Unifying speed limit and Lieb-Robinson bound: Wisdom from optimal transport

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Background and motivation

The study of the speed at which both matter and information propagate is a central focus in quantum mechanics. This subject can be approached through two primary research avenues.

The first avenue can be traced back to the seminal work by Mandelstam and Tamm [J. Phys. USSR (1945)], where they established a constraint on the minimum time required for transitions between orthogonal states in closed quantum dynamics. This work led to the development of the concept of quantum speed limits, which has found applications in various areas of physics [Lloyd, Nature (2000); Giovannetti et al., PRL (2006); Becker et al., PRL (2021), QIP2021; Van Vu and Saito, PRX (2023)]. The second avenue of research emerged with Lieb and Robinson's pioneering work [CMP (1972)], which primarily focused on the speed at which information propagates in quantum spin systems. They introduced the notion of an effective light cone, beyond which information propagation exponentially decays with increasing distance. This concept, known as the Lieb-Robinson bound, has proven to be a powerful tool for analyzing quantum many-body systems and has found applications in diverse fields, including entanglement theory [Hastings, J. Stat. Mech. (2007)], correlation function theorems [Hastings and Koma, CMP (2007); Kuwahara and Saito, PRX (2022), QIP2022], classical simulation of many-body systems [Osborne, PRL (2006)], circuit complexity of quantum dynamics [Haah et al., FOCS (2018), QIP2019], Hamiltonian learning complexity [Anshu et al., Nat. Phys. (2021), QIP2021], and quantum information scrambling [Robert and Swingle, PRL (2016)].

The two previously discussed speed limit concepts have evolved independently, each with its own strengths and weaknesses. Quantum speed limits are primarily concerned with determining the shortest time required for state changes but may not provide accurate predictions for many-body systems [Bukov et al., PRX (2019)]. On the other hand, the Lieb-Robinson bound offers valuable insights into many-body systems but does not furnish information regarding the speed of transitions inside the light cone. Consequently, it seems reasonable to anticipate that unifying these two concepts could yield a novel bound that takes advantage of their respective strengths and establishes new constraints that neither concept can fully encompass on its own. To achieve this, a significant challenge lies in integrating geometric structures pertaining to the underlying dynamics into quantum speed limits, allowing for meaningful limits to be derived, even when dealing with large-scale systems.

In this research, we address the issue mentioned above by harnessing the capabilities of optimal transport theory, a well-established framework widely employed in various interdisciplinary fields [Chakrabarti et al., NeurIPS 2019; De Palma et al., IEEE Trans. Inf. Theory (2021), QIP2021, TQC2021; Rouzé and França, TQC2021; De Palma et al., PRX Quantum (2023)]. By employing the discrete Wasserstein distance, we establish a universal, rigorous, and saturable unified speed limit that can be applied to a broad spectrum of dynamics. To demonstrate the significant implications of our discovery in the realm of quantum many-body physics, we apply our finding to address a long-standing unresolved issue related to bosonic transport. More precisely, we elucidate the optimal shape of the light cone concerning bosonic transport in systems characterized by long-range hopping and long-range interactions.

Main result and applications

Setting.—We begin with a very general setup, which is applicable to both classical and quantum systems. We consider a time evolution of a vector state $\boldsymbol{x}_t \coloneqq [x_1(t), \dots, x_N(t)]^\top \in \mathbb{R}^N_{\geq 0}$ over an undirected graph $G(\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V} = \{1, \dots, N\}$ and edge set $\mathcal{E}$. For each vertex i, let $\mathcal{B}_i \coloneqq \{j \,|\, \langle i, j\rangle \in \mathcal{E}\}$ denote the set of neighboring vertices of i. For simplicity, we consider conserved states (i.e., $\sum_i x_i$ is invariant), with the generalization to the generic case being straightforward. The dynamics of $\boldsymbol{x}_t$ is described by the following deterministic equation: $\dot{x}_i(t) = \sum_{j\in\mathcal{B}_i} f_{ij}(t)$, where $f_{ij}(t) = -f_{ji}(t)$ denotes the flow exchange between vertices i and j for $i \neq j$.

