Bose-Einstein condensate confined in a 1D ring stirred with a rotating delta link

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We consider a Bose-Einstein condensate with repulsive interactions confined in a 1D ring where a Dirac delta is rotating at constant speed. The spectrum of stationary solutions in the delta comoving frame is analyzed in terms of the nonlinear coupling, delta velocity, and delta strength, which may take positive and negative values. It is organized into a set of energy levels conforming a multiple swallowtail structure in parameter space, consisting in bright solitons, gray and dark solitonic trains, and vortices. Analytical expressions in terms Jacobi elliptic functions are provided for the wave functions and chemical potentials. We compute the critical velocities and perform a Bogoliubov analysis for the ground state and first few excited levels, establishing possible adiabatic transitions between the stationary and stable solutions. A set of adiabatic cycles is proposed in which gray and dark solitons, and vortices of arbitrary quantized angular momenta are obtained from the ground state by setting and unsetting a rotating delta.

I. INTRODUCTION

Bose-Einstein condensates (BECs) constrained in annular traps provide a way to study various phenomena related to superfluidity, including persistent currents and their decay, phase slips, and critical velocity [1\textendash}4]. Persistent currents can be created experimentally by the application of artificial gauge fields [5], or by rotating a localized, tunable repulsive barrier around the ring [6\textendash}9]. Through the later method, hysteresis between different circulation states was observed in [10]. Within the context of atomtronics, a ring condensate is a key atomic circuit element. It has demonstrated its capability as a superconducting quantum interference device [11], entailing the possibility of high precision measurements and applications in quantum information [12\textendash}14].

In the view of a better control of BECs, phase transitions have been analyzed in different ring settings within the mean field approach. They were first studied in a ring under a rotational drive [15], and then through the interplay between rotation and symmetry breaking potentials or rotating lattice rings. One lattice site was studied in [16], a double well in [17], and a more general unified approach of a ring lattice in [18], all involving the possibility to adiabatically connect different quantized states such as persistent currents or solitons.

By solving the Gross-Pitaevskii equation (GPE), various works have studied the energy diagram and metastability of BECs in rings with a rotating defect [19\textendash}21]. In the case of the 1D GPE, stationary solutions can be found through the inverse scattering method [22] or by directly integrating and writing them in terms of Jacobi functions. These solutions have been analyzed under box and periodic boundary conditions [23], under a rotational drive [15], and for some specific constant potentials [24]. The flow past an obstacle in the form of a Dirac delta was studied perturbatively in [25, 26]. In [27] a 1D ring with a rotating Dirac delta was analyzed for some specific rotations, strengths, and nonlinearities.

In this paper, we study a repulsive BEC in a 1D ring where a Dirac delta is rotating at constant speed. The use of analytical solutions, expressed in terms of Jacobi functions, allows us to compute the stationary wave functions and chemical potentials for the ground state and an arbitrary number of excited energy levels. The obtained energy diagram, depending on the delta velocities and strengths, both attractive and repulsive, is analyzed as a function of the coupling strength. This diagram entails a series of critical velocities which, together with a Bogoliubov analysis, lays out the distribution of stable and metastable states in parameter space, and which adiabatic transitions between them are possible. Within these transitions, we propose a few adiabatic cycles in which excited solitonic states and vortices are obtained by setting and unsetting a rotating delta.

This paper is organized as follows. In the next section, II, we introduce the theoretical model, defining the GPE and boundary conditions in the Dirac delta comoving frame, and provide a method to compute the spectrum. The results are in Sec. III, in which we illustrate the main features of the spectrum (III-A), its stability (IIIB), and its dependence on the nonlinearity (III-C). In Sec. IV, we propose a set of adiabatic paths to excite the condensate. We conclude this paper in Sec. V. Mathematical details are in the Appendices.

II. THEORETICAL MODEL

We consider a BEC at zero temperature in a tightly transverse annular trap where a Dirac delta is rotating at constant speed. The point-like potential is chosen instead of finite one such that analytical solutions can be obtained, with the view that the re-
results may not qualitatively change with respect to a very peaked Gaussian. Considering only stationary solutions, and within the mean field approach, we can determine the condensate wave function in the delta comoving frame, $\phi(\theta)$, as the solution of the 1D Gross-Pitaevskii equation,
\[ -\frac{1}{2} \phi''(\theta) + g|\phi(\theta)|^2 \phi(\theta) = \mu \phi(\theta), \tag{1} \]
where $g > 0$ is the reduced 1D coupling, $\mu$ the chemical potential, and $\theta \in [0, 2\pi)$. Here we use units $\hbar = \mathcal{R} = M = 1$, $\mathcal{R}$ being the radius of the ring and $M$ the mass of the atoms. In the rotating frame, where the delta is at a constant position $\theta = 0$, the wave function acquires a phase and a derivative jump (kink), entailing the boundary conditions,
\[ \phi(0) - e^{-i2\pi \Omega} \phi(2\pi) = 0, \tag{2} \]
\[ \phi'(0) - e^{-i2\pi \Omega} \phi'(2\pi) = \alpha \phi(0), \tag{3} \]
with $\Omega$ and $\frac{\phi}{2}$ the velocity and strength of delta (see App A). Renormalizing a wave function $\phi(\theta) \rightarrow \sqrt{N} \phi(\theta)$ amounts to a rescaling of $g \rightarrow gN$. We normalize $\phi(\theta)$ to
\[ \int_0^{2\pi} d\theta |\phi(\theta)|^2 = 1, \tag{4} \]
and study how the spectrum depends on $g$.

