The K-theoretic classification of topological insulators and superconductors

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Abstract

We review the topological classification of free-fermionic Hamiltonians with a spectral gap via real and complex K-theory as popularised by Kitaev. After reviewing complex symmetry classes and the general form of symmetries in quantum mechanics via projective unitary/anti-unitary (PUA) representations, we study the physical symmetry classes of free fermions as specified by Altland and Zirnbauer. In particular, we work in the setting of Nambu space and Bogoliubov–de Gennes (BdG) Hamiltonians as a model for free-fermionic systems. We finally define K-theoretic indices for pairs of BdG Hamiltonians via Clifford module indices or van Daele K-theory. The range of our index depends both on the symmetry class and the allowed deformations of the Hamiltonians under consideration.

Keywords: free-fermions, K-theory, operator algebras, topological insulators and superconductors, topological phases of matter

Key points/objectives:
Our central aims for this note are to:

- Demonstrate that K-theory for operator algebras is both a natural and convenient framework to study topological properties of Hamiltonians on Hibert spaces.

- Explain the mathematical framework to study symmetric (gapped) ground states of free-fermions using Nambu space and Bogoliubov–de Gennes (BdG) Hamiltonians.

- Provide caution to the reader that any 'K-theoretic classification' is always with reference to certain choices related to the setting under consideration.

1 Introduction

The integer quantum Hall effect opened new connections between condensed matter physics and topology, where the quantised and stable Hall conductance can be described using characteristic classes of vector bundles or, more generally, methods from noncommutative geometry [4]. Several years later within the physics literature, ideas from topology were also found to be useful in describing the quantum spin Hall effect [17] and Majorana fermions in p-wave superconductors [20]. Where these systems differ from the integer quantum Hall effect is that extra assumptions or symmetries are imposed on the system such as time-reversal symmetry.

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Schnyder et al. [25] built on these examples and assigned a $\mathbb{Z}$ or $\mathbb{Z}_2$-phase label to Hamiltonians describing free-fermionic systems in dimension ≤ 3 that fall into a symmetry class previously outlined by Zirnbauer et al. [2, 16]. These phase labels are 'topological' in the sense that a change in this label via a symmetric path of Hamiltonians implies a certain discontinuity such as a closing of a spectral gap.

The work of Schnyder et al. was then systematised by Kitaev [21], who noted that the phase labels given to free-fermionic Hamiltonians describe the 8 real and 2 complex K-theory groups of a point. Kitaev then outlined that real and complex K-theory can be used to give a more general classification of free-fermionic systems in any dimension. Kitaev's result created a flurry of work within both the physics and mathematics literature. Indeed, Kitaev's paper provides an outline but precise mathematical details are omitted. Unfortunately this can lead to imprecision or ambiguities when applying or adapting the method. Still, a general picture of a '10-fold way' and 'K-theoretic classification' of free-fermions is accepted within the physics literature.

A mathematically precise K-theoretic classification of free-fermionic Hamiltonians can be presented. But as we will emphasise in this note, it is with reference to certain choices:

- 'Homotopy in what?' If we are considering paths of symmetric Hamiltonians, what types of deformations are allowed and what is considered continuous? If, for example, the Hamiltonians are bounded, then the bounded operators $\mathcal{B}(\mathcal{H})$ with norm-topology is too large as any two (gapped) Hamiltonians can be connected.

- If our Hamiltonians have additional symmetries, what is the precise action of these symmetries on the Hilbert space under consideration?

We will show that once we have set these precise conditions, a topological phase label for free-fermionic Hamiltonians with a spectral gap can be defined using real and complex K-theory for operator algebras. The aim of this work is not to provide an introduction to K-theory of operator algebras, but instead show how the theory can be used to define such a free-fermionic topological phase label. However, we warn that this phase label is either given with reference to a pre-determined 'trivial' system or it is a *relative* phase between a pair of symmetric gapped Hamiltonians.

We start with the relatively simple case of complex symmetry classes in Section 2. Section 3 then gives a framework to understand more general (possibly anti-linear) symmetries in quantum mechanics. We then restrict our setting in Section 4 to free-fermionic systems using the Hartree–Fock–Bogoliubov mean field approximation via dynamics on Nambu space. The free-fermionic symmetry classes as outlined in [2, 16] are then briefly reviewed. Then, under differing assumptions, we define K-theoretic indices for such free-fermionic and symmetric Hamiltonians in Sections 5 and 6. Our exposition closely follows the perspective of Alldridge et al. [1], though the techniques are related to the work of Kellendonk [18]. We conclude with some extensions and limitations of our method.

2 Warm up: complex classes of topological insulators

Before we consider the general classification, let us look at the case of a quantum mechanical system without additional symmetries or with a $\mathbb{Z}_2$-graded structure and odd Hamiltonian (chiral symmetry). In either setting, we can assign a topological index to the system using topological (complex) K-theory for spaces or C^*-algebras.

2.1 Insulators without symmetries (type A)

Assuming the temperature is sufficiently low and many-body interactions negligible, we can mathematically model an insulator via a complex Hilbert space $\mathcal{H}$ and a self-adjoint operator, the Hamiltonian H, whose spectrum can be split into at least 2 disjoint regions. Our aim is to model fermionic systems at low temperature, so the particle statistics are governed by the Fermi-Dirac distribution $f_\beta(H) = (1 + e^{\beta(H-\mu)})^{-1}$ with μ the chemical potential at absolute zero (also called the Fermi energy) and β the inverse temperature. As $\beta \to \infty$, $f_\beta(H) \to P_\mu(H) = \chi_{(-\infty, \mu]}(H)$, the projection onto energies less that μ. Our central hypothesis is that μ lies in a gap of the spectrum of H with spectrum above and below. Namely, we have an insulating system.

In the zero temperature limit, the projection $P_\mu(H)$ determines the ground state and so specifies the physics of the insulating system. Therefore, a classification of free-fermionic insulators can be mathematically recast as a problem of classifying projections $P_\mu(H)$ for H a Hamiltonian and $\mu \notin \sigma(H)$ its Fermi energy. However, to obtain a physically meaningful classification, information of the Hamiltonian and its properties should be reflected in how we declare projections equivalent. Indeed, if we work on an infinite-dimensional Hilbert space, then all projections are Murray–von Neumann equivalent and so all insulators would be considered equivalent without additional information. The hypothesis we work under for this document is that our Hamiltonians of interest are an element of or affiliated to a C^*-algebra of observables A, which we think of as a strict (norm-closed) subalgebra of the $\mathcal{B}(\mathcal{H})$, the bounded operators on $\mathcal{H}$. For simplicity, we will assume that A is unital. The precise nature of A depends on the setting under consideration, though in typical examples it will contain information about the spatial dimension d and locality properties of H.

