Inverse scattering problems in quantum graphs & periphery of Hadamard conjecture

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Introduction

- Interests in **quantum graph** as
  -- solvable playground for **quantum exotica**
  -- model of single-electron **sub-nano** device

- Importance of inverse scattering problem
  -- physical realization of **unitary matrices**
  -- design single electron **device on demand**

- Examine inverse scattering in simplest
  quantum graph ---> **scale-invariant vertex**
Connections in quantum vertex

- boundary vectors
  \[ \Psi = \begin{pmatrix} \psi_1(0+) \\ \vdots \\ \psi_n(0+) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'_1(0+) \\ \vdots \\ \psi'_n(0+) \end{pmatrix} \]

- (self-adjoint extension) flux conservation
  \[ A \Psi + B \Psi' = 0 \]

\[ A = I - U, \quad U : U(n) \]
\[ B = iL_0 (I + U) \]

\[ A = I - U, \quad U : U(n) \]
\[ B = iL_0 (I + U) \]

\[ (Fulop&Tsutsui '00) \]

\[ \text{rank}(A, B) = n \]
\[ AB^\dagger = BA^\dagger \]

\[ (Kostrykin&Schrader '99) \]
Scattering in quantum graphs

- scattering for incoming wave at j-th line

\[ \psi_i^{(j)}(x_i) = e^{-ikx_i} + S_{jj}e^{ikx_i} \quad (i = j) \]
\[ = S_{ij}e^{ikx_i} \quad (i \neq j) \]

- \( S(k) = \{S_{ij}(k)\} \): scattering matrix

\[ A\psi + B\psi' = 0 \]

\[ \implies A\left(S(k) + I\right) + ikB\left(S(k) - I\right) = 0 \]

- \( S(k) = -\frac{1}{A + ikB}(A - ikB) \)
Fulop-Tsutsui vertex

- scale invariant connection condition \( T: m \times (n-m) \)

\[
\begin{pmatrix}
  I^{(m)} & T \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  \Psi'
\end{pmatrix}
=
\begin{pmatrix}
  0 & 0 \\
  -T^\dagger & I^{(n-m)}
\end{pmatrix}
\begin{pmatrix}
  \Psi
\end{pmatrix}
\]

- eg. free connection: \( m=1, \ T=(1 \ldots 1) \)

\[
\psi_1 = \psi_2 = \ldots = \psi_n
\]

\[
\psi_1' + \psi_2' + \ldots + \psi_n' = 0
\]

- in general, constant transmission with fixed branching ratio & w.f. mismatch \(<---- both controlled by \( T \)\)
Inverse scattering for FT vertex

- **Hermitian unitary matrix:** \( \text{rank}(S + I^{(n)}) = m \)

\[
S + I^{(n)} = \begin{pmatrix} I^{(m)} \\ T^\dagger \end{pmatrix} M \begin{pmatrix} I^{(m)} \\ T \end{pmatrix} \quad (S + I^{(n)})^2 = 2(S + I^{(n)}) \\
M = 2(I^{(m)} + TT^\dagger)^{-1}
\]

--> solution of inverse scattering of FT vertex

\[
S = -I^{(n)} + 2 \begin{pmatrix} I^{(m)} \\ T^\dagger \end{pmatrix} \left( I^{(m)} + TT^\dagger \right)^{-1} \begin{pmatrix} I^{(m)} \\ T \end{pmatrix}
\]

- eqv. to a diagonalization

\[
Z_m = \begin{pmatrix} I^{(m)} & 0 \\ 0 & -I^{(n-m)} \end{pmatrix}
\]

\[
S = X_m^{-1} Z_m X_m
\]

\[
X_m = \begin{pmatrix} I^{(m)} & T \\ T^\dagger & -I^{(n-m)} \end{pmatrix}
\]
Alternative procedure to get $T$

