Periodic solutions bifurcated from figure eight choreography: non-planar eight and non-symmetric eight

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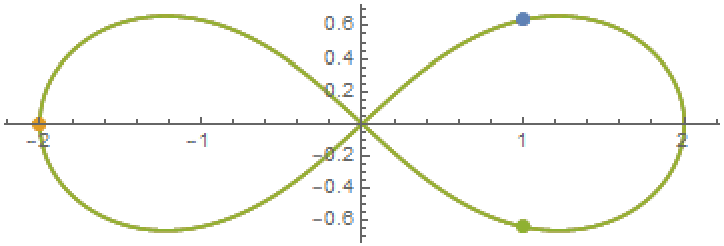
(2023.8.21 15:30) 20min+5min,
#00475 at E (Building 11) 819
ICIAM 2023 TOKYO

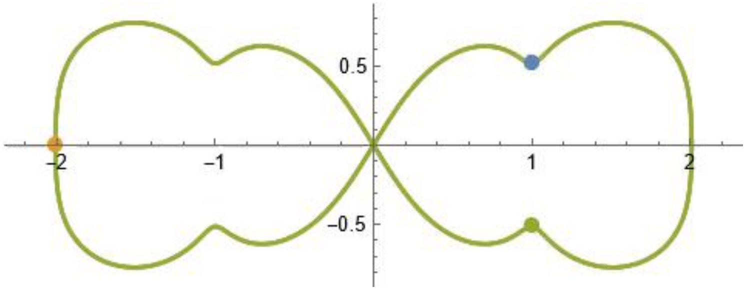
Bifurcations of figure eight choreography

We are investigating bifurcation of the figure eight choreography under homogeneous potential $-\frac{1}{r^a}$ by changing the power a, or Lenard-Jones(LJ) type potential $\frac{1}{r^{12}}-\frac{1}{r^6}$ by period T.



homogeneous potential (a=1)

$$-\frac{1}{r^a}$$

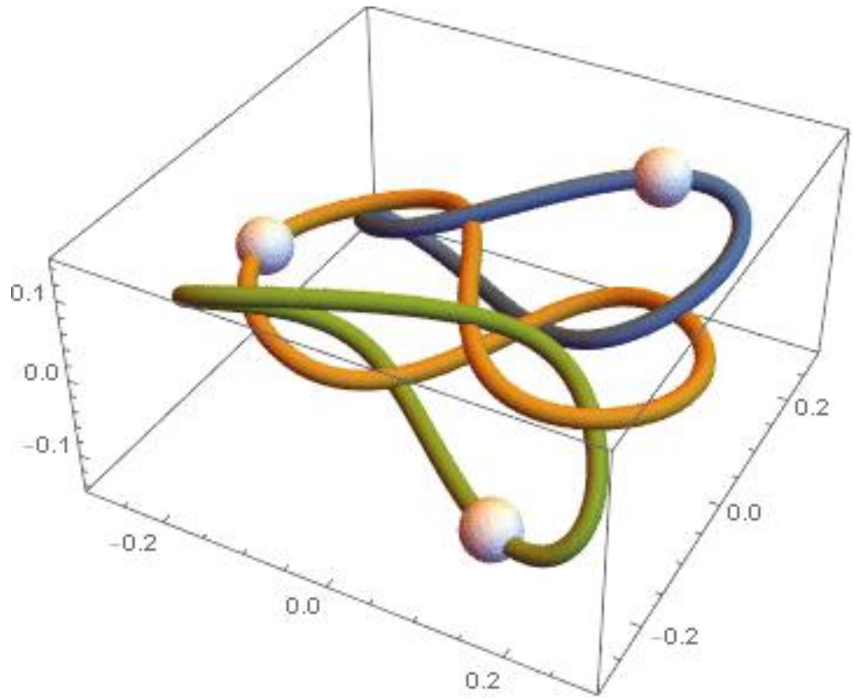


LJ potential (T=50)

$$\frac{1}{r^{12}}-\frac{1}{r^6}$$

Non-planar bifurcation solution

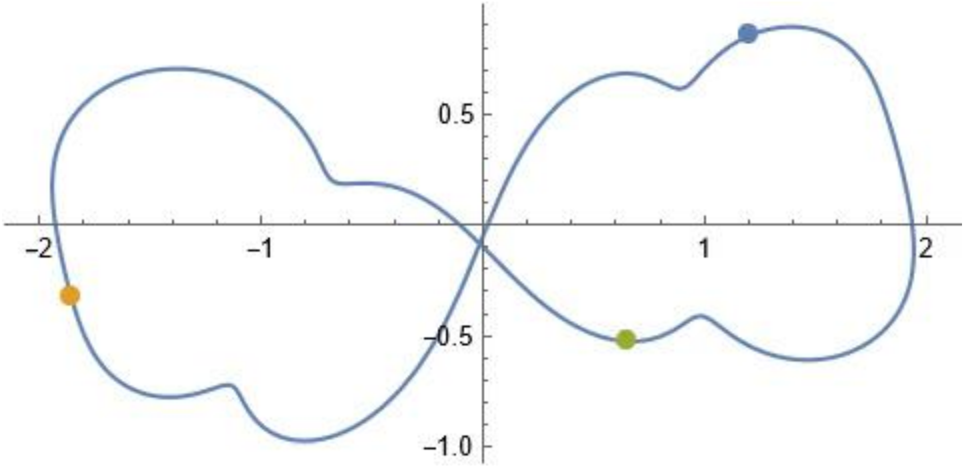
In this talk, we show two interesting bifurcation solutions, a non-planar bifurcation solution under homogeneous system, and



$(a = 0.75)$

Figure-eight with no spatial symmetry

an eight-shaped choreography having no spatial symmetry under Lenard-Jones system.



Contents

My talk consists of the following eight parts:

1. Necessary condition of bifurcation

Necessary condition of bifurcation

At the bifurcation point, eigenvalue λ of linear operator

$$H(q) = -\frac{d^2}{dt^2} + \frac{\partial^2 L}{\partial q^2}$$

crosses zero, where L is Lagrangian for three bodies.