Let us briefly work in the setting of a tight-binding system and periodic Hamiltonian, where $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^n)$ and the bounded Hamiltonian H is a self-adjoint matrix of finite polynomials of the lattice shift operators $\{S_m\}_{m \in \mathbb{Z}^d}$,

$$(S_m \psi)(x) = \psi(x - m), \quad x, m \in \mathbb{Z}^d, \qquad (S_m)^* = S_{-m}.$$

In particular H is periodic under such lattice shits. By the Fourier transform we can decompose both our Hilbert space and Hamiltonian

$$\mathcal{H} = \int_{\mathbb{T}^d}^{\oplus} \mathbb{C}^n(k) \, \mathrm{d}k, \qquad H = \int_{\mathbb{T}^d}^{\oplus} H(k) \, \mathrm{d}k,$$

where we have identified the dual space $\mathcal{F}\mathcal{H}\mathcal{F}^* \cong L^2(\mathbb{T}^d, \mathbb{C}^n)$. The spectral projection $P_\mu(H)$ also decomposes as a direct integral of the finite-dimensional projections $P_\mu(H(k))$. Because $\mu \notin \sigma(H)$, the family $k \mapsto P_\mu(H(k))$ is analytic. By taking the family of vector spaces $\{\mathrm{Ran}(P_\mu(H(k)))\}_{k \in \mathbb{T}^d}$, we obtain a finite-rank complex vector bundle $P_\mu \to \mathbb{T}^d$. Returning to our original goal of classifying insulating systems, in this periodic setting, we can recast this question as a classification of finite-rank vector bundles over $\mathbb{T}^d$, e.g. up to unitary equivalence. It is well known that this question can be answered by studying the Chern classes of complex vector bundles.

A very similar approach can be done by considering the topological K-theory of $\mathbb{T}^d$, which is the Grothendieck completion of the monoid of isomorphism classes of finite-rank complex vector bundles over $\mathbb{T}^d$ with addition via direct sum. The Grothendieck completion means that we declare a *pair* of vector bundles (E_+, E_-) to be equivalent to (F_+, F_-) if there are finite-rank vector bundles G and H such that

$$\left(E_+ \oplus G, E_- \oplus G \right) \sim \left(F_+ \oplus H, F_- \oplus H \right)$$

as pairs of vector bundles. Such an equivalence relation gives pairs of vector bundles an inverse, $\left[(E_+, E_-)\right]^{-1} = \left[(E_-, E_+)\right]$ and one obtains an abelian group $K(\mathbb{T}^d)$. The advantage of working with K-theory as opposed to more standard cohomological methods is that it comes with a variety of tools to help us compute the K-theory group of a given space.

Returning to our more generic picture of a self-adjoint Hamiltonian contained in or affiliated to a C^*-algebra A with $\mu \notin \sigma(H)$. Provided H is bounded from below, the Fermi projection $P_\mu(H)$ is then an element of A. The K-theory group $K_0(A)$ is the Grothendieck completion the semigroup of stable equivalence classes of projections in $M_n(A)$, the $n \times n$ matrices with entries in A. Therefore, in analogy to periodic setting, the group $K_0(A)$ can be used to distinguish equivalence classes of free-fermionic insulators without symmetry with respect to the underlying algebra A that is specifying the physical details of the system. Because any Hamiltonian acting on $\mathcal{H}$ and contained in or affiliated to a C^*-algebra A with $\mu \notin \sigma(H)$ defines an element $[P_\mu(H)] \in K_0(A)$, this K-theory class will give a topological classification of all insulating Hamiltonians affiliated to our observable algebra A.

Returning to the periodic setting with $d = 2$ and H a matrix of polynomials of shift operators acting on $\ell^2(\mathbb{Z}^2, \mathbb{C}^n)$. We can take $A = C^*(\mathbb{Z}^2)$, the group C^*-algebra, which is generated by $\{S_m\}_{m \in \mathbb{Z}^2}$ and $K_0(C^*(\mathbb{Z}^2)) \cong K^0(\mathbb{T}^2)$, the topological K-theory. In this case, an element $[P_\mu(H)] \in K_0(C^*(\mathbb{Z}^2))$ is specified by 2 integers, the (virtual) rank of the bundle $P_\mu \to \mathbb{T}^2$ and the integration of the first Chern class with respect to the orientation element (often called the first Chern number). Thus, the K-theory class $[P_\mu(H)] \in K_0(C^*(\mathbb{Z}^2))$ can be considered as a topological index of the insulator H on $\ell^2(\mathbb{Z}^2, \mathbb{C}^n)$ with $0 \notin \sigma(H)$.

We finish our discussion of quantum mechanical systems without symmetries with a few warnings. Firstly, the case of Chern classes or $K_0(A)$ more generally is quite special as there is a canonical 'trivial' system. Namely, bundles $P_\mu \to \mathbb{T}^d$ that are equivalent to a trivial bundle over $\mathbb{T}^d$ or $[P_\mu(H)] = n[\mathbf{1}_A] \in K_0(A)$. For the case of other symmetries (particularly real symmetries), what is deemed a trivial system/base point needs to be more explicitly stated or instead one considers a relative index from a pair of Hamiltonians.

2.2 Insulators with chiral/sublattice symmetry (type AIII)

We now consider the case of a complex Hilbert space $\mathcal{H}$ with a $\mathbb{Z}_2$-grading $\mathcal{H} \cong \mathcal{H}^0 \oplus \mathcal{H}^1$ such that the Hamiltonian $H = H^*$ acts as an odd operator $H \cdot \mathcal{H}^j \subset \mathcal{H}^{j+1}$, $j \in \mathbb{Z}_2 = \{0, 1\}$ (for simplicity, we will assume H is bounded). An equivalent way to describe this setting is that there is a self-adjoint unitary $\gamma = \gamma^* = \gamma^{-1}$ on $\mathcal{H}$, where $\mathcal{H}^0$ and $\mathcal{H}^1$ are the $+1$ and -1 eigenspaces of γ respectively. Then H is an odd operator if and only if $\gamma H \gamma = -H$. Such a relation is called a chiral or sublattice symmetry of the operator H. Because H anti-commutes with γ, its spectrum must be symmetric about the point 0. Let us then assume that the Fermi energy $\mu = 0$ (taking a shift if necessary) and $0 \notin \sigma(H)$. Therefore H is an invertible and odd operator.

