# Mathematical morphology – fundamentals and advances

**"Le Club des Morphologistes Mathématiques du Japon"**
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*Abstract*— **This invited talk presents the concept of mathematical morphology, which is a mathematical framework of quatitative image manipulations. The basic operations of mathematical morphology, the relationship to image processing filters, the idea of size distribution and its application to texture analysis are explained.**

## I. Introduction

Mathematical morphology treats an effect on an image as an effect on the shape and size of objects contained in the image. Mathematical morphology is a mathematical system to handle such effects quantitatively based on set operations [1]–[5]. The word stem "morpho-" originates in a Greek word meaning "shape," and it appears in the word "morphing," which is a technique of modifying an image into another image smoothly.

The founders of mathematical morphology, G. Mathéron and J. Serra, were researchers of lÉcole Nationale Supérieure des Mines de Paris in France, and had an idea of mathematical morphology as a method of evaluating geometrical characteristics of minerals in ores [6]. Mathéron is also the founder of the *randam closed set theory*, which is a fundamental theory of treating random shapes, and *kriging*, which is a statistical method of estimating a spatial distribution of mineral deposits from trial diggings. Mathematical morphology has relationships to these theories and has been developed as a theoretical framework of treating spatial shapes and sizes of objects.

This invited talk explains the framework of mathematical morphology, especially opening, which is the fundamental operation of describing operations on shapes and sizes of objects quantitatively, in Sec. II. Section III proves "filter theorem," which guarantees that all practical image processing filters can be constructed by combinations of morphological operations. Section IV introduces recent advances in mathematical morphology, by picking up some papers mainly from the 7th International Symposium on Mathematical Morphology. Section V briefly introduces our presentations in the "Mathematical Morphology" organized session in this conference.

## II. Basic operations of mathematical morphology

The fundamental operation of mathematical morphology is "opening," which discriminates and extracts object shapes with respect to the size of objects. We explain opening on binary images at first, and basic operations to describe opening.

### A. Opening

In the context of mathematical morphology, an object in a binary image is regarded as a set of vectors corresponding to the points composing the object. In the case of usual digital images, a binary image is expressed as a set of white pixels or pixels of value one. Another image set expressing an effect to the above image set is considered, and called *structuring element.* The structuring element corresponds to the window of an image processing filter, and is considered to be much smaller than the target image to be processed.

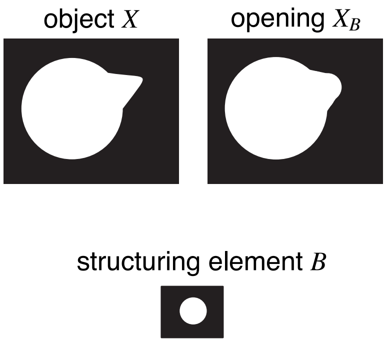
Let the target image set be $X$, and the structuring element be $B$. *Opening* of $X$ by $B$ has a property as follows:

$$X_B = \{B_z \,|\, B_z \subseteq X, z \in \mathbb{Z}^2\}, \tag{1}$$

where $B_z$ indicates the *translation* of $B$ by $z$, defined as follows:

$$B_z = \{b + z \,|\, b \in B\}. \tag{2}$$

This property indicates that the opening of $X$ with respect to $B$ indicates the locus of $B$ itself sweeping all the interior of $X$, and removes smaller white regions than the structuring element, as illustrated in Fig. 1. Since opening eliminates smaller structures and smaller bright peaks than the structuring element, it has a quantitative smoothing ability.




(a)

Fig. 1. Effect of opening.

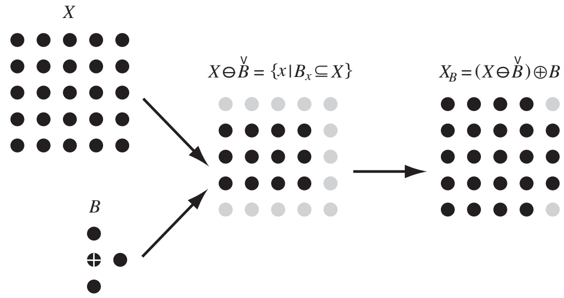



Fig. 2. opening composed of fundamental operations.