- divide $S$ into $m$ & $(n-m)$ sub-matrices $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$

$$S_{11} = -I^{(m)} + 2 \left( I^{(m)} + TT^\dagger \right)^{-1}$$

$$S_{12} = S_{21}^\dagger = 2 \left( I^{(m)} + TT^\dagger \right)^{-1} T$$

$$S_{22} = I^{(n-m)} - 2 \left( I^{(n-m)} + T^\dagger T \right)^{-1}$$

- inverse scattering $S \rightarrow T$ obtained as

$$T = \left( I^{(m)} + S_{11} \right)^{-1} S_{12} = S_{21}^\dagger \left( I^{(n-m)} - S_{22} \right)^{-1}$$

unique apart from re-indexing
Finite approximations

- free & delta: easy-to-realize
- realize FT conditions with jumps
  -- remove the node
  -- calculate
  \[ Q = \begin{pmatrix} T \\ \text{I}^{(n-m)} \end{pmatrix} \begin{pmatrix} -T^\dagger \\ \text{I}^{(m)} \end{pmatrix} = \begin{pmatrix} -TT^\dagger & T \\ -T^\dagger & \text{I}^{(m)} \end{pmatrix} \]
  \[ r_{ij} e^{i\chi_{ij}} = Q_{ij} \]
  \[ V = \frac{1}{d} (2I^{(n)} - F^{(n)}) R \]
  \[ R = \{ r_{ij} \} = \{ |Q_{ij}| \} \]
  -- connect edges \((ij)\) with internal line with \(d/r_{ij}\)
  -- apply magnetic \(A\) on \((ij)\) to give phase shift \(e^{i\chi_{ij}}\)
  -- place delta with strength \(V_j\)
Hermitian unitary matrices

- infos compressible to $m \times (n-m)$ complex matrix $T$

$$S = 2 \left( \frac{1}{T^\dagger \frac{1}{I^{(m)} + TT^\dagger}} \frac{1}{I^{(m)} + TT^\dagger} T \right) - I^{(n)}$$

- eigenvalue $+1/-1$

- $S = X_m^{-1} Z_m X_m$

- Important quantities

- $m = \text{rank}(S + I^{(n)})$

- $\text{Tr}S = \text{Tr}(X_m^{-1} Z_m X_m) = \text{Tr}Z_m = 2m-n$

- examples ----->
Free & free-like connection

- free (m=1) and free’ (m=n-1) are dual
- a free-like connection condition with m=n/2 exists

\[
S(k) = \begin{pmatrix}
1 - \frac{2}{n} & \cdots & -\frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
-\frac{2}{n} & \frac{2}{n} & 1 - \frac{2}{n} & \cdots & \frac{2}{n} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\frac{2}{n} & \cdots & \frac{2}{n} & \cdots & -1 + \frac{2}{n} & \cdots \\
\frac{2}{n} & \cdots & \frac{2}{n} & \cdots & \cdots & -1 + \frac{2}{n}
\end{pmatrix} \quad \rightarrow \quad T = \frac{2}{n} \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}
\]

- an example of “equitransmitting” quantum graph
- diag./nondiag ratio |S_{jj}|/|S_{ij}|=n/2-1
  :maximal attainable value for equitransmitting S
Hadamard Conference

\[ S = \frac{1}{\sqrt{5}} \begin{pmatrix}
0 & -1 & -1 & -1 & 1 & 1 \\
-1 & 0 & -1 & 1 & -1 & 1 \\
-1 & -1 & 0 & 1 & 1 & -1 \\
-1 & 1 & 1 & 0 & 1 & 1 \\
1 & -1 & 1 & 1 & 0 & 1 \\
1 & 1 & -1 & 1 & 1 & 0 \\
\end{pmatrix} \]

\[ \gamma = (\sqrt{5} - 1)/2 \quad \text{golden mean} \]

\[ T = \begin{pmatrix}
1 & 1+\gamma & 1+\gamma \\
1+\gamma & 1 & 1+\gamma \\
1+\gamma & 1+\gamma & 1 \\
\end{pmatrix} \]

Conference, only with \( n = 6, 10, 14, 18, \ldots \)

✧ Reflectionless & Equiscattering \( S \) matrices

Hadamard, only with \( n = 4, 8, 12, 16, \ldots \)

\[ T = \frac{1}{\sigma+1} \begin{pmatrix}
\sigma & 1 & 1 & 1 \\
1 & \sigma & 1 & 1 \\
1 & 1 & \sigma & 1 \\
1 & 1 & 1 & \sigma \\
\end{pmatrix} \]

\[ \sigma = \sqrt{2} - 1 \quad \text{silver mean} \]

\[ S = \frac{1}{\sqrt{8}} \begin{pmatrix}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
\end{pmatrix} \]
Finite graph approx. examples

- free-like
  \[ n=6 \]

- conference
  \[ n=6 \]