By the spectral decomposition of γ, the Hamiltonian can be decomposed

$$\mathcal{H} = (\mathbf{1} + \gamma)\mathcal{H} \oplus (\mathbf{1} - \gamma)\mathcal{H}, \qquad H = \begin{pmatrix} 0 & \frac{1}{2}(1 + \gamma)H\frac{1}{2}(1 - \gamma) \\ \frac{1}{2}(1 - \gamma)H\frac{1}{2}(1 + \gamma) & 0 \end{pmatrix}.$$

We let $Q = \frac{1}{2}(1 - \gamma)H\frac{1}{2}(1 + \gamma)$, which is invertible and generally will not be self-adjoint. Let us emphasise that Q depends on both the Hamiltonian H and the self-adjoint unitary γ. For a fixed self-adjoint unitary γ, we can consider topological properties of chiral symmetric Hamiltonians by considering stable homotopy classes of the invertible operator Q. If H and γ are elements of a unital C^*-algebra A, then so is Q and we can consider the K-theory class $[Q] \in K_1(A)$, where

$K_1(A)$ is built from stable equivalence classes of invertible operators in (matrices over) A.[1]

The element $[Q] \in K_1(A)$ represents a topological obstruction to the existence of a homotopy in $M_N(A)$ that connects $Q = \frac{1}{2}(1 - \gamma)H\frac{1}{2}(1 + \gamma)$ to the multiplicative identity of A, $\mathbf{1}_A$. In the setting

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^{2n}), \qquad \gamma = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix},$$

where H is translation invariant under discrete shifts in $\mathbb{Z}$, we can consider the operator Q as an invertible function $Q : \mathbb{T} \to M_n(\mathbb{C})$. Assuming that $\mathbb{T} \ni k \mapsto Q(k) \in M_n(\mathbb{C})$ is continuous (which holds if H is a finite polynomial of discrete shifts), the topological obstruction to connect Q to the identity is given by the degree (winding number) of the function $Q : \mathbb{T} \to GL_n(\mathbb{C})$. The Noether–Toeplitz Index Theorem then guarantees that a non-trivial winding number implies a non-trivial Fredholm index of the restriction $\widehat{Q}$ of Q to a half-space system $\ell^2(\mathbb{N}, \mathbb{C}^{2n})$. The Fredholm index of $\widehat{Q}$ can thought of as the number of oriented edge states/zero modes of the Hamiltonian restricted to $\ell^2(\mathbb{N}, \mathbb{C}^{2n})$, where the orientation comes from the spectral decomposition of γ.[2]

The winding number of $Q : \mathbb{T} \to GL_n(\mathbb{C})$ also gives the isomorphism $K_1(C^*(\mathbb{Z})) \xrightarrow{\cong} \mathbb{Z}$ and so $\mathrm{Wind}(Q) = \mathrm{Index}(\widehat{Q})$ gives a topological obstruction to connect different chiral symmetric Hamiltonians (with respect to a fixed chiral symmetry γ). Because the group $K_1(A)$ is constructed from stable homotopy equivalences of invertible operators in A, the more general index $[Q] \in K_1(A)$ plays an analogous role.

Another interpretation of the chiral/sublattice symmetry $\gamma H \gamma = -H$ is that γ encodes the ambiguity of choosing a unit cell for H on $\ell^2(\mathbb{Z}, \mathbb{C}^{2n})$ with sublattice symmetry. Namely, there are two choices H_0 and H_1 which correspond choosing a unit cell with pairing within the cell or across different cells. Fixing a boundary $\ell^2(\mathbb{N}, \mathbb{C}^{2n})$ then forces one of these choices to be topologically trivial and the other non-trivial via the Fredholm index of $\widehat{Q}$. The relative obstruction between these two systems is encoded in the class $[Q] \in K_1(C^*(\mathbb{Z}))$ or $K_1(A)$ more generally. See [26] for further details.

3   Symmetries in quantum mechanics

We have briefly explained in two simple cases how K-theoretic indices can be defined for Hamiltonians on a complex Hilbert space. We now want to consider the case of Hamiltonians with a richer class of symmetries. Though to do this task, we must first analyse the nature of symmetries in quantum mechanics. In this section, we will review such a mathematical framework as discussed in [13, 27] for example, but which has close connections to the classical theorem of Wigner.

3.1   Group actions and Wigner's Theorem

Following the basic principles of quantum mechanics, we consider a complex Hilbert space $\mathcal{H}$, whose elements may be regarded as the pure states of the system under consideration. We expect any symmetry to be a map on Hilbert spaces that is norm-preserving and bijective. However, from the perspective of quantum mechanics, expectations of states are the only physically observable quantities. Therefore, given a pure state $\psi \in \mathcal{H}$, the key condition we require for

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[1]The more typical construction of $K_1(A)$ comes from stable homotopy equivalence classes of unitaries in A. As unitaries elements are a retract of invertible elements, the two presentations are equivalent.

[2]Such a result is an example of the bulk-boundary correspondence, which is of fundamental importance to topological insulators and superconductors.

this state to be invariant under a symmetry transformation $\Theta : \mathcal{H} \to \mathcal{H}$ is that

$$|\langle \Theta \psi, \Theta \psi \rangle|^2 = |\langle \psi, \psi \rangle|^2. \tag{1}$$

As noted by Wigner [28], Equation (1) implies that Θ may act as a unitary or anti-unitary operator.

We expect symmetries to emerge via a group representation $G \ni g \to \theta_g$. For simplicity, we will restrict to the case where G is finite. The condition (1) means that there is an ambiguity with respect to the choice of θ_g up to a $U(1)$-phase. That is, the representation of G is instead a *projective* representation. Given a homomorphism $G \to \{\pm 1\}$, we can specify whether the operator θ_g is unitary or anti-unitary if $\varphi(g) = 1$ or -1 respectively. We furthermore have a map $\sigma : G \times G \to \mathbb{T}$ such that $\theta_{g_1} \theta_{g_2} = \sigma(g_1, g_2) \theta_{g_1 g_2}$ satisfying a 2-cocycle relation

$$\sigma(g_1, g_2) \sigma(g_1 g_2, g_3) = \sigma(g_2, g_3)^{g_1} \sigma(g_1, g_2 g_3), \quad g_1, g_2, g_3 \in G,$$

where for $z \in \mathbb{T}$, $z^g = z$ if $\varphi(g) = 1$ and $z^g = \bar{z}$ if $\varphi(g) = -1$. The tuple $(G, \varphi, \theta, \sigma)$ is called a projective unitary/anti-unitary (PUA) representation of G.