Any solution of Eq. (1) can be written in closed form in terms of a Jacobi elliptic function [24]. In particular, the density $r(\theta)^2 \equiv |\phi(\theta)|^2$ depends linearly on the square of one of the twelve Jacobi functions $J_j$,
\[ r_j^2(\theta) = A + B J_j^2(k(\theta - \theta_j), m), \tag{5} \]
where $A$, $B$ are constants, $k$ is the frequency, $\theta_j$ the shift in $\theta$, and $m$ the elliptic modulus, which generalizes the trigonometric and hyperbolic functions into the Jacobi ones. The squares of the six convergent or divergent Jacobi functions are related among themselves linearly and through shifts in $\theta$, and therefore one may consider only a convergent one and a divergent one with general $A$, $B$, and $\theta_j$. The linear coefficients $A$ and $B$ are fixed by the Gross-Pitaevskii equation (1) and normalization (4) in terms of $k$, $m$, and $\theta_j$, and then $\theta_j$ is determined by the continuity of the density, $r(0) = r(2\pi)$,
\[ \theta_j = \pi - \frac{j}{k} K(m), \quad j = 0, 1; \tag{6} \]
where $K$ is the elliptic integral of the first kind. In the case of a convergent Jacobi function, two shifts are allowed, $j = 0, 1$ (forth and back), while for the divergent one, $\theta_j$ is fixed to only one of them in order to avoid the singularity. This leaves three possible solutions, the convergent function with two possible shifts, and the divergent one, each parametrized by $k > 0$ and $0 < m < 1$. The specific functions chosen and their explicit expressions are shown in App. B. We label them as $\alpha$, $\beta$, and $\gamma$.

Stationary solutions in the comoving frame have constant current $\gamma = r(\theta)^2 \beta(\theta)$, where $\beta(\theta)$ is the phase of the condensate wave function, and therefore, for any given density $r(\theta)^2$, $\beta(\theta)$ is determined up to a constant. Both the chemical potential $\mu$ and the current $\gamma$ are also fixed by the Gross-Pitaevskii equation in terms of $k$ and $m$. The current $\gamma$ must be real and this further constrains the elliptic modulus $m$. At these more constraining bounds of $m$, the current $\gamma$ is zero (see App. B). For a similar treatment of the Jacobi functions and a complete derivation of the solutions see e.g. Ref. [23]. The main difference between [23] and our work is that in [23] $k$ and $m$ are not taken as parameters to account for a phase jump and a kink, but adjusted such that periodic boundary conditions are obtained.

In brief, all solutions $\phi = r e^{i\beta}$ and the corresponding chemical potential $\mu$ satisfying Eqs. (1)-(4) can be obtained by running $k$ and $m$ in their allowed ranges in any of the three Jacobi functions, two convergent and one divergent. Then, for each $k$ and $m$, the delta strength and velocity are obtained from Eqs. (2) and (3) as
\[ \alpha_j(k, m) = \frac{r_j'(0) - r_j'(2\pi)}{r_j(0)}, \tag{7} \]
\[ \Omega_j(k, m) = \frac{1}{2\pi} [\beta_j(2\pi) - \beta_j(0)]. \tag{8} \]

The spectrum $\mu(\alpha, \Omega)$ is thus given in parametric form, $\{\alpha(k, m), \Omega(k, m), \mu(k, m)\}$. Running $k$ and $m$ in a systematic way we obtain $\mu(\alpha, \Omega)$ as a series of surfaces which fold onto each other — energy levels which cross and become degenerate at specific lines $\Omega(\alpha)$. Depending on the region in the space defined by $\alpha$ and $\Omega$, the surface will represent the ground state or an excited one. In the left panel of Fig. 1 a part of the spectrum $\mu(\alpha, \Omega)$ with $g = 10$ is shown from the top, together with three sample lines $\{\alpha(k_i, m), \Omega(k_i, m)\}$, each line $i = 1, 2, 3$ with $k$ fixed to a different value $k = k_i$ and with $0 < m < 1$ a parameter. The lines are divided at a certain point into two parts, each belonging to a different surface. This point is shared by another line $\{\alpha(k, m), \Omega(k, m)\}$ with fixed $m$ and parametrized by $k > 0$, as plotted on the right panel of Fig. 1. Both lines share the same tangent at this point,
\[ \frac{\partial}{\partial k} \left( \alpha(k, m), \Omega(k, m) \right) \approx \frac{\partial}{\partial m} \left( \alpha(k, m), \Omega(k, m) \right), \tag{9} \]
and therefore, any degeneracy line may be computed.
by solving
\[ \frac{\partial \Omega(k,m)}{\partial k} \frac{\partial \Omega(k,m)}{\partial m} = \frac{\partial \Omega(k,m)}{\partial m} \frac{\partial \Omega(k,m)}{\partial k}. \tag{10} \]

The spectrum obtained by scanning \( k > 0 \) and \( m \) maps onto a series of regions in the parameter space defined by \( \alpha \) and \( \Omega \), each region spanning a space \( \Omega \geq \frac{n}{2} \), where \( n \) is an integer. For example, out of all the solutions with constant amplitude (for \( \alpha = 0 \)),
\[ \phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{i(\Omega+n)\theta}, \tag{11} \]
\[ \mu_n = \frac{g}{2\pi} + \frac{1}{2} (\Omega + n)^2, \tag{12} \]
only the state \( n = 0 \) and for \( \Omega > 0 \) is obtained. To compute the complete spectrum we boost the solutions to \( \Omega \to \pm \Omega + n \).

### III. STATIC PROPERTIES

Our goal is to analyze the possible stable and adiabatic changes of the condensate as one varies the strength and velocity of the Dirac delta. For this we first study the structure of the spectrum \( \mu(\alpha, \Omega) \), i.e. the regions \( \{\alpha, \Omega\} \) in which stationary solutions exist for the ground and first excited states, and how the chemical potential depends on \( \alpha \) and \( \Omega \) (Sec. IIIA). Then we analyze whether the solutions at each region are stable or metastable against a perturbation through a Bogoliubov analysis (Sec. III B). The results in Sec. III A and III B are analyzed and illustrated for \( g = 10 \), and in Sec. III C we study how they depend on \( g \).