- Hadamard
  \[ n=8 \]

\[ n=10 \]
Scattering by depth-one graphs

- Finite graph; w.f. on internal line of length $d/r$

\[
\begin{pmatrix}
\phi'(0)
\\
e^{ix} \phi'(\frac{d}{r})
\end{pmatrix} = -\frac{r}{d} \begin{pmatrix}
F(\frac{d}{r})
\\
G(\frac{d}{r})
\end{pmatrix} - \begin{pmatrix}
G(\frac{d}{r})
\\
-F(\frac{d}{r})
\end{pmatrix}
\]

\[F(x) = x \cot x\]
\[G(x) = x \cosec x\]

\[d\psi'_i = (v_i d + \sum_{l \neq i} r_{il} F_{il}) \psi_i - \sum_{l \neq i} e^{ix_{ij}} r_{il} G_{il} \psi_l,\]

- converge at $d \to 0$ to desired $S$
Equitransmitting matrix

- Consider **Hermitian unitary** matrix of the form

\[
S = \frac{1}{N} \begin{pmatrix}
  d & e^{i\phi_{12}} & \cdots & e^{i\phi_{1n}} \\
  e^{i\phi_{21}} & d & \cdots & e^{i\phi_{2n}} \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{i\phi_{n-1,1}} & \cdots & -d & e^{i\phi_{n-1,n}} \\
  e^{i\phi_{n1}} & \cdots & e^{i\phi_{nn-1}} & -d
\end{pmatrix}
\]

- \( \rho = \# [+d] \), \( m = \text{rank}(S+I^{(n)}) \)

- \( d = (2m-n)/(2\rho-n) \)

- \( \text{Tr}S = \text{Tr}(X_{m}^{-1}Z_{m}X_{m}) = \text{Tr}Z_{m} \)

- \( \rightarrow \) discrete \( d \) only, if \( \rho \neq m \)

- value of \( d \) not limited by above for \( m = n/2 \) (=\( \rho/2 \))
Existence problem

- what kind of equitransmitting graph exists?
- -> for equitransmitting $S$ of FT vertex

$$S = \frac{1}{N} \begin{pmatrix}
    d & e^{i\phi_{12}} & \cdots & e^{i\phi_{1n}} \\
    e^{i\phi_{21}} & d & \cdots & e^{i\phi_{1n}} \\
    \vdots & \ddots & \ddots & \vdots \\
    e^{i\phi_{n-11}} & \cdots & -d & e^{i\phi_{n-1n}} \\
    e^{i\phi_{n1}} & \cdots & e^{i\phi_{nn-1}} & -d
\end{pmatrix}$$

determine the conditions for

-- existence of $S$ with given $d$
-- existence of $S$ with real $d$
-- existence of $S$ with integer $d$
Some examples

- $n=6$

$$F_6 = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & 1 \\
-1 & 2 & -1 & 1 & 1 & 1 \\
-1 & -1 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & -2 & 1 & 1 \\
1 & 1 & 1 & 1 & -2 & 1 \\
1 & 1 & 1 & 1 & 1 & -2
\end{pmatrix}$$

$$H_6 = \begin{pmatrix}
1 & -1 & -1 & i & 1 & 1 \\
-1 & 1 & -1 & 1 & i & 1 \\
-1 & -1 & 1 & 1 & 1 & i \\
-i & 1 & 1 & -1 & 1 & 1 \\
1 & -i & 1 & 1 & -1 & 1 \\
i & 1 & -i & 1 & 1 & -1
\end{pmatrix}$$

$$C_6 = \begin{pmatrix}
0 & -1 & -1 & -1 & 1 & 1 \\
-1 & 0 & -1 & 1 & -1 & 1 \\
-i & -1 & 0 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 0
\end{pmatrix}$$

- $n=8$

$$\begin{pmatrix}
A & B \\
B^\dagger & -A
\end{pmatrix}$$

- 

$$\begin{pmatrix}
3 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 3 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 3 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 3 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -3 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -3 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -3 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -3
\end{pmatrix}$$

$$\begin{pmatrix}
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{pmatrix}$$

$$\begin{pmatrix}
0 & -i & i & -i & i & -i & i & -i \\
i & 0 & i & i & i & i & i & i \\
i & -i & 0 & i & -i & i & i & i \\
i & i & -i & 0 & i & -i & i & i \\
i & -i & i & -i & 0 & i & -i & i \\
i & i & -i & i & -i & 0 & i & -i \\
i & -i & i & -i & i & -i & 0 & i \\
i & i & -i & i & -i & i & -i & 0
\end{pmatrix}$$
More examples

- $n=10$

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Some theorems and conjectures

- $n/2-1 \geq d \geq 0$ : possible range  
  $S = \frac{1}{N} \begin{pmatrix} 
  d & \cdots & e^{i\phi_1n} \\
  \vdots & \ddots & \vdots \\
  e^{i\phi_{n-1}} & \cdots & -d 
\end{pmatrix}$

- only $m=n/2$ (?) -> $n=$ even?