We have yet to speak of dynamics in our quantum mechanical system. Of most interest to us is the time evolution and its generator, the self-adjoint Hamiltonian H acting on $\mathcal{H}$. One typically expects symmetries to commute with this operator, though it will also be useful to consider unitary/anti-unitary operators that anti-commute with H. One may consider these possibilities as the symmetries and constraints of the Hamiltonian. An example of an anti-commuting operator is the charge-conjugation/particle-hole involution on single-particle Hilbert space. We say that H is compatible with G if there is a homomorphism $c : G \to \{\pm 1\}$ such that

$$\theta_g H (\theta_g)^* = c(g) H \quad \text{for all } g \in G. \tag{2}$$

We now restrict ourselves to the setting of insulators or *gapped* systems. Taking a constant shift if necessary, we assume $0 \notin \sigma(H)$ and so H is invertible. This allows us to consider the spectrally flattened Hamiltonian $\mathrm{sgn}(H) = H|H|^{-1} = 2P_{(0,\infty)}(H) - \mathbf{1}$ with $P_{(0,\infty)}(H)$ the projection onto all positive energies of H. The ± 1 eigenspaces of $\mathrm{sgn}(H)$ give a $\mathbb{Z}_2$-splitting of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We see that Equation (2) implies that the homomorphism $c : G \to \{\pm 1\}$ is such that the PUA representation $(G, \varphi, \theta, \sigma)$ is a $\mathbb{Z}_2$-graded representation under this splitting $\mathcal{H}_+ \oplus \mathcal{H}_-$. To summarise our discussion, symmetries in single-particle Hilbert spaces that commute or anti-commute with an invertible (gapped) Hamiltonian have a mathematical description as a $\mathbb{Z}_2$-graded PUA representation of a symmetry group G.

3.2 The CT-symmetry group, Clifford algebras and the 10-fold way

Now that we have a general method to understand finite-group actions on complex Hilbert spaces with respect to an invertible Hamiltonian, let us now restrict our attention to the symmetries that are of particular interest for topological insulators and superconductors. The symmetries of interest to us are time-reversal symmetry, charge-conjugation symmetry (also called particle-hole symmetry) and chiral/sublattice symmetry (cf. Section 2.2). Each of these symmetries is given by an involution and we use the notation, $T \equiv$ time-reversal, $C \equiv$ charge-conjugation and $S \equiv$ sublattice. These are not independent but generate the CT-symmetry group $\{1, T, C, CT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where $S = CT = TC$.

We say that H acting on $\mathcal{H}$ respects these symmetries if there exists a unitary operator R_S and anti-unitary operators R_T and R_C on $\mathcal{H}$ such that

$$R_T H R_T^* = H, \qquad R_C H R_C^* = -H, \qquad R_S H R_S^* = -H \tag{3}$$

In the case of R_T and R_C, the Hamiltonian is said to have even (resp. odd) symmetry if $R_\bullet^2 = 1$ (resp. $R_\bullet^2 = -1$). Because R_S is complex-unitary, the sign of its square is irrelevant (in the same way that the complex Clifford algebra $\mathbb{C}\ell_1$ may have a generator that squares to $+1$ or -1). We note that a Hamiltonian may only respect a single symmetry. However, if H is compatible with two symmetries, then by the underlying group structure it is compatible with the third symmetry.

There is no general form that the symmetry operators R_T, R_C and R_S are forced to take and are determined by the example under consideration. We note that the operators R_T, R_C are defined with reference to a chosen complex conjugation/Real involution on $\mathcal{H}$. Because we have already outlined the case of 'complex symmetries' in Section 2, we will focus on the case where R_C or R_T is present. Equation (3) tells us that a symmetric Hamiltonian with respect to a subgroup $G \subset \{1, C, T, S = CT\}$ gives a $\mathbb{Z}_2$-graded PUA representation of G with $\theta_g = R_g$ and

$$(\varphi, c)(T) = (-1, 1), \qquad (\varphi, c)(C) = (-1, -1), \qquad (\varphi, c)(S) = (1, -1).$$

We can 'normalise' this PUA representation so that R_C and R_T commute with $R_C R_T = R_{CT}$. The sign of R_T^2 and R_C^2 will then specify the 2-cocycle σ.

In the particular setting of a subgroup $G \subset \{1, C, T, S = CT\}$, the $\mathbb{Z}_2$-graded PUA representation of G can be linked to a Clifford algebra. Here we are using the convention that the real clifford algebra $C\ell_{r,s}$ is the algebraic span (over $\mathbb{R}$) of the unitary elements $\{\gamma_1, \ldots, \gamma_r, \rho_1, \ldots, \rho_s\}$, where

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{i,j}, \qquad \rho_k \rho_l + \rho_l \rho_k = -2\delta_{k,l}, \qquad \gamma_j \rho_k = -\rho_k \gamma_j$$

for all $i, j \in \{1, \ldots, r\}$ and $k, l \in \{1, \ldots, s\}$. Clifford algebras also come with a natural $\mathbb{Z}_2$-grading, where each generating element is odd, though we will occasionally consider Clifford algebras without a $\mathbb{Z}_2$-grading.

A key result shown in [13, Appendix B] and [27, Section 6] is that each PUA representation of a subgroup $G \subset \{1, C, T, CT\}$ graded by the gapped Hamiltonian H gives rise to a real or complex Clifford algebra representation. Furthermore, all 10 Morita equivalence classes of Clifford algebras are exhausted by the various symmetry types of H. The representations are summarised in Table 1. This result is shown on a case-by-case basis, we highlight a few examples.

- For the full symmetry group $G = \{1, C, T, CT\}$, we consider the real algebra generated by $\{R_C, iR_C, iR_C R_T\}$. One checks that these generators have odd grading under $\text{sgn}(H)$, mutually anti-commute and are self-adjoint (resp. skew-adoint) if they square to $+1$ (resp. -1).

- For the subgroup $\{1, C\}$, we assign the real algebra generated by $\{R_C, iR_C\}$ and graded by $\text{sgn}(H)$.

