Quantum weak value expansion & its application in history-erasing measurements

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June 14, 2014 @KEK

Spectre of weak value

- Weak value $\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$ billed variously as

  - new measurable quantum quantity
  - experimentally accessible
  - signifier of non-classical physics
  - allowing metaphysical insight

- Questions

  - what's really new?
  - proper place in "ordinary" quantum mechanics?

Weak value expansion

- Hermitian operator A acting on Hilbert vectors

$$A = \sum_{\ell,j} |\phi_\ell\rangle \langle\phi_\ell| A |\psi_j\rangle \langle\psi_j|$$ with 2 orthonormal bases

- weak value expansion

$$A = \sum_{\ell,j} (A)_{\ell,j} \, W_{\ell,j} \, \mu_{\ell,j}$$

$$(A)_{\ell,j} = \frac{\langle\phi_\ell| A |\psi_j\rangle}{\langle\phi_\ell|\psi_j\rangle}$$ weak value

$$W_{\ell,j} = \frac{|\phi_\ell\rangle \langle\psi_j|}{\langle\psi_j|\phi_\ell\rangle}$$ W-operator

$$\mu_{\ell,j} = |\langle\phi_\ell|\psi_j\rangle|^2$$ weight

  - all quantities in expansion gauge invariant $|\psi_j\rangle \to e^{i\chi_j} |\psi_j\rangle$

- compare w. normal expansion $A = \sum_{i,j} |\psi_i\rangle \langle\psi_i| A |\psi_j\rangle \langle\psi_j|$

Some useful relations

- The expansion is unique, because
$$\langle\phi_k|\, W_{\ell,j} W^{\dagger}_{j',\ell'}\, |\phi_k\rangle = \delta_{j',j}\delta_{\ell,k}\delta_{\ell',k}\frac{1}{\mu_{k,j}}$$
leads to
$$(A)_{\ell,j} = \mathrm{tr}[A W^{\dagger}_{j,\ell}] = \mathrm{tr}[W_{\ell,j} A]$$
- Weak value as fractional expectation value
$$\langle\psi_k|\, A\, |\psi_k\rangle = \sum_{\ell} (A)_{\ell,k}\ \ \mu_{\ell,k}$$
$$\langle\phi_m|\, A\, |\phi_m\rangle = \sum_{j} (A)_{m,j}\ \ \mu_{m,j}$$
- Weak value for mixed states: $W_{q,p} = \sum_{\ell,j} q_\ell p_j \frac{|\phi_\ell\rangle\langle\psi_j|}{\langle\psi_j|\phi_\ell\rangle}$
$$(A)_{q,p} = \mathrm{tr}[W_{q,p} A] = \sum_{\ell,j} q_\ell p_j (A)_{\ell,j}$$

Example; spin one-half

- pre-states $|\psi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ post-states $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

  $|\psi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ $|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

- W-operators

$$W_{1,1} = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} \qquad W_{1,2} = \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}$$

$$W_{2,1} = \begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix} \qquad W_{2,2} = \begin{pmatrix}0 & -1\\0 & 1\end{pmatrix}$$

- Expansion

$$\sigma_x = (W_{1,1} + W_{1,2} - W_{2,1} - W_{2,2})\,\frac{1}{2}$$

$$\sigma_y = (\mathrm{i}W_{1,1} - \mathrm{i}W_{1,2} - \mathrm{i}W_{2,1} + \mathrm{i}W_{2,2})\,\frac{1}{2}$$

$$\sigma_z = (W_{1,1} - W_{1,2} + W_{2,1} - W_{2,2})\,\frac{1}{2}$$

Example; spin one-half (2)

- pre-states $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ post-states $|\phi_1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$

  $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $|\phi_2\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$

- W-operators

$$W_{1,1} = \begin{pmatrix} 1 & 0 \\ \tan\frac{\theta}{2} & 0 \end{pmatrix} \qquad W_{1,2} = \begin{pmatrix} 0 & \cot\frac{\theta}{2} \\ 0 & 1 \end{pmatrix}$$

$$W_{2,1} = \begin{pmatrix} 1 & 0 \\ -\cot\frac{\theta}{2} & 0 \end{pmatrix} \qquad W_{2,2} = \begin{pmatrix} 0 & -\tan\frac{\theta}{2} \\ 0 & 1 \end{pmatrix}$$

