DYNAMICAL GALAM MODEL

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MODELING DEMOCRATIC DEBATE

➤ How collective decisions are achieved?
➤ *Majority principle* ubiquitous from bee to human

\[
\text{with some twist}
\]

➤ Interpret democracy as assertive minorities in search of majority support and try to build mathematical model

➤ Dynamical systems theory (toy model) of *political cycle*
POLYA URN
POLYA URN

- \( t \) balls in two colors (\( m \) black/ \( t-m \) white) in an urn at time \( t \)
- At update \( t \rightarrow t+1 \), a ball randomly drawn, put back with an additional ball with same color
- What is the ratio of black balls \( p_t = \frac{m}{t} \) at \( t \rightarrow \infty \) ?

\[ \text{no(all) fixed points} \rightarrow \text{random walk} \]
POLYA URN

- At update $t \rightarrow t+1$, $r$ ball randomly drawn, put back with an additional ball with majority color.

- What is the ratio of black balls $p_t = m/t$ at $t \rightarrow \infty$?

$r=3$

attractor at $p^*=0 \& 1$
separator at $p_s=1/2$
**POLYA URN**

- At update \( t \rightarrow t+1 \)
  - add a black / white ball with prob. \( a / b \)
  - with prob. \( 1-a-b \), count all \( t \) balls,
    - add a black ball with prob. \( Q(p_t) \), white with \( 1-Q(p_t) \)

\[
Q(p) = \Theta(p-1/2)
\]
At update $t \to t+1$

- Count all $t$ balls, add a ball with majority color, except...
- If $i$ hard-black found, add a hard-black with prob $a = i/t$
- If $j$ hard-white found, add a hard-white with prob $b = j/t$

**POLYA URN**

$Q(p) = \Theta(p - 1/2), \quad p_t = m/t$

Ref: Hisakado & Mori (2010)
Two-state agents evolving by group-majority rule (size $r$) with the presence of *inflexible agents* (ratios $a$, $b$)

Critical $a^*$ ($b^*$) beyond which, *minority dominates* in any initial configuration

$\sim 17\%$ for $r=3$

Ref: Glam (2004), Galam & Cheon (2017)
Group-majority rule \((r=3)\) with inflexible agents (ratios \(a, b\))

- Critical \(a^* \sim 17\%\) beyond which, inflexible minority dominates
Group-majority rule \((r=3)\) with inflexible agents (ratios \(a, b\))

\[ p_t: \text{ratio of black at time } t \]

\[ p_{t+1} = a - 2ap_t + (3 + a - b)p_t^2 - 2p_t^3 \]

\(p^*(a,b)\) : fixed point as function of \((a,b)\)

- Critical \(a^* = 3 - 2\sqrt{2}\) beyond which inflexible minority dominates
Two-state agents evolving by group-majority rule (size $r$) with the presence of **inflexible agents** (ratios $a$, $b$)...

\[ \beta \ (1-g) < \beta \quad r \quad \alpha \ (1+f) > \alpha \]

Vocal minority **grows in hostile environment**
EXTREMISTS AND MODERATES

➤ Committed few (extremists) drives political movement

➤ Extremists thrive in hostile environment

➤ Extremists lose their edge after success
  (moderates tend to suppress them in dominance)

➤ —> Increase/decrease rate of hard-black
  in friendly environ \((1-g)\) ; in hostile environ \((1+f)\)

➤ —> Increase/decrease rate of hard-white
  in friendly environ \((1-g)\) ; in hostile environ \((1+f)\)
“DYNAMICAL” OPINION DYNAMICS

➤ Evolution equation for majority and assertive minorities

\[
\begin{align*}
p_{t+1} &= P_+^{(r)}(p_t, a_t, b_t) \\
a_{t+1} &= P_A^{(r)}(p_t, a_t, b_t) \\
b_{t+1} &= P_B^{(r)}(p_t, a_t, b_t)
\end{align*}
\]

➤ Increase/decrease rate of hard-black in friendly environ \((1-g)\); in hostile environ \((1+f)\)

➤ Increase/decrease rate of hard-white in friendly environ \((1-g)\); in hostile environ \((1+f)\)

➤ Inflex-black/white appear in all-white/all-black env. w \(h\)
“DYNAMICAL” OPINION DYNAMICS

Evolution equation for majority and assertive minorities

\[ p_{t+1} = -2p_{t}^3 + 3p_{t}^2 + (1 - f) \left[ (1 - p_{t})^2 a_t - p^2 b_t \right] \]
\[ + \frac{1}{3} h \left[ (1 - p_{t} - b_{t})^3 - (p_{t} - a_{t})^3 \right] \]
\[ a_{t+1} = a_t \left\{ 1 + g + (f - g)(1 - p_{t})^2 \right\} + \frac{1}{3} h(1 - p_{t} - b_{t})^3 \]
\[ b_{t+1} = b_t \left\{ 1 + g + (f - g)p_{t}^2 \right\} + \frac{1}{3} h(p_{t} - a)^3 \]