- The case of the subgroup $\{1, T\}$ is a little different as R_T commutes with $\text{sgn}(H)$. For the case that $R_T^2 = 1$, R_T defines a Real structure on the Hilbert space and gives no additional Clifford generators. If $R_T^2 = -1$, then R_T defines a quaternionic structure on $\mathcal{H}$ under the identification $\{i, j, k\} \sim \{i, R_T, iR_T\}$. There is an equivalence between a graded quaternionic vector space and a graded action of $C\ell_{4,0}$ on $\mathcal{H} \oplus \mathcal{H}$. Specifically, we take $\mathcal{H} \oplus \mathcal{H}$ with grading $\begin{pmatrix} \text{sgn}(H) & 0 \\ 0 & -\text{sgn}(H) \end{pmatrix}$ and the real span of the Clifford generators

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -R_T \\ R_T & 0 \end{pmatrix}, \begin{pmatrix} 0 & -iR_T \\ iR_T & 0 \end{pmatrix} \right\},$$

Therefore, the subgroup $\{1, T\}$ gives rise to a graded representation of $C\ell_{0,0}$ or $C\ell_{4,0}$.

Symmetry generators	R_C^2	R_T^2	Graded Clifford representation (up to Morita equivalence)
T		$+1$	$C\ell_{0,0}$
C, T	$+1$	$+1$	$C\ell_{1,0}$
C	$+1$		$C\ell_{2,0}$
C, T	$+1$	-1	$C\ell_{3,0}$
T		-1	$C\ell_{4,0}$
C, T	-1	-1	$C\ell_{5,0}$
C	-1		$C\ell_{6,0}$
C, T	-1	$+1$	$C\ell_{7,0}$
N/A			$\mathbb{C}\ell_0$
S	$R_S^2 = 1$		$\mathbb{C}\ell_1$

Table 1: Symmetry types and their corresponding graded Clifford representations [27, Table 1].

Because we are interested in the link between Clifford representations and K-theory, we may choose representations up to Morita equivalence, where $C\ell_{r,s} \sim C\ell_{r+1,s+1}$ or real Clifford algebras and $\mathbb{C}\ell_n \sim \mathbb{C}\ell_{n+2}$ for complex algebras.

The algebra generated by $\{R_C, iR_C, iR_{CT}\}$ give rise to four different Clifford algebras determined by the sign of R_C^2 and R_T^2. The graded PUA representations of $\{1, C\}$ and $\{1, T\}$ give $2 + 2$ Clifford algebras determined by R_C^2 and R_T^2. Graded representations of $\{1, S\}$ correspond to the Clifford algebra $\text{span}_{\mathbb{C}}\{R_S\} \cong \mathbb{C}\ell_1$, which is the same whether $R_S^2 = \pm 1$ (again, these representations come with the grading $\text{sgn}(H)$). In total, we have nine possible representations of symmetry subgroups as *distinct* Clifford algebras and a lack of any symmetry gives us one more possibility. This equivalence between Hamiltonians symmetric with respect to a subgroup of $\{1, C, T, CT\}$ and the 10 distinct Clifford algebras has been named the '10-fold way'.

4   Free-fermionic symmetries in Nambu space

In this section, we review the Hartree–Fock–Bogoliubov free-fermionic approximation of many-body ground states using Nambu space, a theoretical framework that is particularly adept for considering mathematical models of (topological) superconductors. Using this framework, we then briefly introduce the fundamental symmetry classes of free-fermions as outlined in [2, 16].

4.1   Nambu space and ground states

We set $\mathcal{V}$ to be the complex Hilbert space of fermions and its dual $\mathcal{V}^*$ to be the space of holes. The sum $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*$ is a complex Hilbert space of the 'doubled' system. So as not to introduce artificial degress of freedom into the system, the Nambu space $\mathcal{V} \oplus \mathcal{V}^*$ comes with a Real structure $\Gamma = \begin{pmatrix} 0 & \mathfrak{R}^{-1} \\ \mathfrak{R} & 0 \end{pmatrix}$, a self-adjoint anti-unitary with $\mathfrak{R} : \mathcal{V} \xrightarrow{\cong} \mathcal{V}^*$ the Riesz isomorphism. For a quantity to be physically relevant, we require that it is well defined on the subspace $\mathcal{H}_{\mathbb{R}} = \{w \in \mathcal{H} : \Gamma w = w\}$, which is a real Hilbert space. We will call any bounded operator on $\mathcal{V} \oplus \mathcal{V}^*$ that commutes with Γ Real (with capitalised R).

Some care must be taken when applying our framework of single-particle quantum mechanical symmetries in the Nambu space setting. Firstly, so as to not introduce fictitious symmetries, we expect any group representation to commute with the Real structure Γ. While our analysis

of symmetries as PUA representations in the previous section is well-suited to representations on a Hilbert space of states, Nambu space is an artificially doubled space of particles and holes. As such, symmetries in Nambu space are made with more direct reference to a given dynamics and Hamiltonian, which then specifies a (many-body) state.

Dynamics and ground states are considered via a Hamiltonian $H = H^*$ acting on $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*$. However, we restrict to self-adjoint operators such that the time e^{itH} is well-defined on the real subspace $\mathcal{H}_\mathbb{R}$. This amounts to the condition that $\Gamma H \Gamma = -H$, which we have already seen as the particle-hole/charge-conjugation symmetry/constraint. We will call such Hamiltonians Bogoliubov–de Gennes (BdG) Hamiltonians. The condition $\Gamma H \Gamma = -H$ implies that the spectrum of H is symmetric about the point 0. In particular, our spectral gap condition is precisely that $0 \notin \sigma(H)$. In such a case, the positive spectral projection $P_{>0} = \chi_{(0,\infty)}(H)$ will define a vacuum (ground) state in the fermionic second quantization, $\bigwedge^* P_{>0}\mathcal{H}$. The ground state specified by $P_{>0}$ and its dynamics provides a reasonable approximation of low-energy properties of superconductors, which are intrinsically many-body systems.

To summarise our construction, the BdG Hamiltonian $\Gamma H \Gamma = -H$ on the Nambu space $\mathcal{H} = \mathcal{V}^* \oplus \mathcal{V}^*$ with $0 \notin \sigma(H)$ defines a fermionic (gapped) ground state. Taking the spectral flattening $\mathrm{sgn}(H)$, the particle-hole symmetry/constraint can be equivalently stated by requiring $J = i\,\mathrm{sgn}(H)$ to commute with Γ. Namely, J is Real and so restricts to a skew-adjoint orthogonal operator on the real subspace $\mathcal{H}_\mathbb{R}$.