### A. Spectrum

The spectrum is characterized by a series of regions in \( \{\alpha, \Omega\} \) in which solutions continuously change with an adiabatic variation of the Dirac delta strength and velocity. These regions are bounded by a critical velocity \( \Omega_{\alpha_{\text{cr}}} \) which is determined by either the limits of \( m \to 0, 1 \), or the ones fixed by \( \gamma_j = 0 \); or by the line in which solutions from top and bottom levels become degenerate, satisfying Eq. (10). Fig. 2 shows these regions for the ground state and first excited levels and some particular boosts \( \Omega \to \pm \Omega + n \). This figure can also be interpreted as part of the spectrum \( \mu(\alpha, \Omega) \) viewed from the top. Five sections of \( \mu(\alpha, \Omega) \) with constant \( \alpha \) are plotted in Fig. 3.

The main structure of the lower part of the spectrum at \( \alpha < 0 \) is determined by the lines uniting the points \( P_i - (\cdots) - P_i \) and \( P_i - P_j - P_k \) bounding the adiabatic regions. Each segment of these bounds is listed in Table I together with the function and limit of \( m \) they represent. Note that the line \( P_2 - (\cdots) - P_5 \) and its continuation at \( \Omega = \frac{1}{2} \) is the only one not defining the degeneracy of two energy levels.

<table>
<thead>
<tr>
<th>Region</th>
<th>( J )</th>
<th>Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom 1st ST ( \alpha &lt; 0 )</td>
<td>( P_1 - P_2 - P_3 )</td>
<td>( P_2 - P_3, m = 1 )</td>
</tr>
<tr>
<td>Bottom 1st ST ( \alpha &lt; 0 )</td>
<td>( P_3 - P_4 - P_5 - P_\infty )</td>
<td>( P_3 - P_4 - P_5, m = 0 )</td>
</tr>
<tr>
<td>Top 1st ST ( \alpha &gt; 0 )</td>
<td>( P_1 - P_2, \text{Eq. (10)} )</td>
<td></td>
</tr>
<tr>
<td>Top 1st ST ( \alpha &gt; 0 )</td>
<td>( P_2 - P_6, m = 1 )</td>
<td></td>
</tr>
<tr>
<td>Bottom 2nd ST ( \alpha &lt; 0 )</td>
<td>( P_1 - P_2, \text{Eq. (10)} )</td>
<td></td>
</tr>
<tr>
<td>Bottom 2nd ST ( \alpha &gt; 0 )</td>
<td>( \text{Eq. (10)} )</td>
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Table I. Relation between the adiabatic regions, the Jacobi function with which the solutions in the region are computed, and the constraints of the function at the boundaries. The regions are defined according to the swallowtail (ST) structure of Fig. 3 and through the lines bounding them. In the first swallowtail at \( \alpha < 0 \), these lines are described by the union of the various points \( P_i \), as in Fig. 2, and where \( P_\infty \equiv (\alpha \to -\infty, \Omega = \frac{1}{2}) \).
trains, which have, for each level $n$,
\[
\mu_{\alpha=0} = \frac{1}{4\pi} \left[ 3g + \frac{2(n-1)^2(m-2)}{\pi} K(m)^2 + \frac{6(n-1)^2}{\pi} K(m)E(m) \right],
\]
where $K$ is the elliptic integral of the first kind. Both expressions correspond to periodic boundary conditions (solved in Ref. [23]) and coincide at $m = 0$ and $k = \frac{n-1}{\pi} K(0) = \frac{n-1}{2}$. These values determine the critical velocity,
\[
\Omega_{cr,\alpha=0} = \sqrt{\frac{g}{2\pi} + \frac{n^2}{4}},
\]
that bounds both regions $n$ and $n + 1$ at $\alpha = 0$. Points $P_1$, $P_3$, and $P_7$ correspond to $n = 1$, $n = 0$, and $n = 2$, respectively. The values of $P_4$, $P_5$ are constrained by their current and elliptic modulus being zero, $\gamma_J = m = 0$, while $P_6$ has $\gamma_J = 0$ and $m \to 1$. Point $P_2$ is obtained by minimizing $\alpha$ with $m = 1$. All these constrains fix $P_i$ to the values shown in Table II.

The spectrum presented in Fig. 3 can be organized into a set of swallowtail diagrams from lower to higher energy. These type of diagrams are characteristic of hysteresis and were analyzed in the context of ring condensates in [28]. Accordingly, the regions of Fig. 2 can be defined as the bottom and top of the first swallowtail, and the bottom of the second one. Their relation to the specific Jacobi functions used is listed in Table I. These three regions are also characterized by the eigenfunctions they represent. A sample density and phase for each region at $\alpha < 0$ and $\alpha > 0$ are plotted in Fig. 4. The bottom of the first swallowtail region entails densities $|\phi|^2$ with one upward (downward) kink for attractive (repulsive) potentials. At $\alpha < 0$, as the delta becomes more attractive, the height of the upward kink becomes larger, and the corresponding bright soliton more peaked. The same holds for the top part, but with the bright solitons much more constrained by the bounds at $\alpha < 0$, and with deeper notches and larger currents at $\alpha > 0$ (implying larger phase gradients for the same values of the density). As for the densities of the bottom part of the second swallow-
<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$\alpha_{P_i}$</th>
<th>$\Omega_{P_i}$</th>
</tr>
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<tbody>
<tr>
<td>$P_1$</td>
<td>0</td>
<td>$\sqrt{\frac{2}{\pi}}$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$\alpha_{dn}(k_2, 1)$, $2k_2 (-3\pi g + 8\pi^2 k_2^2 + 2)$ $+ (4\pi k_2^2 - 3g) \sinh(4\pi k_2) + 2(\pi g - 2)k_2 \cosh(4\pi k_2)$ $- 6g \sinh(2\pi k_2)$ $- 4\pi g k_2 \cosh(2\pi k_2) = 0$</td>
<td>$\Omega_{P_2} = \Omega(k_2, 1)$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-g$</td>
<td>0</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$\alpha_{dc}(k_5, 1)$, $-2k_5 \tan(k_5 \pi)$ $+ g + 2\pi k_5^2 = 0$</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$P_6$</td>
<td>$\alpha_{dn}(k_6, 1)$, $2k_6 \tanh(k_6 \pi)$ $+ g - 2\pi k_6^2 = 0$</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$P_7$</td>
<td>0</td>
<td>$\sqrt{\frac{2}{\pi}}$ + 1</td>
</tr>
</tbody>
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Table II. Expressions of $\alpha$ and $\Omega$ for the critical points $P_i = (\alpha_i, \Omega_i)$, $i = 1, \cdots, 7$.