- $d = n/2-1$ : free-like matrix : real $S$ for even $n$

- real $S$ possible only for $d = n/2-1, n/2-3, ..., 0$ or $1$

- $d = 1$ : Hadamard matrix 
  real $S$ conjectured to exist for evenly even $n$

- $d = 0$ : conference matrix 
  real $S$ conjectured to exist for some oddly even $n$
Constructing equitransmitting $S$

- Basic idea is extending Paley's 2nd construction:

\[
\mathcal{M} = \begin{pmatrix} A & B \\ B^\dagger & -A \end{pmatrix} \quad A^\dagger = A \quad AB - BA = 0 \quad A^2 + BB^\dagger = aI^{(m)}
\]

then $\mathcal{M}$ is Hermitian and $\mathcal{M}^2 = aI^{(n)}$

- $A = (d+1)I^{(n)} - J^{(n)}$
- $A = dI^{(n)} - C$ \quad $C$: conference matrix or its core
- $B = e^{i\lambda}I^{(n)} + (J^{(n)} - I^{(n)})$, $A^\dagger = A$, $AB - BA = 0$ guaranteed
- $B = e^{i\phi C}$
Full-house & conference matrix

- full house matrix $J$ with any $m$
  \[ M = \begin{pmatrix}
    (d + 1)I^{(m)} - J^{(m)} & (e^{i\chi} - 1)I^{(m)} + J^{(m)} \\
    (e^{-i\chi} - 1)I^{(m)} + J^{(m)} & -(d + 1)I^{(m)} + J^{(m)}
  \end{pmatrix} \]
  \[ M^2 = (d^2 + 2m - 1)I^{(n)} + 2(m - 2 + \cos \chi - d) \begin{pmatrix}
    J^{(m)} - I^{(m)} & 0 \\
    0 & J^{(m)} - I^{(m)}
  \end{pmatrix} \]
  \[ \frac{n}{2} - 3 \leq d \leq \frac{n}{2} - 1 \quad (n = \text{even}) \]

- conference matrix $C$ with even $m$
  \[ M = \begin{pmatrix}
    dI^{(m)} + C & -e^{i\chi}I^{(m)} + C \\
    -e^{-i\chi}I^{(m)} + C & -dI^{(m)} - C
  \end{pmatrix} \]
  \[ M^2 = (d^2 + 2m - 1)I^{(n)} + 2(d - \cos \chi) \begin{pmatrix}
    C & 0 \\
    0 & C
  \end{pmatrix} \]
  \[ 0 \leq d \leq 1 \quad (n = 4, 8, 12, 16, 20, \ldots) \]
Hadamard/conference Core

- Core of Hadamard/conference matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & ie^{i\varphi} & -1 & -ie^{i\varphi} \\
1 & -1 & 1 & -1 \\
1 & -ie^{i\varphi} & -1 & ie^{i\varphi}
\end{pmatrix}
\]

Hadamard core \((m=3)\)

\[
H_c^cH_c^{c\dagger} = (m+1)I^{(m)} - J^{(m)}
\]

- Use conference core with any \(m\), \(C_c^cC_c^{c\dagger} = mI^{(m)} - J^{(m)}\)

\[
\mathcal{M} = 
\begin{pmatrix}
(d+1)I^{(m)} - J^{(m)} & C_c^c \\
C_c^{c\dagger} & -(d+1)I^{(m)} + J^{(m)}
\end{pmatrix}
\]

\[
\mathcal{M}^2 = (d^2+2m-1)I^{(n)} + 2(m-6-2d)
\begin{pmatrix}
J^{(m)} - I^{(m)} & 0 \\
0 & J^{(m)} - I^{(m)}
\end{pmatrix}
\]

- \(d = \frac{n-6}{4} \) \((n = \text{even})\)
Symmetric conference core (1)

- for \( m = 5, 9, 13, 17, 21, \ldots \)