4.2 Free-fermionic symmetries

Returning to symmetries of our system. We expect any unitary/anti-unitary operator R on $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*$ that is compatible with the Hamiltonian H to be such that it extends to a unitary/anti-unitary on the fermionic second quantization, $\bigwedge^* P_{>0}\mathcal{H}$ that leaves the vacuum invariant. A sufficient condition for this to occur is that R commutes with H and Γ. This is in contrast to the case of commuting and anti-commuting operators in the single particle Hilbert space setting.

In the paper [2], Altland and Zirbauer showed that fundamental (physical) symmetries of the Nambu space can be classified into 10 distinct symmetry classes, which can be distinguished by the Cartan label of certain symmetric spaces. See also [16]. Such symmetries include time-reversal, charge, particle-hole and spin symmetries. Briefly, the time-reversal symmetry, particle-hole and spin symmetries are first defined on space of particles $\mathcal{V}$. These operators are then extended to unitary or anti-unitary operators on the Nambu space $\mathcal{V} \oplus \mathcal{V}^*$ that square to $\pm\mathbf{1}$ and commute with both the BdG Hamiltonian and Γ. The Charge symmetry operator assigns a charge of 1 (resp. -1) to particles (resp. holes). BdG Hamiltonians with charge symmetry must act as a block matrix on $\mathcal{V} \oplus \mathcal{V}^*$ and so the Hamiltonian can be determined by its restriction to $\mathcal{V}$, taking us back to the more standard picture of complex Hilbert spaces. A basic summary is provided in Tables 2 and 3. While the picture of symmetries presented on Nambu space is quite different to the CT-symmetries considered in Section 3.2, the two presenations are compatible. See [1, 19] for more details.

The Hamiltonian and ground state can be equivalently characterised by the skew-adjoint unitary $J = i\,\mathrm{sgn}(H)$ that commutes with the real structure Γ on $\mathcal{V} \oplus \mathcal{V}^*$. A key result by Kennedy and Zirnbauer [19] is the following: there is a one-to-one equivalence of physical symmetry operators commuting with the BdG Hamiltonian with an ungraded representation of the Clifford algebra $C\ell_{0,n}$ on $\mathcal{V} \oplus \mathcal{V}^*$ whose generators $\{\kappa_j\}_{j=1}^n$ commute with the Real structure Γ and anti-commute with the skew-adjoint unitary J. If the BdG Hamiltonian has complex symmetry type, then the Clifford representation is $\mathbb{C}\ell_n$ on $\mathcal{V}$. The construction is similar to the case of CT-symmetries in Section 3.2, where the generators of the Clifford algebras are

n	class	symmetries	comments
0	D	none	
1	DIII	T	time-reversal
2	AII	T, Q	charge
3	CII	T, Q, C	particle-hole
4	C	S_1, S_2, S_3	spin rotations
5	CI	S_1, S_2, S_3, T	
6	AI	S_1, S_2, S_3, T, Q	
7	BDI	S_1, S_2, S_3, T, Q, C	

Table 2: Real symmetry classes

n	class	symmetries	comments
0	A	Q	charge
1	AIII	Q, C	particle-hole

Table 3: Complex symmetry classes

constructed from the symmetry operators.

5   Rough classification: symmetric ground states and the Atiyah–Bott–Shapiro construction

Working in Nambu space $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*$ with Real structure Γ, we can describe free-fermionic and symmetric ground states as a skew adjoint Real unitary J acting on $\mathcal{H}$ that anti-commutes with the generators of an ungraded and Real representation of $C\ell_{0,n}$ or $\mathbb{C}\ell_n$, where n depends on the symmetry. We can therefore compare different symmetric ground states with the aim of finding a topological classification. As was emphasised in the introduction, our classification will rely on various choices and different choices will lead to different topological invariants. In this section, our aim is to give a rough (0-dimensional) classification by considering only the symmetries of the fermionic ground states and without further information of the system. Because of the richer structure that arises, we will focus on the case of real symmetries (cf. Table 2). The complex case can be done via analogous arguments.

5.1   Classification principles

Given a fixed BdG Hamiltonian H that is invertible, the ground state that it determines in the second quantisation will be pure. Because any two pure ground states are either unitarily equivalent or disjoint, we may for example ask for a classifcation of BdG Hamiltonians where H_0 is deemed equivalent to H_1 if the corresponding ground states are unitarily equivalent. Certainly this will give a classification of free-fermionic ground states. However, if $\mathcal{V} \oplus \mathcal{V}^*$ is infinite-dimensional, then there are uncountably many inequivalent pure ground states and such a classification is therefore of limited value. Similarly, any two Real skew-adjoint unitaries J_0 and J_1 on $\mathcal{H}$ are unitarily equivalent via a unitary that commutes with Γ.

Instead we will consider the following setting: (gapped) BdG Hamiltonians H_0 and H_1 of the same free-fermionic symmetry type whose ground are equivalent, but where the ground states might not be connected in a way that respects the free-fermionic symmetries. In the case of symmetry class D, where H_0 and H_1 have no symmetries, we instead ask whether the ground

n	symmetry class	$KO_{n+2}(\mathbb{R})$
0	D	$\mathbb{Z}_2$
1	DIII	0
2	AII	$2\mathbb{Z}$
3	CII	0
4	C	0
5	CI	0
6	AI	$\mathbb{Z}$
7	BDI	$\mathbb{Z}_2$

Table 4: K-theory groups of the real C^*-algebra $\mathbb{R}$.

states can be connected in a way that respects the fermionic parity (particle number modulo 2). We will clarify what we mean by 'connected' below.

5.2 Clifford modules and the Atiyah–Bott–Shapiro construction

We now suppose that H_0 and H_1 are gapped BdG Hamiltonians of the same symmetry type and such that H_0 and H_1 give equivalent ground states in the second quantization. Because the ground states are equivalent, by the Shale–Stinespring Theorem, the Real skew-adjoint unitaries $J_0 = i\,\mathrm{sgn}(H_0)$ and $J_1 = i\,\mathrm{sgn}(H_1)$ are such that $J_0 - J_1$ is a Hilbert–Schmidt operator. Using this fact, one can conclude that space $\mathrm{Ker}(J_0 + J_1)$ is finite-dimensional. Because J_0 and J_1 are Real, the space $\mathrm{Ker}(J_0 + J_1)$ can be considered as a real vector subspace of $\mathcal{H}_{\mathbb{R}}$, the elements in $\mathcal{H}$ fixed by the Real structure Γ. Furthermore, because H_0 and H_1 are of the same symmetry type, there are Clifford generators $\{\kappa_j\}_{j=1}^n$ such that $J_0\kappa_j = -\kappa_j J_0$ and $J_1\kappa_j = -\kappa_j J_1$ for all $j \in \{1, \ldots, n\}$. Therefore, the finite-dimensional space $\mathrm{Ker}(J_0 + J_1)$ will be invariant by the action of these Clifford generators. Furthermore, noting the simple relation

$$J_1(J_1 + J_0) = (J_0 + J_1)J_0,$$

we see that J_0 will also leave $\mathrm{Ker}(J_0 + J_1)$ invariant. Therefore the ungraded Clifford generators $\{J_0, \kappa_1, \ldots, \kappa_n\}$ act on the finite-dimensional real vector space $\mathrm{Ker}(J_0 + J_1)$. Namely, $\mathrm{Ker}(J_0 + J_1)$ is an ungraded $C\ell_{0,n+1}$-module.