tail, they contain two depressions. At $\alpha < 0$ the two depressions are formed by a valley at $\theta = 0$ with an upward kink in the middle, while at $\alpha > 0$ there is a downward kink at $\theta = 0$ and a valley at $\theta = \pi$ (see Fig. 4). At $\Omega = \frac{1}{2}$ in both the top part of the first swallowtail and the bottom of the second, the solutions correspond to dark solitons: the densities become zero at a point where the phases acquire a jump of $\pi$. In general, at precisely $\alpha = 0$, solutions from $\alpha < 0$ and $\alpha > 0$ merge into the same wave function, consisting of either plane waves (constant densities and constant phase gradients) or gray solitons. In the later case the wave functions obtained from the limits at $\alpha < 0$ and $\alpha > 0$ differ in a shift of $\Delta \theta = \pi$. Note however that at $\alpha = 0$ there is no Dirac delta and the system is symmetric with respect to any shift in $\theta$.

B. Metastability

A Dirac delta with fixed strength and rotating at a constant speed allows for an infinite set of solutions organized in chemical potential values. The stability of these solutions can be studied by adding a small perturbation to the stationary wave function

$$\Psi = e^{-i\mu t}(\phi + u e^{-i\omega t} - v* e^{i\omega t}),$$

and analyzing how it evolves. Replacing this function in the time-dependent Gross-Pitaevskii equation (Eq. (A7)) and linearizing in $u$ and $v$, we obtain the corresponding Bogoliubov system of equations [29],

$$\frac{1}{2} u'' + 2g |\phi|^2 u - \mu u - g\phi^2 v = w u,$$

$$\frac{1}{2} v'' - 2g |\phi|^2 v + \mu v + g\phi^2 u = w v. \quad (16)$$

Both perturbations must satisfy the boundary conditions separately, which read,

$$u(0) = e^{-i2\pi \Omega} u(2\pi) = 0,$$

$$u'(0) = e^{-i2\pi \Omega} u'(2\pi) = \alpha u(0), \quad (19)$$

$$v(0) = e^{i2\pi \Omega} v(2\pi) = 0,$$

$$v'(0) = e^{i2\pi \Omega} v'(2\pi) = \alpha v(0). \quad (21)$$

In order to work with continuous functions we change variables to

$$u(\theta) = e^{i\Omega \theta} \tilde{u}(\theta),$$

$$v(\theta) = e^{-i\Omega \theta} \tilde{v}(\theta), \quad (23)$$

the differential equations and boundary conditions becoming,

$$w \tilde{u} = -\frac{1}{2} \tilde{u}'' + i \Omega \tilde{u}' + \frac{1}{2} \Omega^2 \tilde{u} + 2g |\phi|^2 \tilde{u} - \mu \tilde{u} - g\phi^2 \tilde{v}, \quad (24)$$

$$w \tilde{v} = \frac{1}{2} \tilde{v}'' + i \Omega \tilde{v}' - \frac{1}{2} \Omega^2 \tilde{u} - 2g |\phi|^2 \tilde{v} + \mu \tilde{v} + g\phi^2 \tilde{u},$$

Figure 4. (Color online). Densities (solid lines) and phases (dashed lines) of the eigenfunctions for six sample points in the regions corresponding to the bottom and top of the first swallowtail diagram (first and second rows) and bottom of the second one (third row) at $\alpha < 0$ and $\alpha > 0$ (left and right columns). All eigenfunctions are computed for $g = 10$, and the specific values of $\alpha$ and $\Omega$ are explicitly written in each plot.
\[ \dot{u}(0) - \ddot{u}(2\pi) = 0, \quad \dot{v}(0) - \ddot{v}(2\pi) = 0, \]
\[ \dot{\alpha}(0) - \dot{\alpha}(2\pi) = 0, \]
with \( \alpha\) being the introduced parameter.

We solve this system of equations by expanding \( \dot{u} \) and \( \dot{v} \) in an orthonormal basis, thus converting it into a matrix eigenvalue problem, and by the Direct and Arnoldi methods integrated in the differential solvers in Mathematica. In the later method the boundary conditions are automatically imposed through a Neumann value, while in the former they constrain the orthonormal basis. This basis does not consist in periodic plane waves, since the derivatives must be discontinuous according to the delta conditions, but in the solutions of Eqs. (1)-(4) with \( g = 0 \) and \( \Omega = 0 \). Imposing these constraints on exponential and trigonometric functions, we obtain the basis,

\[ s_0(\theta) = \frac{e^{\theta}}{\sqrt{k_0}/(-1 + e^{4\pi k_0} + 4\pi k_0 e^{2\pi k_0})}, \]
\[ s_{2n+1}(\theta) = \frac{\cos(k_0(\theta - \pi)}){\sqrt{1 + \sin(2\pi k_n)/(2k_n)}}, \]
\[ s_{2n}(\theta) = \frac{\sin(n \theta)}{\sqrt{\pi}}, \]

with \( n \) a positive integer, and where the element \( w_0(\theta) \) is only used for \( \alpha < 0 \). \( \dot{u} \) and \( \dot{v} \) expanded in this set of functions solve Eqs. (26) and (28), and Eqs. (27) and (29) are satisfied as long as \( k_0 \) and \( k_n \) are the solutions of, respectively,

\[ \alpha = \frac{2k_0}{e^{2\pi k_0} - 1}, \]
\[ \alpha = \frac{2k_0}{\tan(k_0 \pi)}. \]

Both methods yield the same eigenvalues, and analyzing whether they are real or complex, we split each region in stable and unstable parts.