### *B. Fundamental operations of mathematical morphology*

Although the property of opening in (1) is intuitively understandable, this is not a pixelwise operation. Thus opening is defined by a composition of simpler pixelwise operations. In order to define opening, *Minkowski set subtraction* and *addition* are defined as the fundamental operations of mathematical morphology.

$$X \ominus B = \bigcap_{b \in B} X_b, \tag{3}$$

$$X \oplus B = \bigcup_{b \in B} X_b. \tag{4}$$

Minkowski set subtraction has the following property: It follows from $x \in X_b$ that $x - b \in X$. Thus the definition of Minkowski set subtraction in Eq. (3) can be rewritten to the following pixelwise operation:

$$X \ominus B = \{x | x - b \in X, b \in B\}. \tag{5}$$

The *reflection* of $B$, denoted $\check{B}$, is defined as follows:

$$\check{B} = \{-b | b \in B\} \tag{6}$$

Using the above expressions, Minkowski set subtraction is expressed as follows:

$$X \ominus B = \{x | \check{B}_x \subseteq X\}. \tag{7}$$

Since we get from the definition of reflection in Eq. (6) that $\check{B}_x = \{-b + x | b \in B\}$, it follows that $\check{B}_x = \{x - b | b \in B\}$. We get the relationship in Eq. (7) by substituting it into Eq. (5). This relationship indicates that $X \ominus B$ is the locus of the origin of $\check{B}$ when $\check{B}$ sweeps all the interior of $X$.

For Minkowski set addition, it follows that

$$\bigcup_{b \in B} X_b = \{x + b | x \in X, b \in B\}. \tag{8}$$

Thus we get

$$X \oplus B = \{b + x | b \in B, x \in X\} = \bigcup_{x \in X} B_x. \tag{9}$$

It indicates that $X \oplus B$ is composed by pasting a copy of $B$ at every point within $X$.

Using the above operations, *erosion* and *dilation* of $X$ with respect to $B$ are defined as $X \ominus \check{B}$ and $X \oplus \check{B}$, respectively.

We get from Eq. (7) that

$$X \ominus \check{B} = \{x | B_x \subseteq X\}. \tag{10}$$

It indicates that $X \ominus \check{B}$ is the locus of the origin of $B$ when $B$ sweeps all the interior of $X$.

The opening $X_B$ is then defined using the above fundamental operations as follows:

$$X_B = (X \ominus \check{B}) \oplus B \tag{11}$$

The above definition of opening is illustrated in Fig. 2. A black dot indicates a pixel composing an image object in this figure. As shown in the above, the erosion of $X$ by $B$ is the locus of the origin of $B$ when $B$ sweeps all the inside of $X$. Thus the erosion in the first step of opening produces every point where a copy of $B$ included in $X$ can be located. The Minkowski addition in the second step locates a copy of $B$ at every point within $X \ominus \check{B}$. Thus the opening of $X$ with respect to $B$ indicates the locus of $B$ itself sweeping all the interior of $X$, as described at the beginning of this section. In other words, the opening removes regions of $X$ which are too small to include a copy of $B$ and preserves the others.

The counterpart of opening is called *closing,* defined as follows:

$$X^B = (X \oplus \check{B}) \ominus B. \tag{12}$$

The closing of $X$ with respect to $B$ is equivalent to the opening of the background, and removes smaller spots than the structuring element within image objects. This is because the following relationship between opening and closing holds:

$$[X^B]^c = (X^c)_B, \tag{13}$$

where $X^c$ indicate the *complement* of $X$ and defined as $\{x | x \notin X\}$. The relationship of Eq. (13) is called *duality* of opening and closing[1].

[1]There is another notation system which denotes opening as $X \circ B$ and closing as $X \bullet B$.

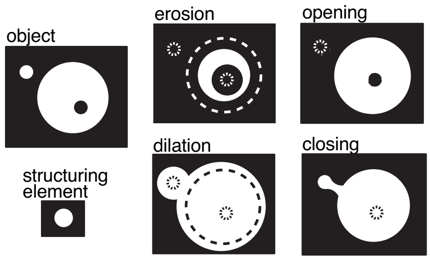



Fig. 3. Effects of erosion, dilation, opening, and closing

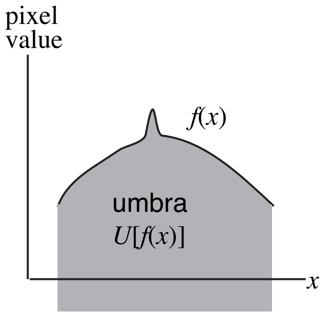



Fig. 4. Umbra. The spatial axis $x$ is illustrated one-dimensional for simplicity.

Figure 3 summarizes the illustration of the effects of basic morphological operations[2].

### C. In the case of gray scale images

In the case of gray scale image, an image object is defined by the *umbra* set. If the pixel value distribution of an image object is denoted as $f(x)$, where $x \in \mathbb{Z}^2$ is a pixel position, its umbra $U[f(x)]$ is defined as follows:

$$U[f(x)] = \{(x, t) \in \mathbb{Z}^3 \mid -\infty < t \leq f(x)\}. \tag{14}$$

Consequently, when we assume a "solid" whose support is the same as a gray scale image object and whose height at each pixel position is the same as the pixel value at this position, the umbra is equivalent to this solid and the whole volume below this solid within the support, as illustrated in Fig. 4.