Next, we introduce the discrete L^1-Wasserstein distance between two distributions $\boldsymbol{x}$ and $\boldsymbol{y}$. Suppose that we have a transport plan that redistributes $\boldsymbol{x}$ to $\boldsymbol{y}$ by sending an amount of $\pi_{ij} \geq 0$ from x_j to y_i with a cost of $c_{ij} \geq 0$ per unit mass for any i and j. Here, $\pi = [\pi_{ij}] \in \mathbb{R}_{\geq 0}^{N\times N}$ is a joint probability distribution of $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\sum_j \pi_{ij} = y_i$ and $\sum_j \pi_{ji} = x_i$, defining an admissible transport plan. The Wasserstein distance is then defined as the minimum transport cost for all feasible plans, given by

$$\mathcal{W}(\boldsymbol{x},\boldsymbol{y}) \coloneqq \min_{\pi\in\mathcal{C}(\boldsymbol{x},\boldsymbol{y})} \sum_{i,j} c_{ij}\pi_{ij}, \tag{1}$$

where $\mathcal{C}(\boldsymbol{x},\boldsymbol{y})$ denotes the set of couplings π. The cost matrix $[c_{ij}]$ can be specified arbitrarily, as long as the two requirements are fulfilled: (i) symmetry ($c_{ij} = c_{ji} \forall i,j$) and (ii) the triangle inequality ($c_{ij}+c_{jk} \geq c_{ik} \forall i,j,k$).

Main result.—Given the setup above, we prove that the minimum time required to transform state $\boldsymbol{x}_0$ into $\boldsymbol{x}_\tau$ is always lower bounded as

$$\tau \geq \frac{\mathcal{W}(\boldsymbol{x}_0,\boldsymbol{x}_\tau)}{\left\langle \sum_{\langle i,j\rangle\in\mathcal{E}} c_{ij}|f_{ij}(t)|\right\rangle_\tau}, \tag{2}$$

where $\langle z_t\rangle_\tau \coloneqq \tau^{-1}\int_0^\tau dt\, z_t$ is the time-average quantity of z_t. The unified speed limit (2) is our main result, which includes all the essence to extend the use of the conventional quantum speed limits to a wide range of quantum many-body problems with various geometric structures. To accomplish this, the proof makes extensive use of optimal transport theory, and a detailed explanation of this can be found at the end of the paper [1]. The crucial step in applying this bound effectively is the thoughtful selection of the cost matrix $[c_{ij}]$. With appropriate choices, our new bound (2) can play roles similar to the Lieb-Robinson bound.

In the following, we demonstrate two applications of our main result (2).

▹ **Application 1: Topological speed limit (attached paper [1])**

The simplest way to incorporate the topological structure is using the shortest-path distances in graph G for the cost matrix $[c_{ij}]$. That is, $c_{ij} = \min_P \mathrm{length}(P)$, where the minimum is over all paths connecting i and j. Since $c_{ij} = 1$ for any $\langle i,j\rangle \in \mathcal{E}$, we immediately obtain the topological speed limit from Eq. (2):

$$\tau \geq \frac{\mathcal{W}(\boldsymbol{x}_0,\boldsymbol{x}_\tau)}{\langle \upsilon_t\rangle_\tau}, \tag{3}$$

where $\upsilon_t \coloneqq \sum_{\langle i,j\rangle\in\mathcal{E}} |f_{ij}(t)|$ is the temporal velocity. Bound (3) utilizes topological information from the graph structure to provide a stringent constraint on the speed of changing states. Notably, it can derive novel speeds of various quantum processes, including bosonic transport in short-range systems, quantum communication through spin chains, and measurement-induced quantum walk.

▹ **Application 2: Optimal light cone for bosonic transport (attached paper [2])**

We consider a generic model of bosons on an arbitrary D-dimensional lattice, wherein bosons can hop between arbitrary sites and interact with each other. The system Hamiltonian is time-dependent and can be expressed in the following form:

$$H_t \coloneqq -\sum_{i\neq j\in\Lambda} J_{ij}(t)\hat{b}_i^\dagger\hat{b}_j + \sum_{Z\subseteq\Lambda} h_Z(t). \tag{4}$$

Here, Λ denotes the set of all the sites in the lattice, $\hat{b}_i^\dagger$ and $\hat{b}_i$ are the bosonic creation and annihilation operators for site i, respectively, and $h_Z(t)$ is an arbitrary function of $\{\hat{n}_i\}_{i\in Z}$, where $\hat{n}_i \coloneqq \hat{b}_i^\dagger\hat{b}_i$ is the number operator. Note that $h_Z(t)$ does not need to be local (i.e., Z can be arbitrarily large). In other words, both *long-range hopping* and *long-range interactions* are allowed in our setup. The hopping terms $J_{ij}(t)$ are upper bounded by a power law in the Euclidean distance, i.e., $|J_{ij}(t)| \le J/\|i-j\|^{\alpha}$, where $\|\cdot\|$ denotes the Euclidean norm and the power decay α satisfies $\alpha > D$.