The energy levels form a three-dimensional multiple swallow-tail structure, as it can be inferred from Fig. 3. The bottom and upper parts of the first swallowtail are found completely stable and unstable, respectively. In contrast, the lower part of the second swallowtail is stable except for regions of metastability in form of stripes at \( \alpha < 0 \), as shown in Fig. 5.

C. Dependence on nonlinearity

The results presented in the previous subsections have been illustrated with the nonlinearity fixed to \( g = 10 \). In Fig. 6 we show the three same adiabatic regions for \( g = 20, 10, 5 \), and 1. The first and third columns correspond to the bottom part of swallowtail diagrams (as in Fig. 3), while the middle column represents the top part. A larger nonlinearity implies a greater span and overlap of all the levels. As \( g \) decreases, and for any fixed \( \alpha \), the tail part of the swallowtail diagrams becomes smaller, vanishing at \( g = 0 \): both the region in the middle column and the overlaps of shifted regions in the others decrease in size. Performing the same metastability analysis we find that the bottom and upper part of the first swallowtail regions remain completely stable and unstable for all the \( g \) tested. The third region is also found completely stable at \( \alpha > 0 \), while the metastable stripes at \( \alpha < 0 \) become thinner (thicker) as \( g \) decreases (increases).

The dependence of the spectrum on \( g \) can be analyzed more quantitatively through the expressions in Table II for the points \( P_1, P_2, P_3, P_4, \) and \( P_5 \) are given in analytical form, and \( P_2, P_5, \) and \( P_6 \) are plotted in Fig. 7. \( \alpha_{P_3} \) approaches zero as \( g \) increases, and the structures bounded by the lines \( P_1 - P_2 - P_3 \) and \( P_3 - P_2 - P_6 \) (right panel in Fig. 2) vanish in the limit \( g \to \infty \). In contrast, \( |\alpha_{P_2}|, |\alpha_{P_1}| \) and \( \Omega_{P_4} \) increase with \( g \), and the region bounded by these points grows at large interactions. At \( g = \frac{-2}{\pi} + 2\pi k_{cr}^2 \simeq 0.280 \), where \( k_{cr} \approx 0.382 \) solves \( \pi k_{cr} \tan(k_{cr} \pi) = 1 \), both Eqs. for \( k_2 \) and \( k_6 \) in Table II are satisfied and points \( P_2 \) and \( P_6 \) coincide. In the limit \( g \to 0 \), \( P_6 \) approaches \( P_5 \) at \( \alpha = -\frac{2}{\pi} \), \( k_5 \) and \( k_6 \) tend to zero, and \( \Omega_{P_1} = \frac{1}{2} \): the parts of the region merge into a flat band. In general, at \( g = 0 \), all levels turn into regions spanning \( \Omega \in [n, n+1] \), where all the solutions are stable (see App. C for the linear solutions).

IV. ADIABATIC PRODUCTION OF VORTICES AND EXCITED SOLITONS

All complex solutions found are continuously connected at \( \alpha = 0 \) (see middle panel of Fig. 3). Once a finite delta strength \( \frac{\alpha}{2} \neq 0 \) is set, a gap between the
The various swallowtail diagrams appears, and depending on the initial rotational velocity, different energy levels can be accessed. This property, together with the stability analysis from the previous section, allows us to propose various adiabatic processes in which by varying $\alpha$ and $\Omega$ different solitonic and vortex solutions are obtained. This is possible because of the circular topology of the $\Omega$ parameter space, and is an example of the exotic quantum holonomy [30].

In Fig. 8 (top panel) we schematically demonstrate three possible adiabatic cycles, defined by the lines

\begin{align*}
A = B - C - D - E - A, \\
A = B - C - D - F - A, \\
A = B - C - D - F - G - H,
\end{align*}

in which a dark soliton, a gray soliton, and a vortex with one quantum of angular momentum are obtained, respectively. The first part of the paths consists in first setting a rotational speed $\Omega_1 < \Omega < \Omega_2$, $A \rightarrow B$ (see Table II), and then turning on a repulsive delta while rotating at $\Omega$, $B \rightarrow C$. At this point, the excited energy level corresponding to densities with two troughs is accessed, and one can adiabatically decrease the rotation down to $\Omega = \frac{1}{2}$, $C \rightarrow D$, where a dark soliton is obtained. By bringing the potential back to zero, $D \rightarrow E$, and the velocity of the observer to $\Omega = 0$, $E \rightarrow A$, the dark soliton is maintained, and the first cycle is completed. If at point $D$ one continues to decrease the velocity to $\Omega = 0$, $D \rightarrow F$, and then reduces the delta magnitude to $\alpha = 0$, $F \rightarrow A$, a gray soliton is obtained instead. By decreasing further the velocity at point $F$, down to $-\Omega + 1 < \Omega < -\Omega_1 + 1$, $F \rightarrow G$, and then adiabatically setting to zero the potential first, $G \rightarrow H$, and second the velocity, $H \rightarrow A$, the vortex is obtained.