Increase/decrease of inflexible-black
in friendly environ \((1-g)\); in hostile environ \((1+f)\)

Increase/decrease of inflexible-white
in friendly environ \((1-g)\); in hostile environ \((1+f)\)

Inflex-black/white appear in all-white/all-black env. w \( h \)
numerics with $r=3$; Phase space trajectories

$a^* \text{ (or } b^*) \sim 3 - 2\sqrt{2}$

$p^* \text{ (or } 1-p^*) \sim (2 - \sqrt{2})/2$
POLITICAL CYCLES

➤ numerics with $r=3$

$$f=0.2, \ g=0.3, \ h=0.0545$$

$a^*$ (or $b^*$) $\sim 3-2\sqrt{2} \quad 17\%$

$p^*$ (or $1-p^*$) $\sim (2-\sqrt{2})/2 \quad 29\%$
POLITICAL CYCLES

➤ numerics with \( r = 3 \)

\[
f = 0.2, \ g = 0.3, \ h = 0.0545
\]

\[
a^* \ (\text{or} \ b^*) \sim 3 - 2\sqrt{2} \quad 17\%
\]

\[
p^* \ (\text{or} \ 1-p^*) \sim \frac{2 - \sqrt{2}}{2} \quad 29\%
\]
ON POLITICAL CYCLES

➤ Two fixed points with different majorities
\[ \{p^*, a^*, b^*\} \sim \{0.3, 0.17, 0\} \]
\[ \{p^*, a^*, b^*\} \sim \{0.7, 0, 0.17\} \]

➤ Minority cycle with ebb and flow of extremists

➤ Majority changing cycle with double period — extremists drive early phase of take-over, then disappear

30% minority with 17% extremists

What our model predicts
Two sections of initial value space \( \{p_0, a_0, b_0\} \) that results in fixed points or majority-minority preserving limit cycles.

\[ f=0.1, g=0.45, h=0.25 \]
Two sections of initial value space \(\{p_0, a_0, b_0\}\) that results mostly in regime-changing limit cycles.

REGIME-CHANGING LIMIT CYCLE

\[ f=0.1, g=-0.31, h=0.15 \]
three sections of initial value space \( \{p_0, a_0, b_0\} \) that results in black/white minority cycles at “critical” parameters

\[ f=0.2, \quad g=0.421, \quad h=0.25 \]
Fixed points \( p^{(1)} = \frac{1}{\sqrt{2}}, \quad a^{(1)} = 0, \quad b^{(1)} = \frac{3 - 2\sqrt{2}}{1 + f} \)

Linear maps around fixed points

\[
\begin{pmatrix}
\delta p_{t+1} \\
\delta a_{t+1} \\
\delta b_{t+1}
\end{pmatrix} = M \begin{pmatrix}
\delta p_t \\
\delta a_t \\
\delta b_t
\end{pmatrix}
\]

\[
M^{(1)} = \begin{pmatrix}
\frac{2f}{f+g} & 0 & -(1 + f)(p^{(1)})^2 \\
0 & (1 + f) \left(\frac{f+2g}{f+g} - 2p^{(1)}\right) & 0 \\
\frac{2(f+g+1)}{1+f} - \frac{4(f+g)p^{(1)}}{1+f} & (f + g + 1) - 2(f + g)p^{(1)} & f - g + 1
\end{pmatrix}
\]
Fixed points

\[ p^{(1)} = \frac{1}{\sqrt{2}}, \quad a^{(1)} = 0, \quad b^{(1)} = \frac{3 - 2\sqrt{2}}{1 + f} \]

Linear maps around fixed points

\[ (f + g + 1 - 2(f + g)p^{(1)}) \]

\[ \frac{f-g}{2} + \frac{3f+g}{2(f+g)} - \frac{\sqrt{D}}{2} \]

\[ \frac{f-g}{2} + \frac{3f+g}{2(f+g)} + \frac{\sqrt{D}}{2} \]

\[ D = (f - g - 1)^2 + 4g + 4p^{(1)}(p^{(1)} - 1)((p^{(1)})^2 + p^{(1)} - 6g) \]
CRITICAL PHENOMENA

- $V_C$: volume of basin of regime-changing cycle
- $V_A$: volume of basin of A-majority attractor
- $V_B$: volume of basin of B-majority attractor

- $V_{Tot} = V_C + V_B + V_A$

- Phase transition: order parameter $= V_C / V_{Tot}$
OSCILLATIONS AROUND FIXED POINTS

➤ Period oscillation

\[ T \approx \frac{2\pi}{\sqrt{3\sqrt{2} - 4 \sqrt{f}}} \]
SUMMARY

➤ Dynamical systems theory of public opinion developed
➤ Complex behavior, butterfly-like structure
➤ Theory amounts to an extension of Galam opinion dynamics
➤ Relation to Mori-Hisakado’s Polya urn model
➤ Political cycles around 2/3-1/3 split with 1/6 extremists
➤ May be capturing some aspect of real-life politics
   — Toward the mathematical theory of politics!
Taksu Cheon, Serge Galam
“Dynamical Galam model”