\( C^c: \) core of conference, \( C^c C^{c \dagger} = mI^{(m)} - J^{(m)} \)

\[ B_{i,j} = e^{i\varphi C^c_{i,j}} \quad \rightarrow \quad \text{write as} \quad B = e^{i\varphi C^c} \]

\[ M = \begin{pmatrix} (d + 1)I^{(m)} - J^{(m)} & e^{i\varphi C^c} \\ e^{-i\varphi C^c} & -(d + 1)I^{(m)} + J^{(m)} \end{pmatrix} \]

\[ M^2 = (d^2 + 2m - 1)I^{(n)} + \frac{3m - 7 + 4\cos \varphi + (m - 1)\cos 2\varphi - 4d}{2} \begin{pmatrix} J^{(m)} - I^{(m)} & 0 \\ 0 & J^{(m)} - I^{(m)} \end{pmatrix} \]

- \[
\frac{n - 6}{4} - \frac{1}{n - 2} \leq d \leq \frac{n}{2} - 1 \quad (n = 10, 18, 26, 34, 42, \ldots)\]
Symmetric conference core (2)

- for \( m = 5, 9, 13, 17, 21, \ldots \)

\[
\mathcal{M} = \begin{pmatrix}
\frac{dI^{(m)} + C_c}{(e^{-i\chi} - 1)I^{(m)} + e^{-i\varphi}C_c} & (e^{i\chi} - 1)I^{(m)} + e^{i\varphi}C_c \\
-dI^{(m)} - C^c & \frac{dI^{(m)} + C^c}{(e^{-i\chi} - 1)I^{(m)} + e^{-i\varphi}C_c}
\end{pmatrix}
\]

\[
\mathcal{M}^2 = (d^2 + 2m - 1)I^{(n)} + (F_+(\varphi, \chi) + d) \begin{pmatrix}
C_c + J^{(m)} - I^{(m)} & 0 \\
0 & C_c + J^{(m)} - I^{(m)}
\end{pmatrix}
\]

\[
+ (F_-(\varphi, \chi) - d) \begin{pmatrix}
-C_c + J^{(m)} - I^{(m)} & 0 \\
0 & -C_c + J^{(m)} - I^{(m)}
\end{pmatrix}
\]

\[
F_\pm(\varphi, \chi) = \frac{m - 5 + 4 \cos(\varphi \mp \chi) + (m - 1) \cos 2\varphi}{4}
\]

- with \( d = \frac{F_+(\varphi, \chi) + F_-(\varphi, \chi)}{2} \), \( F_+(\varphi, \chi) + F_-(\varphi, \chi) = 0 \)

\[
0 \leq d = \leq d^*_s (\approx 0.8) \quad (n = 10, 18, 26, 34, 42, \ldots)
\]
Asymmetric conference core (1)

\[ M = \begin{pmatrix}
(d + 1)I^{(m)} - J^{(m)} & (e^{i\chi} - 1)I^{(m)} + e^{i\phi}C^c \\
(e^{-i\chi} - 1)I^{(m)} + e^{i\phi}C^c & -(d + 1)I^{(m)} + J^{(m)}
\end{pmatrix} \]

\[ M^2 = (d^2 + 2m - 1)I^{(n)} + \frac{3m - 7 + 4\cos \chi \cos \phi + (m - 1)\cos 2\phi - 4d}{2} \begin{pmatrix}
J^{(m)} - I^{(m)} & 0 \\
0 & J^{(m)} - I^{(m)}
\end{pmatrix}
+ 4i \sin \phi \sin \frac{\phi + \chi}{2} \sin \frac{\phi - \chi}{2} \begin{pmatrix}
C^c & 0 \\
0 & C^c
\end{pmatrix} \]

\[ \frac{n - 6}{4} \leq d \leq \frac{n}{2} - 1 \quad (n = 6, 14, 22, 30, 38, \ldots) \]
Asymmetric conference core (2)