Three similar cycles are proposed in the bottom panel of Fig. 8, where an attractive delta is set instead of a repulsive one. In both cases the delta strength is limited by the line $\Omega_{\alpha}(\alpha)$ bounding the adiabatic region corresponding to the bottom of the second swallowtail. For the attractive delta, one also needs to avoid the metastable region, which further limits the value of $\alpha$.

The densities and phases of the wave functions at each point defining the cycles of Fig. 8 are plotted in Fig. 9. The first two plots, $A$ and $B$, are shared by all the cycles, and correspond to a flat density with an increase of the phase gradient from zero (plot $A$) to $\beta'(\theta) = 2\pi\Omega_B$ (plot $B$). By setting a repulsive and an attractive delta, plots $C$ and $C'$ are obtained, each with two small depressions in the density. In this region, corresponding to the bottom of the second swallowtail diagrams, the solutions are symmetric with respect to $\Omega = \frac{1}{2}$. This is because the solutions at $\Omega = \frac{1}{2} + \Delta \Omega$ are obtained by boosting the solutions at $\Omega = \frac{1}{2}$ to $\Omega = \frac{1}{2} + \Delta \Omega$. As the rotation decreases in the repulsive case, the depression at $\theta = \pi$ becomes deeper, while the downward kink at $\theta = 0$ smaller. For the attractive delta, when the rotation

![Figure 7](Image)

Figure 7. (Color online). Dependence of the critical points $P_1 = (\alpha_{P_1}, \Omega_{P_1})$ on the nonlinearity $\alpha$. The plots are only for the values of $\alpha_{P_1}(\alpha)$ and $\Omega_{P_1}(\alpha)$ which do not have closed analytical expressions, and also for $\alpha_{P_4}$ and $\Omega_{P_4}$ for comparison.
is decreased, both depressions surrounding the upward kink at $\theta = 0$ merge, effectively turning into one valley with a small upward kink. At $\Omega = \frac{1}{2}$ both solutions become dark solitons. For the repulsive delta the density contains a dark soliton at $\theta = \pi$ and a small downward kink at $\theta = 0$, while for the attractive potential the dark soliton is at $\theta = 0$, where the derivatives are discontinuous according to the attractive delta conditions (plots $D$ and $D'$). As one continues to decrease the rotation the reverse process happens, the dark solitons becoming shallower gray solitons. The three proposed cycles consist in lowering the delta strength at $\Omega = \frac{1}{2}$ (plots $D$ and $D'$), $\Omega = 0$ (plots $F$ and $F'$), or $-\Omega_2 + 1 < \Omega < -\Omega_1 + 1$ (plots $G$ and $G'$) from a finite $\alpha$ to $\alpha = 0$. Once the delta has vanished the dark solitons (plots $E$ and $E'$), gray solitons (plots $A$ and $A'$ in the lower left), and vortices (plot $H$) are obtained. Note that the dark solitons, together with the condensate, are moving at $\Omega = \frac{1}{2}$ in the lab frame. Plot $A$ in the lower right part of the figure represents a vortex with one quantum of angular momentum, and it’s the same for the the cycles with both attractive and repulsive delta.

Once a vortex is obtained through the third adiabatic cycle, with either a repulsive or attractive delta, one can continue to increase the velocity up to $\Omega = 1$. At this velocity, the condensate that has one quantum of angular momentum in the lab reference frame, is again at rest in the comoving one. The same cycle can then be repeated to obtain a new vortex — with two quanta in the lab frame and one in the comoving one—. A sequence of cycles of the type $A - B - C - D - F - G - H$ (plus the boost) can thus excite the condensate to an arbitrary number of vortices. If any of these cycles is shortened as in the paths $A - B - C - D - E - A$ and $A - B - C - D - F - A$, then a boosted gray and dark soliton are obtained (and analogously for the attractive case). By analyzing higher energy levels and their metastability, one can in principle produce adiabatic excitations of solitonic trains with various troughs, though this is beyond the scope of this paper.

The adiabatic cycles proposed can be reproduced for all $\alpha$ tested. However, the range of velocities at which the Dirac delta strength must be set, decreases with $g$, $\Omega_7 - \Omega_1 \approx \frac{3}{4} \sqrt{\frac{2}{\pi}} + O(g^{-\frac{3}{2}})$. Moreover, in the case of the cycle with an attractive rotating delta, the constrain on the magnitude of the delta potential will depend on the metastability stripes shown in Fig. 6.

V. CONCLUSIONS

The spectrum of a 1D ring condensate with a Dirac delta rotating at constant speed has been analyzed in terms of the nonlinearity $g$, the delta velocity $\Omega$, and the delta strength $\alpha/2$. Analytical expressions are provided for the wave function, the current, and the chemical potential. For a fixed $g$ and $\alpha$, the dependence of the chemical potential on the delta velocity, $\mu(\Omega)$, entails a series of swallowtail diagrams. These diagrams can be organized from smaller to
larger energies, and each one can be split into a bottom and a top part. The lowest diagram at $\alpha < 0$ is an exception, and consists only of the bottom part, with a more complex structure depending on a set of critical points $P_i$. As the magnitude of the delta strength increases, the sizes of the tails in each diagram decrease, while the energy gap among each diagram becomes larger. At $\alpha = 0$, the top parts of the diagrams merge with the bottom parts of the immediate upper ones. The spectrum $\mu(\alpha, \Omega)$ thus consists in a multiple swallowtail 3D structure, each region providing a range of delta strengths and velocities which can be varied adiabatically to access different solitonic solutions. In particular, as a rotating observer sets a delta with a finite strength, the condensate will access different energy levels —bottom parts of swallowtails— depending on the velocity.