- Expansion

$$\sigma_x = \tan\frac{\theta}{2} W_{1,1}\cos^2\frac{\theta}{2} + \cot\frac{\theta}{2} W_{1,2}\sin^2\frac{\theta}{2} - \cot\frac{\theta}{2} W_{2,1}\sin^2\frac{\theta}{2} - \tan\frac{\theta}{2} W_{2,2}\cos^2\frac{\theta}{2}$$

$$\sigma_y = \mathrm{i}\tan\frac{\theta}{2} W_{1,1}\cos^2\frac{\theta}{2} - \mathrm{i}\cot\frac{\theta}{2} W_{1,2}\sin^2\frac{\theta}{2} - \mathrm{i}\cot\frac{\theta}{2} W_{2,1}\sin^2\frac{\theta}{2} + \mathrm{i}\tan\frac{\theta}{2} W_{2,2}\cos^2\frac{\theta}{2}$$

$$\sigma_z = W_{1,1}\cos^2\frac{\theta}{2} - W_{1,2}\sin^2\frac{\theta}{2} + W_{2,1}\sin^2\frac{\theta}{2} - W_{2,2}\cos^2\frac{\theta}{2}$$

Projective State Reconstruction

- Projective measurement

$$\rho = \sum_j |\psi_j\rangle \rho_j^{(\psi)} \langle\psi_j| \quad \longrightarrow \quad \tau = \sum_\ell |\phi_\ell\rangle \tau_\ell^{(\phi)} \langle\phi_\ell|$$

- reconstruct ρ from τ

$$\frac{\langle\phi_m|\rho|\psi_j\rangle}{\langle\phi_m|\psi_j\rangle} = \rho_j^{(\psi)}$$

$$\tau_\ell^{(\phi)} = \rho_{\ell\ell}^{(\phi)}$$

$$\rho_j^{(\psi)} - \sum_{\ell\neq m} \frac{\langle\phi_\ell|\psi_j\rangle}{\langle\phi_m|\psi_j\rangle} \rho_{m\ell}^{(\phi)} = \tau_m^{(\phi)}$$

$$\rho = \sum_{\ell,m} |\phi_\ell\rangle \rho_{\ell m}^{(\phi)} \langle\phi_m|$$

appearance of weak value & weight

$$\sum_j \mu_{mj} \rho_j^{(\psi)} = \tau_m^{(\phi)}$$

$$\mu_{\ell,j} = |\langle\phi_\ell|\psi_j\rangle|^2$$

Unitary matrix $\{\langle\phi_m|\psi_j\rangle\}$ <–> autonomous evolution

Stochastic matrix $\{\mu_{\ell,j}\}$ <–> measurement

\rm sec\,} \theta)

Example; spin one-half

- Pre-state $\rho = |\psi_1\rangle\, \rho_1^{(\psi)}\, \langle\psi_1| + |\psi_2\rangle\, \rho_2^{(\psi)}\, \langle\psi_2|$

$$= |\phi_1\rangle\, \tau_1^{(\phi)}\, \langle\phi_1| + |\phi_2\rangle\, \tau_2^{(\phi)}\, \langle\phi_2| + |\phi_1\rangle\, \rho_{12}^{(\phi)}\, \langle\phi_2| + |\phi_2\rangle\, \rho_{21}^{(\phi)}\, \langle\phi_1|$$

- Post-state $\tau = |\phi_1\rangle\, \tau_1^{(\phi)}\, \langle\phi_1| + |\phi_2\rangle\, \tau_2^{(\phi)}\, \langle\phi_2|$

- weight matrix $\mu^{(2)} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2} \\ \sin^2\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix}$