There, the eigenfunction ϕ represents bifurcation solution as

$$q^{(b)} \to q + h\phi$$

where $q^{(b)}$ is the bifurcation solution, q the original solution, and h a real number.

$$L(q, \dot{q}) = \frac{1}{2}\sum_i \dot{q}_i - U(q)$$

$$U(q) = u(r_{12}) + u(r_{23}) + u(r_{31})$$

$$u(r) = -\frac{1}{r^a} \quad \text{or} \quad -\frac{1}{r^6} + \frac{1}{r^{12}}$$

Necessary condition of bifurcation

At the bifurcation point, eigenvalue λ of linear operator

$$H(q) = -\frac{d^2}{dt^2} + \frac{\partial^2 L}{\partial q^2}$$

crosses zero, where L is Lagrangian for three bodies.

There, the eigenfunction ϕ represents bifurcation solution as

$$q^{(b)} \to q + h\phi$$

where $q^{(b)}$ is the bifurcation solution, q the original solution, and h a real number.

Since bifurcation solution $q^{(b)}$ and original solution q satisfy eq. of motion:

$$\frac{d^2}{dt^2} q^{(b)} = \frac{\partial L}{\partial q}\Big|_{q^{(b)}},$$

$$\frac{d^2}{dt^2} q = \frac{\partial L}{\partial q}.$$

Thus, the difference $\phi = q^{(b)} - q$ satisfies

$$\frac{d^2}{dt^2}\phi = \frac{\partial L}{\partial q}\Big|_{q^{(b)}} - \frac{\partial L}{\partial q}$$

$$\to \frac{\partial^2 L}{\partial q^2}\phi$$

2. Symmetry of figure eight choreography

Eigenfunction of $H(q)$

Suppose the operator

$$H(q) = -\frac{d^2}{dt^2} + \frac{\partial^2 L}{\partial q^2}$$

commute with the all elements of the group $G = \{g_1, g_2, \ldots, g_n\}$.

Then, the eigenspace of eigenvalue λ

$$H(q)\phi = \lambda\phi$$

is represented by unitary irreducible representations of G, especially, orthogonal irreducible representations since the eigenfunctions are real.

Symmetry of figure eight choreography

Actually, figure eight choreography $q(t)$ is invariant under the following three operations C, M, S:

- **C: choreographic**

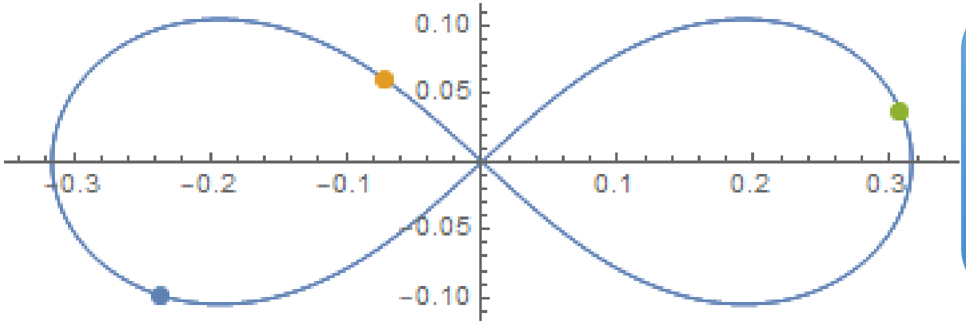
$$Cq(t) = \sigma q(t - \frac{T}{3}),\ \sigma\text{: cyclic permutation of bodies}$$

- M: x inversion with time shift

$$Mq(t) = \mu_x q(t - \frac{T}{2}),\ \mu_x\text{: inversion of } x \text{ coordinates}$$

- S: space and time inversion with exchange two bodies

$$Sq(t) = -\tau q(-t),\ \tau\text{: exchange of body 1 and 2}$$



Choreographic means all bodies move in the same orbit. Thus, time shift T/3 is cyclic permutation of bodies.

Symmetry of figure eight choreography

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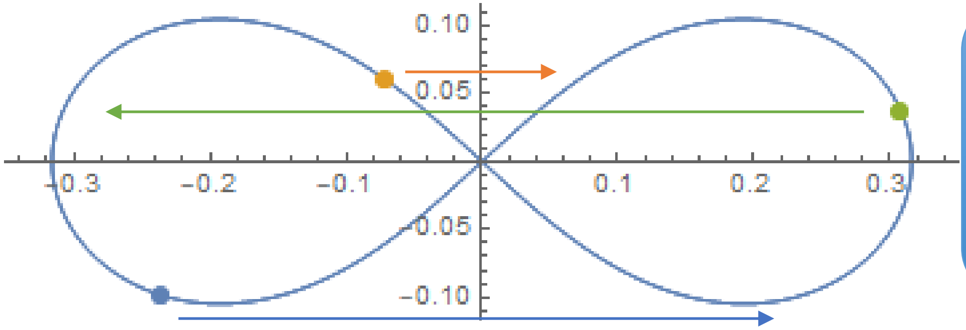

M: x inversion with time shift T/2.