In order to support the possible adiabatic processes allowed by the Gross-Pitaevskii spectrum, we have analyzed the metastability of each solution for the first two swallowtail diagrams. The top parts are found unstable and the bottom ones stable, except for the bottom of the second diagram at $\alpha < 0$, which presents a series of metastable stripes in parameter space.

We have proposed a method to produce dark and gray solitons, and persistent currents of arbitrary quantized angular momentum, by controlling the stirring velocity and strength of the potential. The method consists in setting and unsetting a Dirac delta potential while rotating around the condensate at certain velocities. The cycles in the parameter space defined by $\alpha$ and $\Omega$ corresponding to the various production processes are constrained by the width of the swallowtail diagrams, and in the case of an attractive Dirac delta, also by the metastability stripes.

### Appendix A: Gross-Pitaevskii equation and boundary conditions

The evolution of the condensate wave function in the Lab frame, $\psi_L(\theta_L, t_L)$, is governed by the 1D Gross-Pitaevskii equation,

$$i\hbar \partial_{t_L} \psi_L = -\frac{\hbar^2}{2MR^2} \partial^2_{\theta_L} \psi_L + g|\psi_L|^2 \psi_L + \frac{\alpha}{2} \delta(\theta_L - \Omega t_L) \psi_L,$$

(A1)
where $M$ is the atomic mass, $R$ the radius of the ring, $\theta_L \in (0, 2\pi)$ and $t_L$, the angular and time coordinates in the lab frame, $\alpha/2$ and $\Omega$ the magnitude and angular velocity of the Dirac delta, and $g > 0$ the reduced 1D coupling strength. The circular topology imposes continuity conditions in the wave function,

$$\psi_L(\Omega t_L, t_L) = \psi_L(\Omega t_L + 2\pi, t_L), \quad (A2)$$

and the Dirac delta constrains its derivatives through boundary conditions. These are obtained by integrating Eq. (A1) in a small contour around the delta, $\theta_L \in (\Omega t_L - \epsilon, \Omega t_L + \epsilon)$, and taking the limit $\epsilon \to 0$,

$$\frac{mR^2\alpha}{\hbar^2} \psi(\Omega t_L, t_L) = \left( \partial_{\theta_L} \psi \right)_{\theta_L = \Omega t_L} \quad (A3)$$

$$- \left( \partial_{\theta_L} \psi \right)_{\theta_L = \Omega t_L + 2\pi}.$$ 

We change variables to the delta rotating frame [31],

$$\theta = \theta_L - \Omega t_L, \quad \partial_\theta = \partial_\theta_L, \quad d\theta = d\theta_L, \quad (A4)$$

$$t = t_L, \quad \partial_t = \partial_{t_L} + \Omega \partial_\theta_L, \quad (A5)$$

$$\psi(\theta, t) = e^{i\left(\frac{1}{2}mR^2\Omega^2 t_L + mR\Omega \theta_L\right)} \psi_L(\alpha, t_L), \quad (A6)$$

and use units $\hbar = M = R = 1$, in which $g$, $\alpha$, and $\Omega$ are divided by $\bar{g} = \frac{\hbar^2}{M R^2}$, $\bar{\alpha} = \frac{\hbar}{2M R^2}$, and $\Omega = \frac{\hbar}{M R^2}$. Eqs. (A1), (A2), and (A3) thus become

$$i \partial_t \psi = -\frac{1}{2} \partial_\theta^2 \psi + g|\psi|^2 \psi, \quad (A7)$$

$$\psi(\Omega t, t) = e^{-i\left(\frac{1}{2}mR^2 t + \Omega \theta\right)} \psi(\Omega t + 2\pi, t), \quad (A8)$$

$$\psi(\Omega t, t) = \frac{1}{\sqrt{\alpha}} \left[ (\partial_\theta \psi)_{\theta = \Omega t} - e^{-i\left(\frac{1}{2}mR^2 t + \Omega \theta\right)} \right] \times \left( \partial_{\theta \theta} \psi \right)_{\theta = 2\pi + \Omega t}. \quad (A9)$$

For a stationary solution, $\psi(\theta, t) = e^{-i\mu t}\phi(\theta)$, where $\mu$ is the chemical potential, these equations result in Eqs. (1)-(3).

**Appendix B: Solutions**

To obtain the solutions of Eq. (1) we write the wave function as $\phi(\theta) = r(\theta)e^{i\beta(\theta)}$. Separating into real and imaginary parts, and integrating, the density and phase take the general form,

$$r^2(\theta) = A + B J^2(k(\theta - \theta_j), m), \quad (B1)$$

$$\beta'_j(\theta) = \frac{\gamma_j}{r^2_j(\theta)}, \quad (B2)$$

where $J$ is one of the 12 Jacobi functions and $A$, $B$, $k$, $\theta_j$, $m$, and $\gamma_j$ are constants. Eq. (B2) represents the stationarity condition, with $r^2_j, \beta'_j$ the current. The shift $\theta_j$ is fixed by the continuity condition, $r(0) = r(2\pi)$: the angular length of the condensate, $2\pi$, has to be equal to an integer number of periods $(J T)$ plus twice the shift, $2\pi = J T + 2\theta_j$ (see Fig. 10). The period of a $r(\theta)$ is given in terms of the elliptic integral of first kind $(K(m))$, $T = \frac{2K(m)}{k}$, and therefore

$$\theta_j = \pi - \frac{j}{k}K(m). \quad (B3)$$

Since we take $k$ and $m$ as a parameters, $j$ can be fixed to $j = 1$. Eqs. (1) and (4) fix $A$, $B$, $\gamma_j$ and $\mu_j$ in terms of $\theta_j$, $k$, and $m$. Using the Jacobi functions $dn$ and $dc$ as the convergent and divergent independent solutions, respectively, the amplitudes read,