Letting $\mathcal{M}_n$ denote the Grothendieck group of equivalence classes of ungraded $C\ell_{0,n}$-modules, $\mathrm{Ker}(J_0 + J_1)$ determines an element in $\mathcal{M}_{n+1}$. We can ask if this $C\ell_{0,n+1}$ Clifford module extends to a $C\ell_{0,n+2}$ Clifford module (e.g. via an action of J_1). If such an extension does occur, then $\mathrm{Ker}(J_0 + J_1)$ is more naturally considered as a $C\ell_{0,n+2}$-module and we say that such Clifford modules are trivial. Hence we obtain an equivalence class

$$[\mathrm{Ker}(J_0 + J_1)] \in \mathcal{M}_{n+1}/\mathcal{M}_{n+2} \cong KO_{n+2}(\mathbb{R}),$$

where the isomorphism $\mathcal{M}_{n+1}/\mathcal{M}_{n+2} \cong KO_{n+2}(\mathbb{R})$ is due to Atiyah–Bott–Shapiro [3, Theorem 11.5]. Note that we are considering KO-theory of the real C^*-algebra $\mathbb{R}$. One can identify $KO_{n+2}(\mathbb{R}) \cong KO^{-n-2}(\mathrm{pt})$ with the right-hand side topological K-theory of spaces. The groups $KO_{n+2}(\mathbb{R})$ are given in Table 4.

Our interpretation of the index $[\mathrm{Ker}(J_0 + J_1)] \in KO_{n+2}(\mathbb{R})$ is as a *topological obstruction* to connect the symmetric gapped ground state that comes from H_0 to the symmetric gapped ground states that comes from H_1. Namely, if there is a path of invertible BdG Hamiltonians $\{H_t\}_{t \in [0,1]}$ that connects H_0 and H_1 and respects the symmetry class for all $t \in [0, 1]$, then the K-theoretic index $[\mathrm{Ker}(J_0 + J_1)]$ must be trivial in $KO_{n+2}(\mathbb{R})$. To state the contrapositive, a

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non-trivial index $[\mathrm{Ker}(J_0 + J_1)] \in KO_{n+2}(\mathbb{R})$ implies that the two BdG Hamiltonians H_0 and H_1 can not be connected by a symmetric and invertible path. Namely, there can not be a path of symmetric free-fermionic ground states that joins the two systems.

We close this section by commenting on a few shortcomings of the index $[\mathrm{Ker}(J_0 + J_1)] \cong KO_{n+2}(\mathbb{R})$. While the index can take non-trivial values depending on the symmetry, it is insensitive to other important information such as the dimension of the system or 'local' properties of the BdG Hamiltonian. Indeed, we have imposed no assumptions on the BdG Hamiltonians apart from the existence of a spectral gap at 0. As such, the 'rough index' $[\mathrm{Ker}(J_0 + J_1)]$ can not be used to extract information such as Chern classes as were considered in Section 2.1. To access this more refined information, we must impose additional assumptions on the Hamiltonians and change our notion of equivalence.

6 Refined classification: local Hamiltonians and van Daele K-theory

6.1 Setting and assumptions

We have now seen how certain quantum mechanical symmetries are closely linked to Clifford algebras and defined a K-theoretic obstruction between different gapped BdG Hamiltonians using the Atiyah–Bott–Shapiro construction. We now return to a question posed in the introduction, homotopy in what?

For this section, our operating hypothesis is that within the operators that act on the Nambu space $\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*$, there is a unital C^*-algebra of observables $\mathrm{Mult}(A) \subset \mathcal{B}(\mathcal{H})$ and all BdG Hamiltonians of interest are affiliated to this C^*-algebra. The algebra $\mathrm{Mult}(A)$ is a maximal unitisation of the C^*-algebra A (if A is unital, then $\mathrm{Mult}(A) = A$). We furthermore require compatibility of the Real structure Γ on $\mathcal{H}$ with the algebra A. Namely, $\Gamma \in \mathrm{Mult}(A)$, which then implies that $\mathrm{Ad}_\Gamma(A) \subset A$. A complex C^*-algebra A with a order-2 anti-linear $*$-automorphism $a \mapsto a^{\mathfrak{r}}$ is called a $C^{*,\mathfrak{r}}$-algebra or Real C^*-algebra (with capitalized R). In our setting, we will take $a^{\mathfrak{r}} = \mathrm{Ad}_\Gamma(a)$. The subalgebra $A^{\mathfrak{r}} = \{a \in A : a^{\mathfrak{r}} = a\}$ is a real C^*-algebra (with lower case r), that is, a C^*-algebra over the field $\mathbb{R}$.

- If our Nambu space $\mathcal{H} = \ell^2(\mathbb{Z}^d, \mathbb{C}^{2n})$ and Hamiltonians of interest are constructed as matrices of finite polynomials of the discrete shift operators $\{S_m\}_{m \in \mathbb{Z}^d}$, then a natural choice of observable C^*-algebra A is $C^*(\mathbb{Z}^d)$ (suppressing the matrix degrees of freedom). We have already seen in Section 2.1 that we can detect more refined topological invariants such as Chern classes by studying the K-theory of $C^*(\mathbb{Z}^d)$.

- More generally, for a given dimension d, we take our C^*-algebra observables A to be a separable sub-algebra of the Roe C^*-algebra $C^*_{\mathrm{Roe}}(\mathbb{Z}^d) \cong C^*_{\mathrm{Roe}}(\mathbb{R}^d)$, where we are considering $\mathbb{Z}^d$ and $\mathbb{R}^d$ as metric spaces (in particular, elements in this algebra need not be translation invariant). The Roe algebra can be used to model a very large class of Hamiltonians that are 'local' with respect to the underlying metric space. For more information on the application of the Roe algebra in condensed matter physics, see [22, 11, 1] for example.