Symmetry of figure eight choreography

Actually, figure eight choreography $q(t)$ is invariant under the following three operations C, M, S:

- C: choreographic

$$Cq(t) = \sigma q(t - \frac{T}{3}),\ \sigma\text{: cyclic permutation of body}$$

- M: x inversion with time shift

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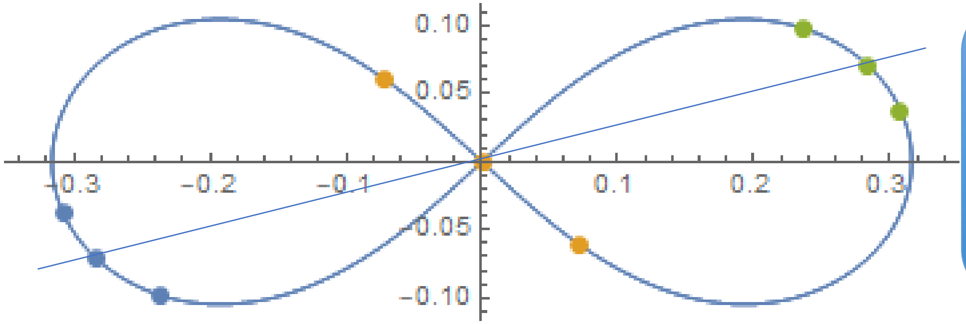
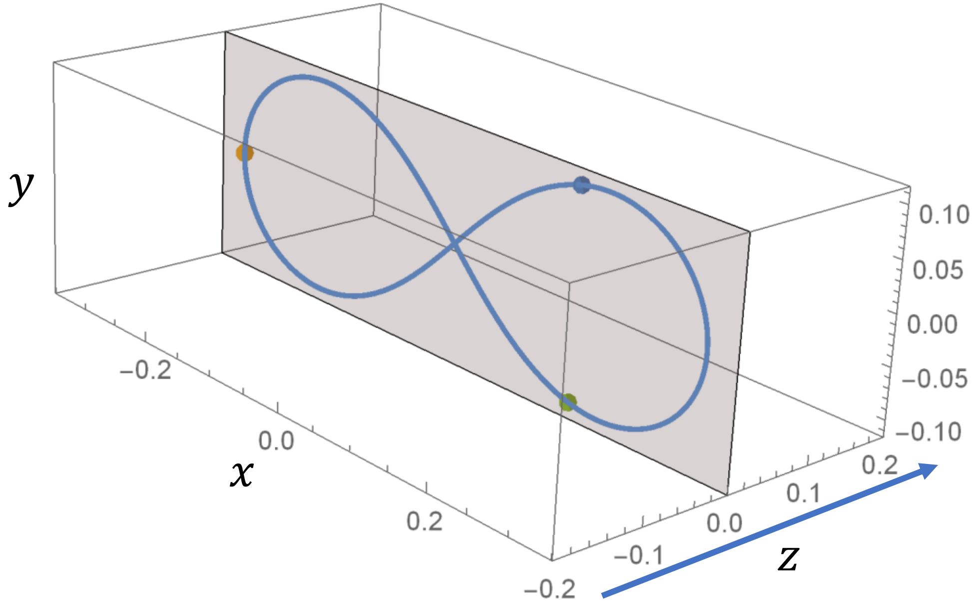


S: space and time inversion with exchange body, since space inversion is time inversion with exchange of two bodies.

Symmetry of figure eight choreography

Moreover, planar motion in x-y plane is invariant under z-inversion μ_z in three dimension.



3. Symmetry group and irreducible representations

Dihedral group D_{6h}

Therefore, $H(q)$ for figure eight choreography q commutes with the four operations

C, M, S and μ_z,

and their combinations (products).

From the algebraic relations

$$C^3 = 1,\ M^2 = 1,\ S^2 = 1,\ \mu_z^2 = 1$$

$$CS = SC^{-1},\ MS = SM,\ CM = MC,$$

$$C\mu_z = \mu_z C,\ S\mu_z = \mu_z S,\ M\mu_z = \mu_z M$$

group generated by $\{C, M, S, \mu_z\}$ is isomorphic to dihedral group D_{6h} with order $|D_{6h}| = 24$.

Irreducible representations of $\boldsymbol{D_{6h}}$

D_{6h} has

8 one dimensional orthogonal irreducible representations and

$(C,\, M,\, S, \mu_z) = (1, \pm 1, \pm 1, \pm 1)$

4 two dimensional orthogonal irreducible representations

$(C,\, M,\, S, \mu_z) = (\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$

Planar and non-planar bifurcation

If representation of μ_z is 1 or unit matrix
⇒ eigenfunctions are symmetric in μ_z, that is, planar. Thus,

4 one dimensional orthogonal irreducible representations and

$$(C,\ M,\ S, \mu_z) = (1, \pm 1, \pm 1, +1)$$

2 two dimensional orthogonal irreducible representations

$$(C,\ M,\ S, \mu_z) = (\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, +\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$$

are planar.

Planar and non-planar bifurcation

The rest half are non-planar:

4 one dimensional orthogonal irreducible representations and

$$(C,\ M,\ S, \mu_z) = (1, \pm 1, \pm 1, -1)$$

2 two dimensional orthogonal irreducible representations

$$(C,\ M,\ S, \mu_z) = (\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \pm\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix}, -\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix})$$

Symmetry of eigenfunction and bifurcation solution

Moreover, symmetry of i'th eigenfunction ϕ_i is also given by the representation:

It is a subgroup $G_e(i)$ consisting of the elements with i'th diagonal element of the representation is one

$$G_e(i) = \{\, g \mid (g)_{ii} = 1\}$$

Then, the symmetry of the bifurcation solution $q^{(b)}$ is given by $G_e(i)$ (Fujiwara et al 2020).

4. Bifurcation of figure eight to the planar motion

4 one-dimensional planar representations

$(1,1,-1,1)$

$(1,-1,1,1)$

$(1,-1,-1,1)$

$(1,1,1,1)$

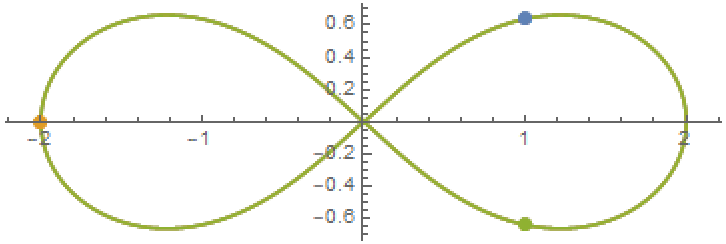
2 two-dimensional planar representations

$$\left(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, +\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