- for \( m = 3, 7, 11, 15, 19, \ldots \)

\[
\mathcal{M} = \begin{pmatrix}
    dI^{(m)} - iC^c & (e^{i\chi} - 1)I^{(m)} + e^{i\varphi}C^c \\
    (e^{-i\chi} - 1)I^{(m)} + e^{i\varphi}C^c & -dI^{(m)} + iC^c
\end{pmatrix}
\]

\[
\mathcal{M}^2 = (d^2 + 2m - 1)I^{(m)} + G_+(\varphi, \chi) \begin{pmatrix}
    J^{(m)} - I^{(m)} & 0 \\
    0 & J^{(m)} - I^{(m)}
\end{pmatrix}
\]

\[
+ (G_-(\varphi, \chi) - 2d)i \begin{pmatrix}
    C^c & 0 \\
    0 & C^c
\end{pmatrix}
\]

\[
G_+(\varphi, \chi) = \frac{1}{2} (m - 5 + 4 \cos \varphi \cos \chi + (m - 1) \cos 2\varphi)
\]

\[
G_-(\varphi, \chi) = 2(\cos \chi - \cos \varphi) \sin \varphi
\]

- with \( d = \frac{G_-(\varphi, \chi)}{2}, \quad G_+(\varphi, \chi) = 0. \)

\[
0 \leq d \leq d_0^* \approx 1.2 \quad (n = 6, 14, 22, 30, 38, \ldots)
\]
Trying to fill the gap

for \( m = 3, 7, 11, 15, 19, \ldots \)

\[
\mathcal{M} = \begin{pmatrix}
(d - 1)I^{(m)} - e^{i\xi}C^c & (e^{i\chi} - 1)I^{(m)} + e^{i\varphi}C^c \\
(e^{-i\chi} - 1)I^{(m)} + e^{i\varphi}C^c & -dI^{(m)} + e^{i\xi}C^c
\end{pmatrix}
\]

\[
\mathcal{M}^2 = (d^2 + 2m - 1)I^{(n)} + (G_+(\varphi, \chi, \xi) - 2d \cos \xi) \begin{pmatrix}
J^{(m)} - I^{(m)} \\
0
\end{pmatrix}
\]

\[
+ (G_-(\varphi, \chi, \xi) - 2d \sin \xi)i \begin{pmatrix}
C^c \\
0
\end{pmatrix}
\]

\[
G_+(\varphi, \chi, \xi) = \frac{1}{2} (2m - 6 + 4 \cos \varphi \cos \chi + (m - 1)(\cos 2\xi + \cos 2\varphi))
\]

\[
G_-(\varphi, \chi, \xi) = 2 \sin \varphi \cos \chi - \sin 2\xi - \sin 2\varphi
\]

with

\[
d = \frac{1}{2} (G_+(\varphi, \chi, \xi) \cos \xi + G_-(\varphi, \chi, \xi) \sin \xi)
\]

\[
G_+(\varphi, \chi, \xi) \sin \xi - G_-(\varphi, \chi, \xi) \cos \xi = 0
\]

\[
0 \leq d \leq d_b^* \quad (\approx 1.5) \\
(n = 6, 14, 22, 30, 38, \ldots)
\]
Current state

$\alpha = 1.8 i / 1.8 i$

Diag./Non-diag. Ratio of Equitransmitting Quantum Graph

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<th>$\alpha / \pi$</th>
<th>6</th>
<th>8</th>
<th>10</th>
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<th>14</th>
<th>16</th>
<th>18</th>
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<th>24</th>
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<th>28</th>
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<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
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<tr>
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<td>2</td>
<td>3</td>
<td>5</td>
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<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>Had/Contrav. $n - \frac{5}{2}$</td>
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<td>3</td>
<td>4</td>
<td>5</td>
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</table>

As of Feb 8, 2011
Some observations & questions

- \( n = 8k + 2 \)
  -- real integer \( d \) found at \( n/2 - 1, n/2 - 3, 1 \).
  -- whole range probably covered by complex \( S \)

- \( n = 8k + 6 \)
  -- real integer \( d \) found at \( n/2 - 1, n/2 - 3, n/4 - 3/2, 0 \).
  -- not clear whether whole range covered

- \( n = 4k \) (\( n = 8k, 8k + 4 \))
  -- real integer \( d \) found at \( n/2 - 1, n/2 - 3, 1 \).
  -- gap b/w lower and higher \( d \) regions? or disprove

- \( \rightarrow \) prove gap exists / prove these exhaust reals / etc.
Summary

- Inverse scattering of Fulop-Tsutsui vertex solved as diagonalization of Hermitian unitary matrix
- Physical realization of $S$ given as depth-1 graphs -- realization of Hadamard and conference matrices
- Consideration of equitransmitting quantum graphs offers generalization of Hadamard matrix -- Intriguing pattern emerges that may be useful in dealing Hadamard conjecture-related problems
References