If a BdG Hamiltonian H is invertible and affiliated to $\mathrm{Mult}(A)$ is invertible, then $J = i\,\mathrm{sgn}(H) \in \mathrm{Mult}(A)$ and $J^{\mathfrak{r}} = J$, so J is an element of the real sub-algebra $\mathrm{Mult}(A)^{\mathfrak{r}}$. If the BdG Hamiltonian also has free-fermionic symmetries (time-reversal, charge, spin, . . .), we also assume that these symmetry operators are elements of $\mathrm{Mult}(A)$. This will then imply that the ungraded Clifford generators $\{\kappa_j\}_{j=1}^n$ that anti-commute with J are also contained in $\mathrm{Mult}(A)^{\mathfrak{r}}$.

6.2 van Daele K-theory and a local K-theoretic obstruction

Our aim is to construct a topological index for pairs of gapped BdG Hamiltonians H_0, $H_1 \in \text{Mult}(A)$ of the same symmetry type. In Section 5 we worked under the assumption that the Real skew-adjoint unitaries J_0 and J_1 were such that $J_0 - J_1$ is a Hilbert–Schmidt operator. We now weaken this assumption to instead require that $J_0 - J_1 \in A$. Namely, the algebra A provides the space of possible deformations that we can consider for a given BdG Hamiltonian. The condition $J_0 - J_1 \in A$ does not necessarily imply that $\text{Ker}(J_0 + J_1)$ is finite-dimensional as in Section 5. Instead, we will define a topological index by looking at stable homotopy classes of Real skew-adjoint unitaries J that anti-commute with $\{\kappa_j\}_{j=1}^n$. If we restrict our space of possible stable homotopies to that of a fixed C^*-algebra A, then one may consider the operator algebraic K-theory of A. A presentation of operator algebraic K-theory that is particularly amenable for the setting under consideration is a due to van Daele [9, 10], who defines the group $DK(B)$ of a $\mathbb{Z}_2$-graded C^*-algebra B via stable homotopy classes of odd self-adjoint unitaries. See also [24, 7].

We can easily build an odd self-adjoint unitary in $\text{Mult}(A) \otimes C\ell_{0,1}$ from a real skew-adjoint unitary $J \in \text{Mult}(A)$ by the map $J \mapsto J \otimes \rho$ with ρ the odd generator of $C\ell_{0,1}$. As such, from a pair of skew-adjoint unitaries J_0 and J_1 that anti-commute with the ungraded Clifford generators $\{\kappa_j\}_{j=1}^n$ and with the property $J_0 - J_1 \in A$, we can associate the odd self-adjoint unitaries $J_0 \otimes \rho$ and $J_1 \otimes \rho$ that anti-commute with the generators $\{\kappa_j \otimes \rho\}_{j=1}^n$ of a $\mathbb{Z}_2$-graded representation of $C\ell_{n,0}$. Such a setting precisely determines a class

$$\big[J_0 \otimes \rho\big] - \big[J_1 \otimes \rho\big] \in DK_n(A^{\mathfrak{r}} \otimes C\ell_{0,1}) \cong KO_{n+2}(A^{\mathfrak{r}}),$$

where the isomorphism $DK_n(A^{\mathfrak{r}} \otimes C\ell_{0,1}) \cong KO_{n+2}(A^{\mathfrak{r}})$ is detailed in [24, 7, 6]. We can again think of this K-theoretic index as a topological obstruction to connecting the two BdG Hamiltonians H_0 and H_1 via a symmetric path that is local with respect to the C^*-algebra A. Namely, if there is a path of skew-adjoint unitaries $\{J_t\}_{t \in [0,1]} \subset \text{Mult}(A)$ from J_0 to J_1 that anti-commutes with $\{\kappa_j\}_{j=1}^n$ and $J_t - J_0 \in A$ for all $t \in [0,1]$, then $\big[J_0 \otimes \rho\big] - \big[J_1 \otimes \rho\big]$ is trivial in $KO_{n+2}(A^{\mathfrak{r}})$. Hence our index is a topological obstruction to the existence of a symmetric path of gapped free-fermionic Hamiltonians that is 'local' with respect to the auxiliary algebra A.

7 Conclusion

In this short note, we have reviewed a framework to understand symmetries in quantum mechanical systems via $\mathbb{Z}_2$-graded projective unitary/anti-unitary (PUA) representations. In the case of free-fermionic systems modeled via dynamics on Nambu space, we have defined various topological indices that can be associated to symmetric and gapped BdG Hamiltonians. In addition to the symmetry type of system under consideration, the space of allowed perturbations will change the possible range of these topological indices. By fixing a C^*-algebra of observables A, which will then fix our notion of homotopy and equivalence, we can define indices that take value in $KO_{n+2}(A^{\mathfrak{r}})$ or $K_n(A)$. The different symmetry types of BdG Hamiltonians exhaust all 8 real and 2 complex K-theory groups for A.

While this document is titled '*The K-theoretic classification of topological insulators and superconductors*', we have emphasised any general classification is both:

1. Relative, in the sense that our topological indices either encode a topological obstruction between a pair of Hamiltonians or a fixed base point/trivial system.

2. Dependent on the space of Hamiltonians we consider and the allowed paths that one can take to connect different Hamiltonians.

Lastly, let us mention some important topics which we have not covered here.

- We have focused on the symmetries of free-fermions as specified by Altland and Zirnbauer. Using the picture of $\mathbb{Z}_2$-graded PUA representations, a variety of other spatial or internal symmetries may be considered. Of particular interest is the case of systems with translation symmetry coming from a (possibly non-symmorphic and magnetic) space group. Such systems have been treated in detail in [13, 14] for example.

- Throughout our exposition, we have asked for a spectral gap of the Hamiltonian under consideration. In order to model more realistic systems, it would be desirable to extend such topological indices to Hamiltonians with dynamical localisation. That is, the spectral gap is instead filled with almost-sure pure point spectrum. Indeed, disorder and localisation play a fundamental role to explain the plateaus of the Hall conductance in the integer quantum Hall effect. Some progress has been made for understanding topological invariants for strongly disordered systems, see [23, 8, 15, 5] for example, but the picture is still far from complete.

- Another key assumption we have worked under is that the effect of electron-electron interactions is negligible. Indeed, the topological indices defined for free-fermions may break down under perturbations by higher-order interactions [12]. Understanding the topological phases of many-body (gapped) ground states such as the fractional quantum Hall effect moves us into the theme of symmetry protected topological (SPT) phases and topological ordered ground states.

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