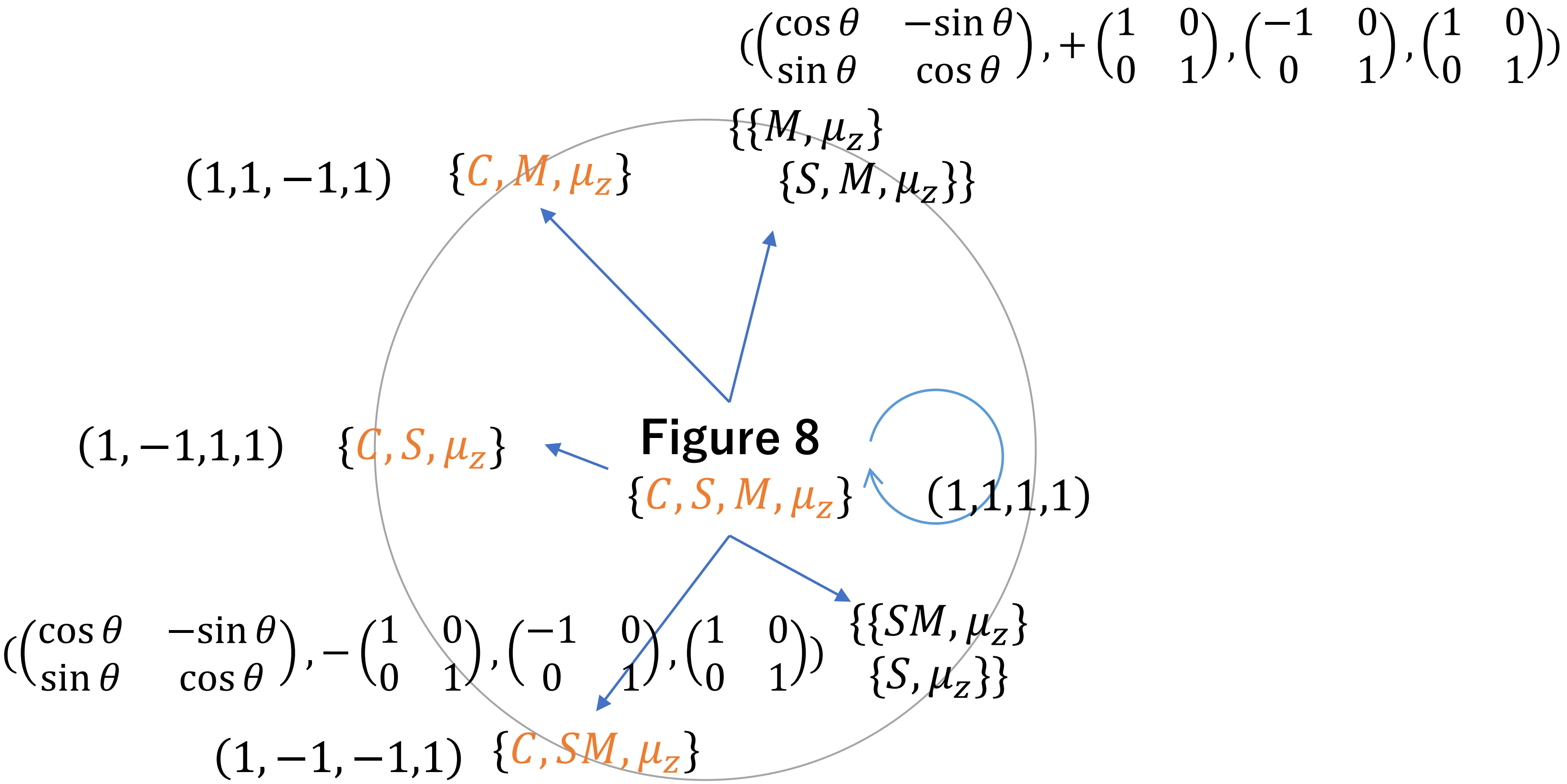
$$\left(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

Figure 8

$\{C, S, M, \mu_z\}$



Thus, the figure 8 choreography bifurcates to planar solutions represented by four one dimensional and two two-dimensional representations.



$$(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, +\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})$$

$\{\{M,\mu_z\}$
$\{S,M,\mu_z\}\}$

$(1,1,-1,1)$ $\{C,M,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$(1,1,1,1)$

$(1,-1,1,1)$ $\{C,S,\mu_z\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$$(\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}, -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix})$$

$(1,-1,-1,1)$ $\{C,SM,\mu_z\}$

In this figure, symmetries are shown in curly parenthesis,

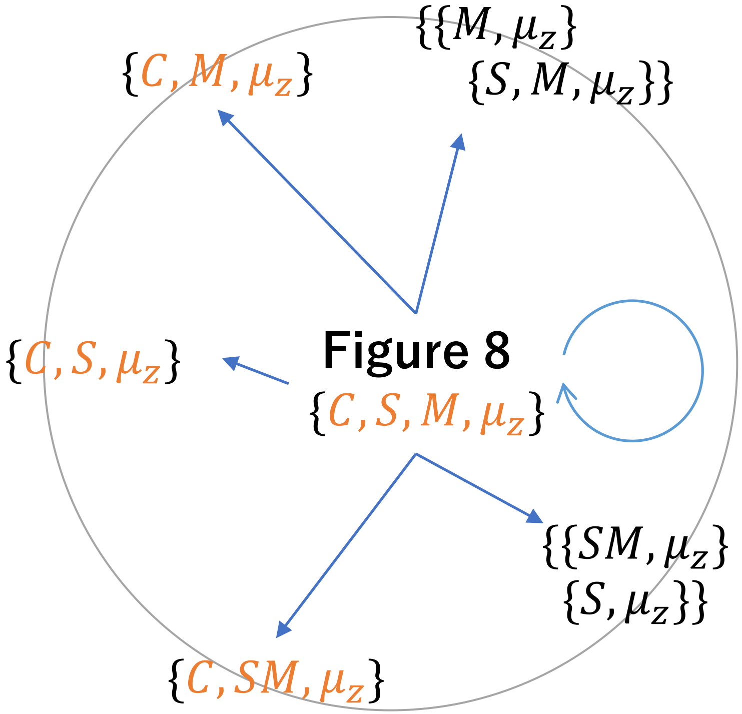


Figure 8

$\{C, S, M, \mu_z\}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}$
$\{S, M, \mu_z\}\}$

$\{C, S, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{C, SM, \mu_z\}$

allows show bifurcations, and choreographies are emphasized by orange.

