PETR SEBA IN KOCHI, JAPAN
OR THE STORY OF TEA

• Green

• Black
MODELING DEMOCRATIC DEBATE

• How collective decisions are achieved?

• Majority principle ubiquitous from bee to human

  with some twist

• Interpret democracy as
  assertive minorities in search of majority support
  and try to build mathematical model

• Dynamical systems theory of political cycle obtained
POLYA URN
WHICH ONE?
POLYTA URN

- $t$ balls in two colors ($m$ black/ $t-m$ white) in an urn at time $t$
- At update $t \rightarrow t+1$, a ball randomly drawn, put back with an additional ball with same color
- What is the ratio of black balls $p_t = m/t$ at $t \rightarrow \infty$?
Polya Urn

- $p_{\text{Final}}$
- no fixed points random walk

\begin{itemize}
  \item $p(0) = 0.3$
  \item $p(0) = 0.7$
\end{itemize}

$p(t+1)$ vs $p(t)$

Graphs showing the evolution of probabilities over time for different initial probabilities.
POLYA URN
EXTENDED

• At update $t \rightarrow t+1$, $r$ ball randomly drawn, put back with an additional ball with **majority color**

• What is the ratio of black balls $p_t = m/t$ at $t \rightarrow \infty$?

$r=3$
POLYA URN
EXTENDED

• $p_{\text{Final}}$

• stable fixed points

$r = 3$

$p(0) = 0.3$

$p(0) = 0.7$

all black

all white
At update $t \to t+1$
- add a black / white ball with prob. $a / b$
- with prob. $1-a-b$, count all $t$ balls,
  add a black ball with prob. $Q(p_t)$, white with $1-Q(p_t)$
• $p_{\text{Final}}$

mostly white

• Phase transition at $a = a^*$
At update $t \rightarrow t+1$

- count all $t$ balls, add a ball with majority color, except...
- if $i$ hard-black balls found, add a hard-black with prob. $a = i/t$
- if $j$ hard-white balls found, add a hard-white with prob. $b = j/t$

$Q(p) = \Theta(p^{-1/2}), \quad p_t = m/t$
At update $t \rightarrow t+1$

- Sample $r$ balls, add a ball with majority color, except...
  - If $\alpha$ hard-black balls found, add a hard-black with $s_a = (1 + f_\pm)\alpha/r$
  - If $\beta$ hard-white balls found, add a hard-white with $s_b = (1 + g_\pm)\beta/r$

$Q(q) = \Theta(q-1/2), \quad q = \mu/r$

$\pm$: black/white majority

$\alpha = 0..\mu$

$\mu = 0..r$

$\beta = 0..r-\mu$
“DYNAMICAL” OPINION DYNAMICS
OUR MODEL AS EXTENDED GALAM MODEL

• Two-state agents evolving by *group-majority rule* (size $r$) with the presence of inflexible agents
EXTREMISTS AND MODERATES
EBB AND FLOW

- Committed few (extremists) drives political movement
- Extremists thrive in hostile environment
- Extremists normally lose their edge after success (moderates tend to suppress them in dominance)
- \( \rightarrow \) Increase/decrease rate of hard-black
  in friendly environ \((1+f_+) < 1\); in hostile environ \((1+f_-) > 1\)
- \( \rightarrow \) Increase/decrease rate of hard-white
  in friendly environ \((1+g_-) < 1\); in hostile environ \((1+g_+) > 1\)
“DYNAMICAL” OPINION DYNAMICS

- Evolution equation for majority and assertive minorities
  
  \[ p_{t+1} = P_+^{(r)}(p_t, a_t, b_t) \]
  \[ a_{t+1} = P_A^{(r)}(p_t, a_t, b_t) \]
  \[ b_{t+1} = P_B^{(r)}(p_t, a_t, b_t) \]

- Increase/decrease rate of hard-black
  
  - in friendly environ \((1+f_+)<1\); in hostile environ \((1+f_-)>1\)

- Increase/decrease rate of hard-white
  
  - in friendly environ \((1+g_-)<1\); in hostile environ \((1+g_+)>1\)

- Hard-black/hard-white appearance in all white/black env: \(f_A / g_B\)
GENERAL FORMULA FOR OPINION UPDATE
FOR ARBITRARY GROUP SIZE

(generalization of Cheon-Galam 2017)

\[ P_+^{(r)} = \sum_{\mu=0}^{r} P_+^{(r,\mu)}, \quad P_A^{(r)} = \sum_{\mu=0}^{r} P_A^{(r,\mu)}, \quad P_B^{(r)} = \sum_{\mu=0}^{r} P_B^{(r,\mu)}. \]