Unistochastic matrix n=2
(parameter space: line)

$$\begin{pmatrix} \rho_1^{(\psi)} \\ \rho_2^{(\psi)} \end{pmatrix} = \begin{pmatrix} \frac{\tau_1^{\phi}+\tau_2^{\phi}}{2} + \frac{\tau_1^{\phi}-\tau_2^{\phi}}{2}\sec\theta \\ \frac{\tau_1^{\phi}+\tau_2^{\phi}}{2} - \frac{\tau_1^{\phi}-\tau_2^{\phi}}{2}\sec\theta \end{pmatrix}$$

- History-erasing "core": $\det \mu^{(2)} = 0$

a point $\theta = \frac{\pi}{2}$ on line $\theta \in [0, \pi]$

Unistochastic Matrices

- Stochastic matrix $\mu = \{\mu_{\ell,j}\}$ (n x n dim)

$$\mu_{\ell,j} \geqslant 0 \; , \quad \sum_{\ell} \mu_{\ell,j} = \sum_{j} \mu_{\ell,j} = 1$$

- Unistochastic matrix: $\mu_{\ell,j} = |\langle \phi_\ell | \psi_j \rangle|^2$

- Parameter space of entire stochastic matrix <— Birkhoff Polytope (Birkhoff, 1944)

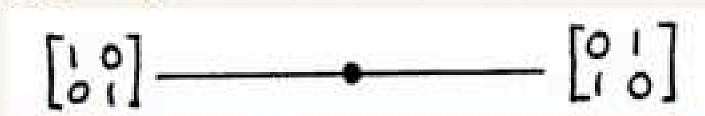
$$D(A,B) = \sqrt{\mathrm{Tr}(A-B)(A^\dagger - B^\dagger)}$$

- Unistochastic space within Birkhoff polytope: *nontrivial* mathematical problem

N=3 unistochastic matrix & its core

- 2dim unistochastic = 2dim stochatic : line $\theta \in [0, \pi]$ degenerate core $\det \mu^{(2)} = 0$: a point (midpoint)

$$\mu^{(2)} = p_0 P_0^{(2)} + p_1 P_1^{(2)} \qquad P_0^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

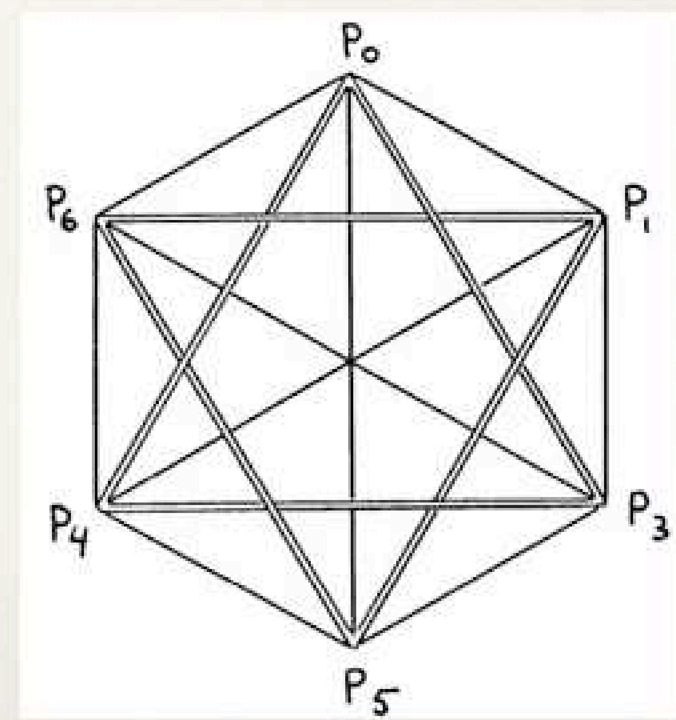


- 3dim unistochastic: hypercycloid within polytope degenerate core $\det \mu^{(3)} = 0$: codim. 1 hypersurface