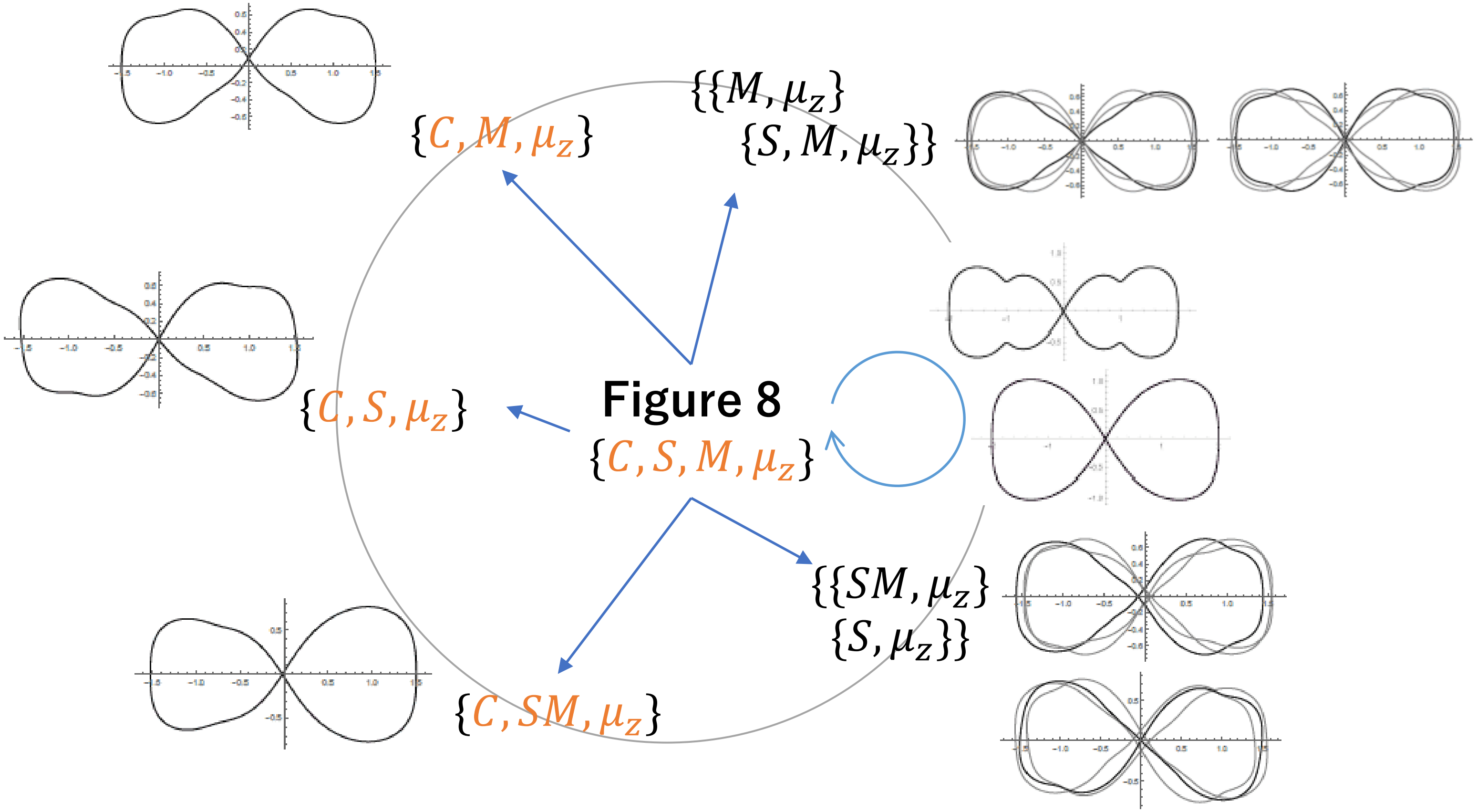


Figure 8
$\{C,S,M,\mu_z\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}$
$\{S,M,\mu_z\}\}$

$\{C,S,\mu_z\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{C,SM,\mu_z\}$

In 2019, we found all these bifurcations numerically in LJ system.