A gray scale structuring element is also defined in the same manner. Let $f(x)$ be the gray scale pixel value at pixel position $x$ and $g(y)$ be that of the structuring element. Erosion of $f$ by $g$ is defined for the umbrae similarly to the binary case, and reduced to the following operation [4], [5]:

$$\{f \ominus \check{g}\}(x) = \inf_{b \in w(g)} \{f(x+b) - g(b)\}. \tag{15}$$

Dilation is also reduced to

$$\{f \oplus \check{g}\}(x) = \sup_{b \in w(g)} \{f(x+b) + g(b)\}. \tag{16}$$

where $w(g)$ is the support of $g$. These equations indicate that the logical AND and OR operations in the definition for binary images are replaced with the infimum and supremum operations (equivalent to minimum and maximum in the case of digital images), respectively. The gray scale opening using the umbra representation is illustrated in Fig. 5.

[2] There is another definition of morphological operations which denotes the erosion in the text as $X \ominus B$ and call the Minkowski set addition in the text "dilation." The erosion and dilation are not dual in this definition, however, it has an advantage that opening is simply denoted as $(X \ominus B) \oplus B$, i. e. "erosion followed by dilation."

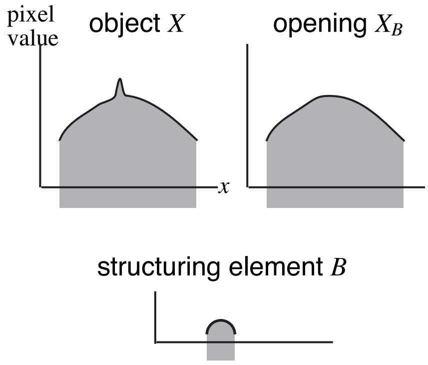



Fig. 5. Gray scale opening using the umbra representation.

Expanding the idea, mathematical morphological operations can be defined for sets where the infimum and supremum among the elements are defined in some sense. For example, morphological operations for color images cannot be defined straightforwardly from the above classical definitions, since a color pixel value is defined by a vector and the infimum and supremum are not trivially defined. The operations can be defined if the infimum and supremum among colors are defined [8], [9]. Such set is called *lattice,* and mathematical morphology is generally defined as operations on a lattice [10].

## III. Mathematical morphology and image processing filter

Are the morphological operations explained in the above *really* fundamental operations? The *filter theorem* shown below gives an answer that *any translantion-invariant increasing* image transformation is decomposed into morphological operations with various structuring elements combined by logical operations.

### A. Morphological filter

Image processing filter is generally an operation at each pixel by applying a calculation to the pixel and the pixels in its neighborhood and replacing the pixel value with the result of calculation, for the purposes of noise removal, etc. Morphological filter in broader sense is restricted to *translation-invariant* and *increasing* one. An operation $\Psi$ on a set (image) $X$ is translation-invariant if

$$\Psi(X_b) = [\Psi(X)]_b. \tag{17}$$

In other words, it indicates that the effect of the operation is invariant wherever the operation is applied. An operation $\Psi$ is increasing if

$$X \subset Y \Rightarrow \Psi(X) \subset \Psi(Y). \tag{18}$$

In other words, the relationship of inclusion is preserved before and after applying the operation.

Let us consider a noise removing filter for example; Since noise objects in an image should be removed wherever it is located, the translation-invariance is naturally required for noise removing filters. An increasing operation can express an operation that removes smaller objects and preserves larger objects, but cannot express an operation that removes larger and preserves smaller. Noise objects are, however, usually smaller than meaningful objects. Thus it is also natural to consider increasing operations only[3].

Morphological filter in narrower sense is all translation-invariant, increasing, and *idempotent* operations. The filter $\Psi$ is idempotent if

$$\Psi[\Psi(X)] = \Psi(X). \tag{19}$$

Consequently, iterative operations of $\Psi$ is equivalent to one operation of $\Psi$ . The opening and closing are the most basic morphological filters in narrower sense.

### *B. Filter theorem*

The *filter theorem* states that all morphological filters (in broader sense) can be expressed by OR of erosions and AND of dilations. It guarantees that almost all practical filters can be expressed by morphological operations, i. e. mathematical morphology is really a fundamental operation set of image object manipulations. Let $\Psi(X)$ be a filter on the image $X$. The theorem states that there exists for all $\Psi(X)$ a set family $\mathrm{Ker}[\Psi]$ satisfying the following.

$$\Psi(X) = \bigcup_{B \in \mathrm{Ker}[\Psi]} X \ominus \check{B} \tag{20}$$

It also states that there exists for all $\Psi(X)$ a set family $\mathrm{Ker}[\Psi]$ satisfying the following.

$$\Psi(X) = \bigcap_{B \in \mathrm{Ker}[\Psi]} X \oplus \check{B} \tag{21}$$

Here the set family $\mathrm{Ker}[\Psi]$ is called *kernel* of filter $\Psi$, defined as follows:

$$\mathrm{Ker}[\Psi] = \{X \mid 0 \in \Psi(X)\}. \tag{22}$$

where "0" indicates the origin of $X$.