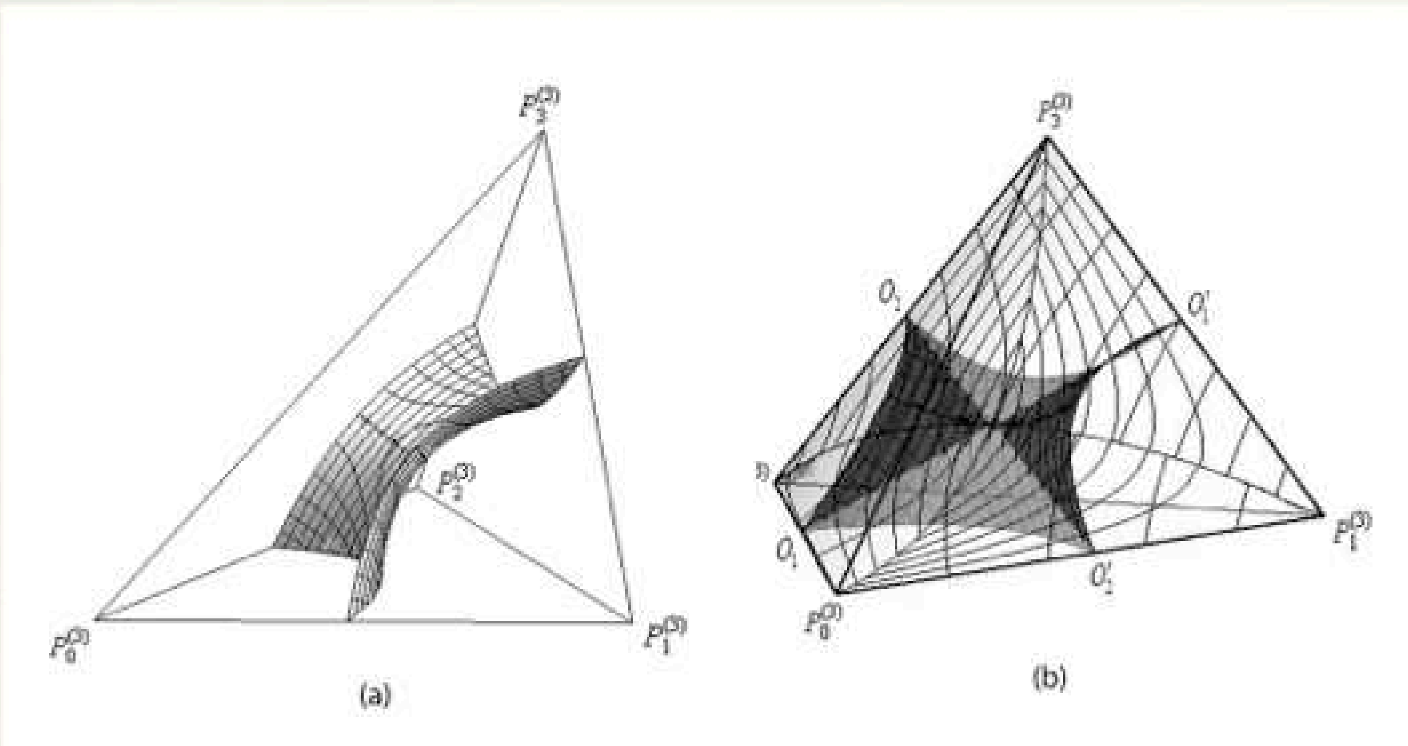
$$\mu^{(3)} = \sum_{i=0}^{5} p_i P_i^{(3)}$$

$$P_0^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, P_2^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_3^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, P_4^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, P_5^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