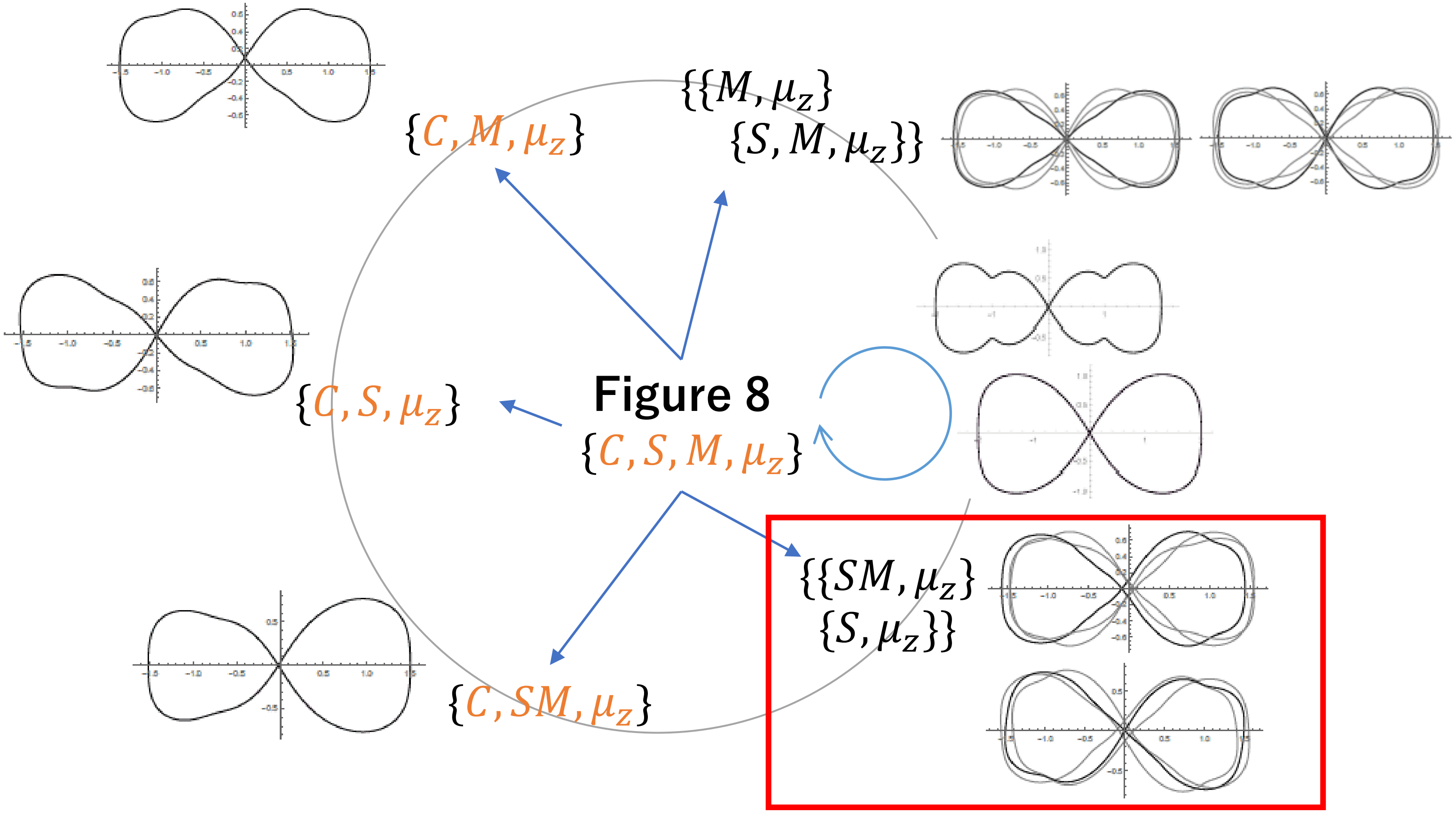


Figure 8
$\{C, S, M, \mu_z\}$

$\{C, M, \mu_z\}$

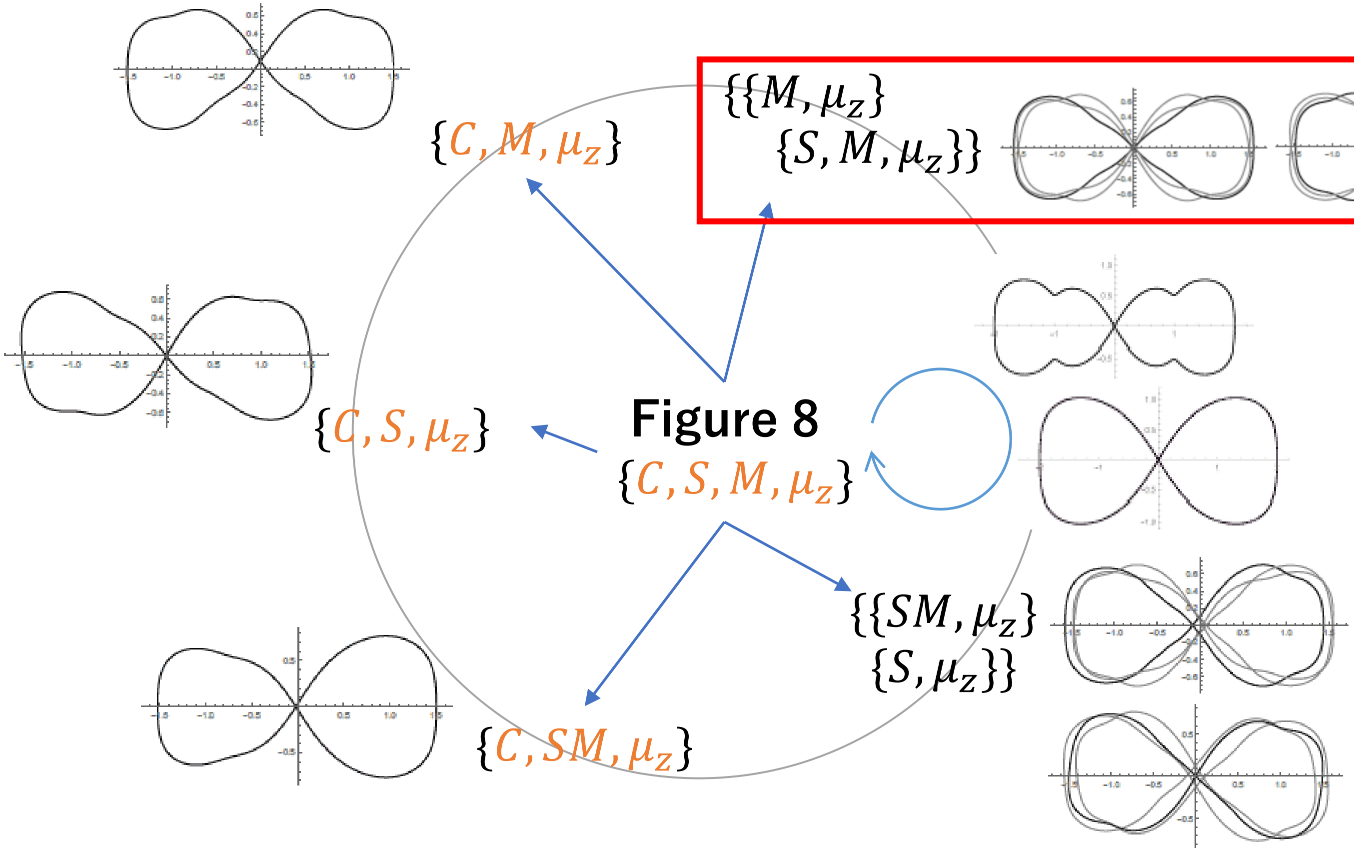
$\{\{M, \mu_z\}$
$\{S, M, \mu_z\}\}$

$\{C, S, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{C, SM, \mu_z\}$

Note that two dimensional representations yield two bifurcations having two symmetries in curly parenthesis like $\{SM, \mu_z\}$ and $\{S, \mu_z\}$.