$$r_{dn}(\theta) = \frac{\sqrt{g + k \eta_{dn} - 2\pi k^2 dn^2(k(\theta - \theta_j), m)}}{\sqrt{2\pi g}}, \quad (B4)$$

$$r_{dc}(\theta) = \frac{\sqrt{g + k \eta_{dc} - 2\pi k^2 dc^2(k(\theta - \theta_j), m)}}{\sqrt{2\pi g}}, \quad (B5)$$

where

$$\eta_{dn} = E[J A(k(2\pi - \theta_j), m), m],$$

$$E[J A(k \theta_j, m), m], \quad (B6)$$

$$\eta_{dc} = \eta_{dn} + dn(k \theta_j)sc(k \theta_j), \quad (B7)$$

with $E$ the elliptic integral of the second kind, $JA$ the Jacobi amplitude, $sc$ the Jacobi function, and where $dc$ allows only for $j = 0$ and $k < K(m)/\pi$ in order to be convergent in $\theta [0, 2\pi]$. The phases $\beta_{dn}$ and $\beta_{dc}$ are then integrated from the Eq. (B2).

For the types of Jacobi functions chosen, $J = dn$, $dc$, the corresponding currents and chemical potentials read,

$$\gamma_j = \frac{1}{g(2\pi)^{3/2}} \sqrt{g + k \eta_j} \sqrt{g - 2\pi k^2 + k \eta_j}$$

$$\times \sqrt{g - 2\pi k^2(1 - m) + k \eta_j}, \quad (B8)$$

$$\mu_j = \frac{1}{4\pi} (3g + 2k^2(m - 2) + 3k \eta_j). \quad (B9)$$

![Figure 10: Example of a density $r(\theta)^2$ with a shift $\theta_j$ such that satisfies periodic boundary conditions at $\theta = 0, 2\pi$.](image)
This leaves the frequency $k$ and elliptic modulus $m$ as the only free parameters. They constrain $\alpha$ and $\Omega$ through the boundary conditions in Eqs. (2) and (3). $k$ and $m$ are either real, $k > 0$, $m \in [0, 1]$, or, in the case of real solutions, may also take complex values with $|m| = 1$ and $k \propto 1/\sqrt{1 + m}$. For the real solutions (with general real boundary conditions) we refer to [32]. The elliptic modulus is further constrained by the condition that $\gamma_J \in \mathbb{R}$, which is satisfied when and odd number of radicants in Eq. B8 are positive. Note that the transition to an even number of radicants being negative, where $\gamma_J$ is not real, happens at $\gamma_J = 0$.

We label the three types of solutions, $dn$ with both shifts and $dc$, as

$$\phi_{dn} = r_{dn}^{(j=0)}(\theta) e^{i\theta_{dn}^{(j=0)}}, \quad (B10)$$
$$\phi_{dc} = r_{dc}^{(j=0)} e^{i\theta_{dc}^{(j=0)}}, \quad (B12)$$

**Appendix C: Linear limit**

Solutions of Eqs. (1)-(4) with $g = 0$ can be found analytically proceeding analogously to App. B and replacing Jacobi functions by trigonometric and hyperbolic ones. They read,

$$r_c^2 = A_c \left[ 1 + B_c \cos(k(\theta - \pi))^2 \right], \quad (C1)$$
$$r_{ch}^2 = A_{ch} \left[ 1 + B_{ch} \cosh(k(\theta - \pi))^2 \right], \quad (C2)$$

where,

$$A_c = \frac{\gamma^2 [2\pi k + \sin(2\pi k)]}{k^3 \pm k \sqrt{k^4 - (2\pi k \gamma)^2 + \gamma^2 \sin^2(2\pi k)}}, \quad (C3)$$

$$B_c = \frac{2k}{2\pi k + \sin(2\pi k)} \left( \frac{1}{A_c} - 2\pi \right), \quad (C4)$$

$$A_{ch} = \frac{\gamma^2 [2\pi k + \sinh(2\pi k)]}{k^3 \pm k \sqrt{k^4 - (2\pi k \gamma)^2 + \gamma^2 \sinh^2(2\pi k)}}, \quad (C5)$$

$$B_{ch} = \frac{2k}{2\pi k + \sinh(2\pi k)} \left( \frac{1}{A_{ch}} - 2\pi \right), \quad (C6)$$

and the frequency $k$ is real and the current $\gamma$ positive. Note that $r_c$ and $r_{ch}$ solutions are related by $k \to i k$.

For a given $k$, $\gamma$ is limited by the square roots in $A_c$, $A_{ch}$ being real. The phases, $\alpha$ and $\Omega$, are computed according to Eqs. (B2), (7), and (8), respectively, where now $k$ and $\gamma$ are taken as parameters, and the chemical potentials read,

$$\mu_c = \frac{k^2}{2}, \quad (C7)$$
$$\mu_{ch} = -\frac{k^2}{2}. \quad (C8)$$

![Figure 11](image_url)

Figure 11. (Color online). Sections $\alpha = -1, 0$ and 1 of the first two energy levels in the spectrum $\mu(\alpha, \Omega)$ for $g = 0$.

The spectrum consists in a series of layered levels, each one spanning all $\alpha$ and $\Omega \in \left[ \frac{\pi}{2}, \frac{\pi}{2} + \frac{1}{2} \right]$, given that $|k| \in \left[ \frac{\pi}{2}, \frac{\pi}{2} + \frac{1}{2} \right]$, where $n = 0, 1, 2$, etc., see Fig. 11. Adding a perturbation to these solutions in the form of Eq. (15) must satisfy the same linear equations with $\mu \to \mu \pm w$, as stated by Eqs. (16) and (17) with $g = 0$. The solutions only satisfy the boundary conditions for real eigenvalues, and therefore the frequencies $w$ are not imaginary and all the solutions stable.

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