- \( \mu < r/2 \)
  \[ P_+^{(r,\mu)}(p, a, b; f_-) = \binom{r}{\mu} p^{\mu-1} (1 - p)^{r-\mu} \cdot \frac{\mu}{r} a(1 + f_-), \]
  \[ P_A^{(r,\mu)}(p, a; f_-) = \binom{r}{\mu} p^{\mu-1} (1 - p)^{r-\mu} \cdot \frac{\mu}{r} a(1 + f_-), \]
  \[ P_B^{(r,\mu)}(p, b; g_j) = \binom{r}{\mu} p^{\mu} (1 - p)^{r-\mu-1} \cdot \frac{r - \mu}{r} b(1 + g_-), \]

- \( \mu > r/2 \)
  \[ P_+^{(r,\mu)}(p, a, b; g_+) = \binom{r}{\mu} p^{\mu} (1 - p)^{r-\mu} - \binom{r}{\mu} p^{\mu} (1 - p)^{r-\mu-1} \cdot \frac{r - \mu}{r} b(1 + g_+), \]
  \[ P_A^{(r,\mu)}(p, a; f_+) = \binom{r}{\mu} p^{\mu-1} (1 - p)^{r-\mu} \cdot \frac{\mu}{r} a(1 + f_+), \]
  \[ P_B^{(r,\mu)}(p, b; g_+) = \binom{r}{\mu} p^{\mu} (1 - p)^{r-\mu-1} \cdot \frac{r - \mu}{r} b(1 + g_+). \]
"DYNAMICAL" OPINION DYNAMICS

- Evolution equation for majority and assertive minorities
  \[ P_+^{(3)}(p, a, b; f, g) = 3p^2 - 2p^3 + (1 + f_-)(1 - p)^2a - (1 + g_+)p^2b \]
  \[ + \frac{1}{3}f_A(1 - p - b)^3 - \frac{1}{3}g_B(p - a)^3 \]

  \[ P_A^{(3)}(p_t, a_t, b_t; f, g) = \{1 + f_+ + (f_- - f_+)(1 - p)^2\} a + \frac{1}{3}f_A(1 - p - b)^3 \]

  \[ P_B^{(3)}(p_t, a_t, b_t; f, g) = \{1 + g_- + (g_+ - g_-)p^2\} b + \frac{1}{3}g_B(p - a)^3 \]

  For full explicit expression for general \( r \), see T.Cheon 2017 (in draft)

- Increase/decrease of hard-black
  in friendly environ \((1+f_+)<1\); in hostile environ \((1+f_-)>1\)

- Increase/decrease of hard-white
  in friendly environ \((1+g_-)<1\); in hostile environ \((1+g_+)>1\)
FIXED POINT AND LIMIT CYCLE

• numerics with $r=3$; Phase space trajectories

$a^*(\text{or } b^*) \approx 3-2\sqrt{2}$ 17%  
$p^* (\text{or } 1-p^*) \approx (2-\sqrt{2})/2$ 29%

$f_-g_+ = 0.2, f_+ - g_- = -0.3, f_A = g_B = 0.00$

$f_-g_+ = 0.2, f_+ - g_- = -0.3, f_A = g_B = 0.05$
POLITICAL CYCLES
MAJORITY-ALTERNATING CYCLE

- numerics with $r=3$

\[ f_+ = g_+ = 0.2, f_- = g_- = -0.3, f_A = g_B = 0.0545 \]

\[ a^* (\text{or } b^*) \sim 3 - 2\sqrt{2} \quad 17\% \]

\[ p^* (\text{or } 1-p^*) \sim (2 - \sqrt{2})/2 \quad 29\% \]
POLITICAL CYCLES
CYCLE WITHIN MINORITY

• numerics with $r=3$

$$f_+=0.2, f_-=0.3, f_A=g_B=0.0545$$

$$a^* \text{ (or } b^*) \sim 3-2\sqrt{2} \quad 17\%$$

$$p^* \text{ (or } 1-p^*) \sim (2-\sqrt{2})/2 \quad 29\%$$
ON POLITICAL CYCLES

• Two fixed points with different majorities
  \( \{p^*,a^*,b^*\} \sim \{0.3, 0.17, 0\} \)
  \( \{p^*,a^*,b^*\} \sim \{0.7, 0, 0.17\} \)  

30% minority with 17% extremists

What our model predicts

• Minority cycle with ebb and flow of extremists

• Majority changing cycle with double period — extremists driving the early phase of take-over, then disappears

What our model predicts
COMPLEXITIES

BASINS OF ATTRACTION OF TWO MINORITY-CYCLES

- three sections of initial value space \{p_0, a_0, b_0\} that results in black/white minority cycles at “critical” parameters

\[
f_-=g_+=0.2, \quad f_+=g_-=0.46, \quad f_A=g_B=0.0757, \quad x=y=0.02
\]
SUMMARY

- Tea, green & black
- Dynamical systems theoretical model public opinion developed
- The theory unifies opinion dynamics models of Galam and Mori- Hisakado with analytical expressions in the form of Polya urn
- Existence of political cycles in the model discovered which may be capturing some aspect of real politics

—> Toward the mathematical theory of politics!
Dedicated to our great Tea Mater, Petr Seba, for the occasion of his 60th birthday.