Due to the critical importance of the Lieb-Robinson bound, determining the optimal shape of the light cone has become a vital challenge in quantum information science. To date, researchers have successfully characterized the light cone shape comprehensively in long-range interacting quantum spin and fermionic systems [Kuwahara and Saito, PRX (2020), TQC2019; Tran et al., PRX (2021), QIP2021]. However, the exploration of bosonic systems has remained challenging due to the unbounded nature of interactions within such systems. While the speeds of bosonic transport and information propagation have been extensively studied in systems with short-range hopping [Schuch et al., PRA (2011), QIP2011; Kuwahara and Saito, PRL (2021), TQC2021; Yin and Lucas, PRX (2022); Kuwahara et al., QIP2023], the investigation of long-range cases remains at an early developmental stage. Concerning macroscopic bosonic transport, recent work by Faupin and colleagues [Faupin et al., PRL (2022)] has demonstrated the finite velocity of macroscopic particle transport by establishing the existence of a linear light cone for cases where $\alpha > D+2$. Nevertheless, a complete classification of the effective light cone for particle transport remains an ongoing endeavor.

By assigning the vector of boson numbers to $\boldsymbol{x}_t$ and considering the cost matrix $c_{ij} = \|i-j\|^{\min(1,\alpha-D-\varepsilon)}$ in our main result (2), we obtain the following theorem.

Theorem 1. *Consider a situation where a fraction $\mu \in (0,1]$ of all bosons is transported from region X to a distant region Y. Then, the operational time τ required for this macroscopic bosonic transport is lower bounded by the distance between the two regions as*

$$\tau \ge \kappa_1^{\varepsilon} d_{XY}^{\min(1,\alpha-D-\varepsilon)}. \tag{5}$$

Here, $0 < \varepsilon < \alpha - D$ is an arbitrary number and $\kappa_1^{\varepsilon} > 0$ is a system-independent constant.

Remarks: First, the result holds for arbitrary initial states, including both pure and mixed states, and for the entire range of the power decay $\alpha > D$. Second, bound (5) implies that bosonic transport always takes time at least proportional to the distance, remaining meaningful even in the thermodynamic limit. Third, bound (5) is optimal in the sense that there exist transport protocols such that bosonic transport can be accomplished within time $\tau = O(d_{XY}^{\min(1,\alpha-D)})$ [Tran et al., PRX (2020)] (see Theorem 11 therein). Fourth, this result reveals the optimal form of the light cone for bosonic transport, thus completely resolving the open problem for generic long-range systems. Moreover, our bound (2) possesses the same advantage as the Lieb-Robinson bound in its capability to handle the probability of observing a specific number of bosons within the target region Y (see Theorem 2 in [2]). Last, it is worth emphasizing that our proof technique is considerably simpler, requiring only one page of exposition, in contrast to previous analyses that relied on the Lieb-Robinson bound.

Summary and outlook

Through the application of optimal transport theory, we have introduced a unified speed limit that integrates two fundamental concepts: quantum speed limits and the Lieb-Robinson bound. As a practical demonstration of our result, we have successfully resolved a long-standing problem in the field of bosonic transport within quantum information science. Moreover, our discovery holds significant promise for a wide array of applications. This includes investigating the speed of various quantum information processes, such as quantum adiabatic algorithms [Kieu, Proc. R. Soc. A (2019); Chen, PRR (2023)], quantum annealing [Garc'ia-Pintos et al., PRL (2023)], performance of quantum devices [Ness et al., Sci.Adv.(2021)], and machine learning [Seroussi et al., arXiv (2023)].

[1] T. Van Vu and K. Saito, Topological speed limit, Phys. Rev. Lett. **130**, 010402 (2023).
[2] T. Van Vu, T. Kuwahara, and K. Saito, Optimal form of light cones for bosonic transport in long-range systems, arXiv:2307.01059 (2023).