$\{\{M,\mu_z\}$
$\{S,M,\mu_z\}\}$

$\{C,M,\mu_z\}$

$\{C,S,\mu_z\}$

Figure 8
$\{C,S,M,\mu_z\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{C,SM,\mu_z\}$

However, if one symmetry is subset of the other as $\{M,\mu_z\}$ and $\{S,M,\mu_z\}$, both bifurcations have the same larger symmetry, as $\{S,M,\mu_z\}$.

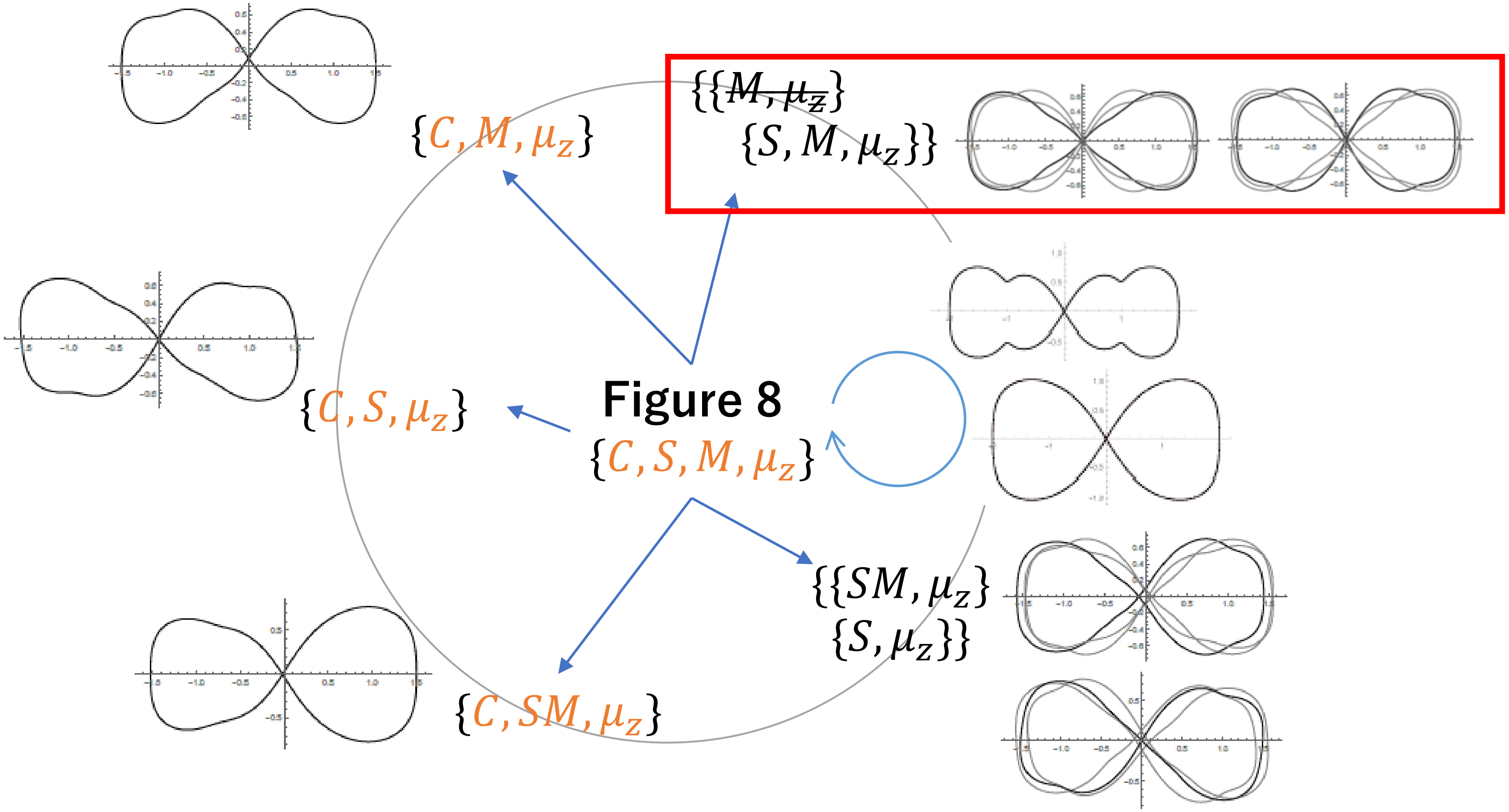


Figure 8

$\{C, S, M, \mu_z\}$

$\{C, M, \mu_z\}$

$\{\{\cancel{M, \mu_z}\}$
$\{S, M, \mu_z\}\}$

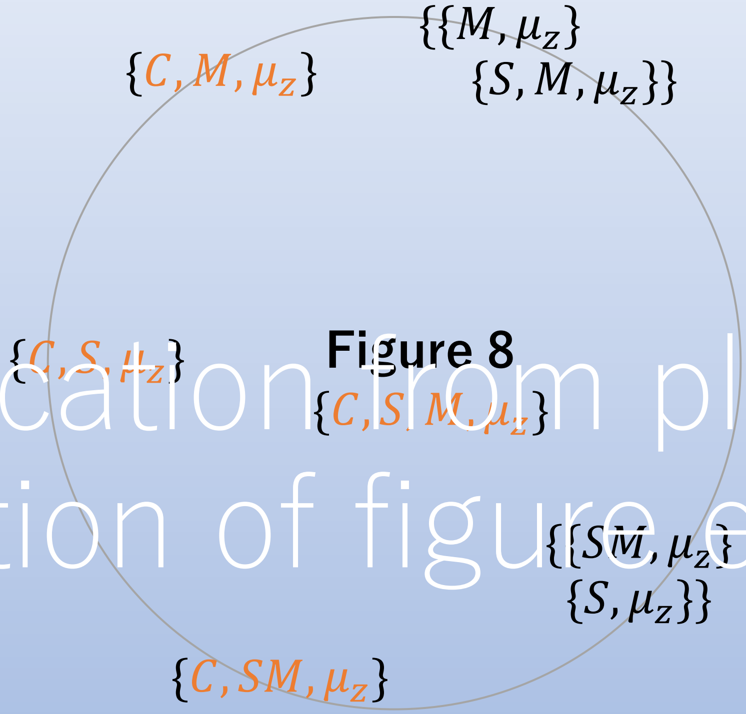
$\{C, S, \mu_z\}$

$\{C, SM, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

Thus, we strike out smaller symmetry.