The proof of the filter theorem in (20) is presented in the following. A more general proof is found in Chap. 4 of [10].

Let us consider an arbitrary vector (pixel) $h \in X \ominus \check{B}$ for a structuring element $B \in \mathrm{Ker}[\Psi]$. From the definition of $X \ominus \check{B}$, $B_h \subseteq X$. Consequently, $B \subseteq X_{-h}$. Since $\Psi$ is increasing, the relationship $B \subseteq X_{-h}$ is invariant by filter $\Psi$. Thus we get $0 \in \Psi(B) \Rightarrow 0 \in \Psi(X_{-h})$. Since $\Psi$ is translation-invariant, we get $0 \in \Psi(B) \Rightarrow h \in \Psi(X)$ by translating $0 \in \Psi(X_{-h})$ by $h$. From the above discussion, $h \in X \ominus B \Rightarrow h \in \Psi(X)$ for all structuring element $B \in \mathrm{Ker}[\Psi]$. Thus $\Psi(X) \supseteq \bigcup_{B \in \mathrm{Ker}[\Psi]} X \ominus \check{B}$.

Let us consider an arbitrary vector $h \in \Psi(X)$. Since $\Psi$ is translation-invariant, $h \in \Psi(X) \Rightarrow 0 \in \Psi(X_{-h})$. Thus we get $X_{-h} \in \mathrm{Ker}[\Psi]$. Since $X \ominus \check{X}_{-h} = \{h' \mid (X_{-h})_{h'} \subseteq X\}$, and $\{(X_{-h})_{h'} \subseteq X\}$ is satisfied if $h' = h$, we get $h \in X \ominus \check{X}_{-h}$. By denoting $X_{-h}$ by $B$, we get $h \in X \ominus \check{B}$.

Consequently, there exists a structuring element $B \in \mathrm{Ker}[\Psi]$ such that $h \in \Psi(X) \Rightarrow h \in X \ominus \check{B}$, i. e. any pixel in $\Psi(X)$ can be included in $X \ominus \check{B}$ by using a certain structuring element $B \in \mathrm{Ker}[\Psi]$. Thus $\Psi(X) \subseteq \bigcup_{B \in \mathrm{Ker}[\Psi]} X \ominus \check{B}$.

From the above discussion, it holds that $\Psi(X) = \bigcup_{B \in \mathrm{Ker}[\Psi]} X \ominus \check{B}$.

[3]Edge detecting filter is not increasing, since it removes all parts of objects except edges regardlessly of the sizes.

## IV. Advances in ISMM'05

The International Symposium on Mathematical Morphology (ISMM), which is the topical international symposium focusing on mathematical morphology only, has been organized almost every two years, and its seventh symposium was held in April 2005 in Paris as a cerebration of 40 years anniversary of mathematical morphology [7].

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## V. Organized session in this conference

We established a group of researchers of mathematical morphology, called "Le Club des Morphologistes Mathématiques du Japon [22] [4]," and organized a session specialized to mathematical morphology in this conference. We present various application and theoretical studies of the members in the session. The followings are outlines of our presentations.

A. Asano and M. Muneyasu [23] present an application of morphological size distribution for the evaluation of skin rejuvenation methods. The morphological size distribution measures the fineness of particles in an image, and it is applied to the evaluation of the fineness of collagen in the dermis after the rejuvenation treatments.

C. Muraki Asano, A. Asano, and M. Muneyasu [24] present an application of mathematical morphology for *kansei* engineering, which is an approach to connect the human sensibility to engineering applications. This research evaluates human impressions to black cloths and other texture images using mathematical morphological methods.

A. Asano and M. Muneyasu [25] also present a method of optimizing the structuring element of the opening for removing noise in texture images. This method can estimate the structure of the target texture even if it is corrupted by noise, and extract the optimal structuring element which can preserve the structure by opening.

H. Nobuhara [26] presents rather a theoretical work on morphological wavelet transform, which is a wavelet transform based on the max-plus algebra. This work investigates a generalization of the morphological wavelet by using various sampling windows, and shows applications to image and video codings.

M. Fujio [27] presents a pure theoretical work. As described at the end of Sec. II, mathematical morphology is generalized to a set of operations on a lattice. This work applies the concept of morphological operations to an analysis of logic, which is a typical example of the lattice.

[4]The group has been named in French because of our respect to the founders of mathematical morphology.

## VI. Conclusions

This paper

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