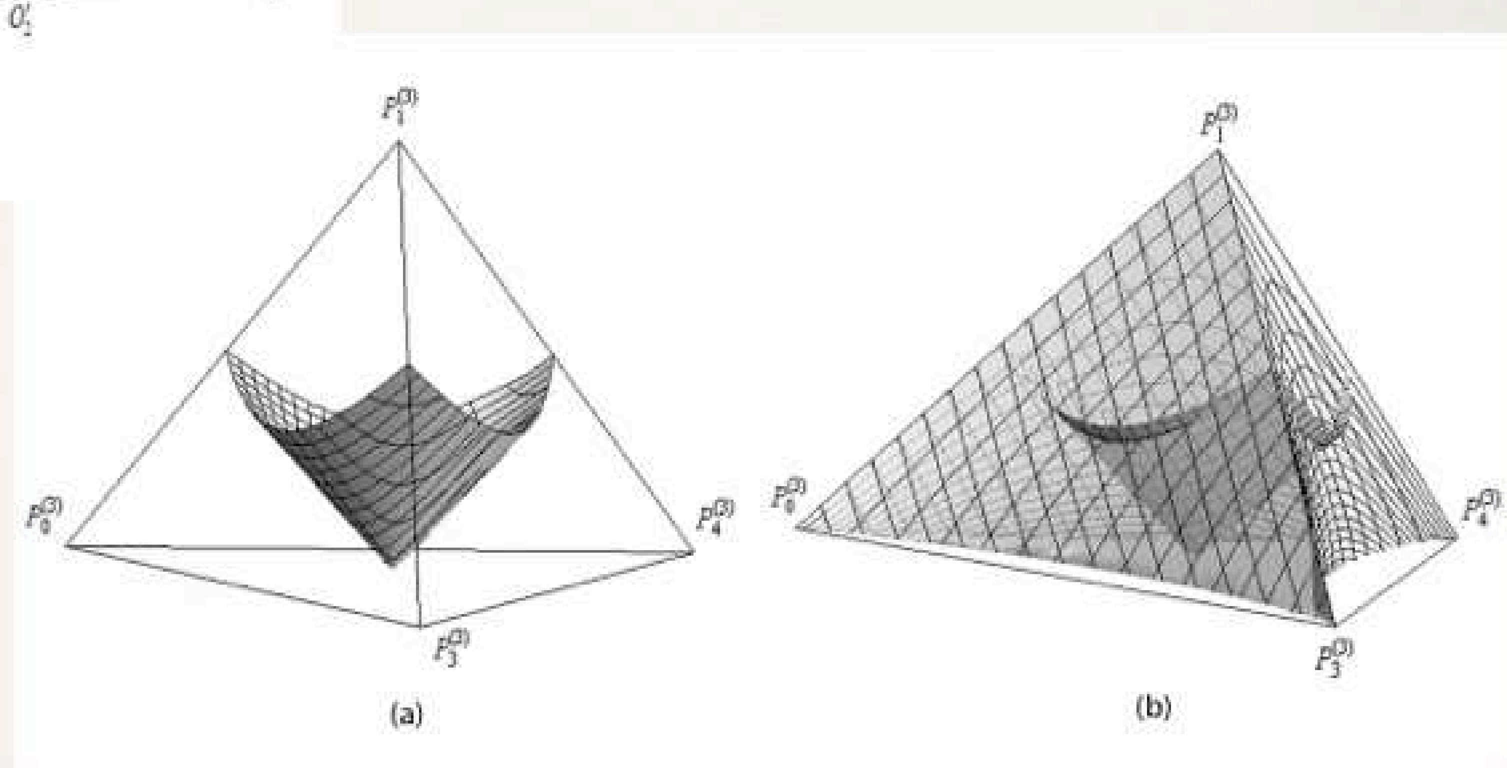


N=3 unistochastic matrix & its core



<—— P_0, P_1, P_2, P_3 surface

P_0, P_1, P_3, P_4 polytope ——>



- N=4 and beyond: still incomplete

Answers to the initial questions

- Complete set of weak values describes Hermitian operator in full
- Weak value expansion with two bases
  - all relevant quantities are gauge invariant
  - possibly measurable
  - don't miss the role of "weight"
- Weak value expansion is singular at one-basis limit
  - new aspects of quantum system possibly revealed

Summary

- Weak values as a whole set placed in the context of "normal" formulation of quantum mechanics
- Unistochastic subspace of Birkhoff polytope and its irreversible core analyzed for 3 dim Hilbert space
- Evolution by unitary sphere & stochastic kompeto
- References:
  Taksu Cheon & Sergey Poghosyan, (June 2013)
  Weak value expansion of quantum operators and its application in stochastic matrices (to appear in arXiv)