the dimension of the lattice, ψ is an arbitrary quantum state and each quantum channel $\{\mathbb{F}_s\}_{s=1}^{D+1}$ is composed of quasilocal CPTP maps that act on $\mathcal{O}(\log^D n)$ spins. This implies the sampling algorithm which requires only quasipolynomial computational time. Also, for $\beta < \beta_c$, this implies non-existence of the topological order under the quantum-cirquit definition [27].

[Proof Ideas on the result 1 (see Sec IV of the attached paper [21])] The primary problem is how to construct the approximate-ground-state-projection (AGSP) operators [23, 24] with appropriate properties. For the purpose, we first perform the truncation of long-range interactions. Here, we do NOT simply truncate all the long-range interactions in the entire region, but truncate the long-range interaction only around the boundary we are interested in. We decompose the total system into multiple block subsystems and truncate the interactions except the nearest-neighbor block-block interactions. Second, we introduce a multi-energy cut-off to construct an effective Hamiltonian for the AGSP operator. We consider energy spectrums of multiple block subsystems which has been introduced above and perform the energy cut-off in each of the energy spectrums. This is contrast to the standard construction of the effective Hamiltonian [23, 28] which supposes the cut-off of single energy spectrum. The multi-energy cut-off is crucial to prove the long-range area law even in the long-range power-exponent regimes (i.e., $\alpha \le 3$).

[Proof Ideas on the result 2 (see Appendix A of the attached paper [22])] The proof comes from a kind of the high-temperature expansion. The difficulty now lies in the point that the standard cluster expansion technique cannot be applied to the logarithm of the reduced density matrix (e.g., ρ^X with $X \subset \Lambda$). We introduce a new technique of the generalized cluster expansion which allows us to systematically treat logarithmic operators. Here, we parametrize the Hamiltonian as follows: $H_{\vec{a}} = \sum_{Z \in \Lambda} a_Z h_Z$ with $\vec{a} = \{a_Z\}_{Z \in \Lambda}$. By using this parametrization, we parametrize a target function of interest by $f_{\vec{a}}$ and directly expand it with respect to $\vec{a}$, where $f_{\vec{a}}$ can be chosen not only as a scholar function but also as a operator function. Here, we choose the conditional mutual information as the function $f_{\vec{a}}$. The technical difficulty in the generalized cluster expansion is to estimate the convergence radius of the Taylor expansion, where we need to consider multiple derivative of the operators like $\log[\mathrm{tr}_{X^c}(e^{-\beta H_{\vec{a}}})]$. Our technical contributions are the systematical expression of the multiple derivative in the generalized cluster expansion and the estimation of the convergence radius.

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