5. Bifurcation from planar bifurcation of figure eight



$\{\{M, \mu_z\}$
$\{S, M, \mu_z\}\}$

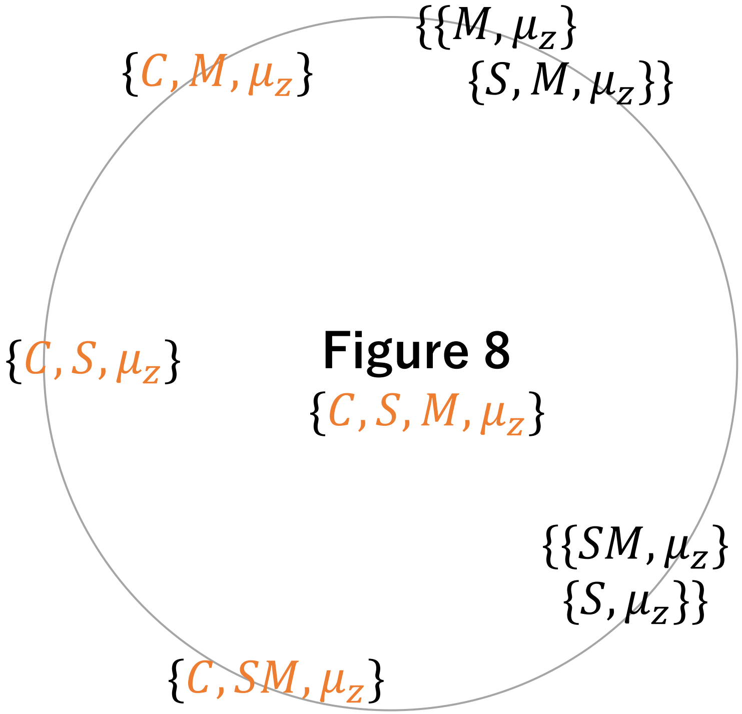
$\{C, M, \mu_z\}$

$\{C, S, \mu_z\}$

Figure 8
$\{C, S, M, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{C, SM, \mu_z\}$



$\{\{M, \mu_z\}$

$\{S, M, \mu_z\}\}$

$\{C, M, \mu_z\}$

$\{C, S, \mu_z\}$

Figure 8

$\{C, S, M, \mu_z\}$

$\{\{SM, \mu_z\}$

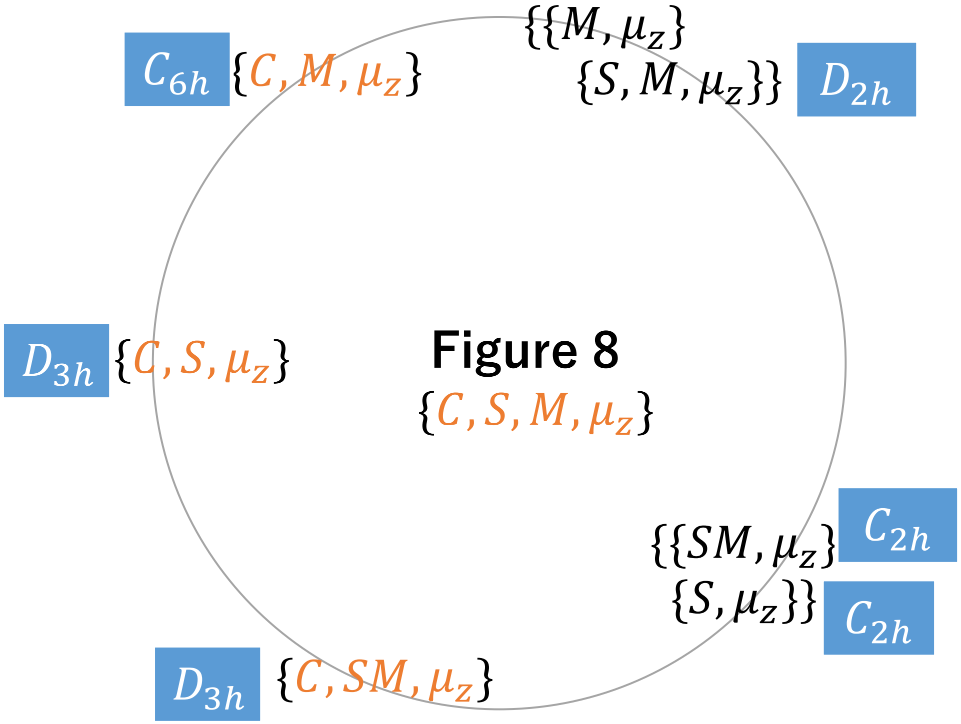
$\{S, \mu_z\}\}$

$\{C, SM, \mu_z\}$

Next, we consider the bifurcation from these bifurcations. We first consider the symmetry group for each solutions,

Figure 8

$\{C, S, M, \mu_z\}$



C_{6h} $\{C, M, \mu_z\}$

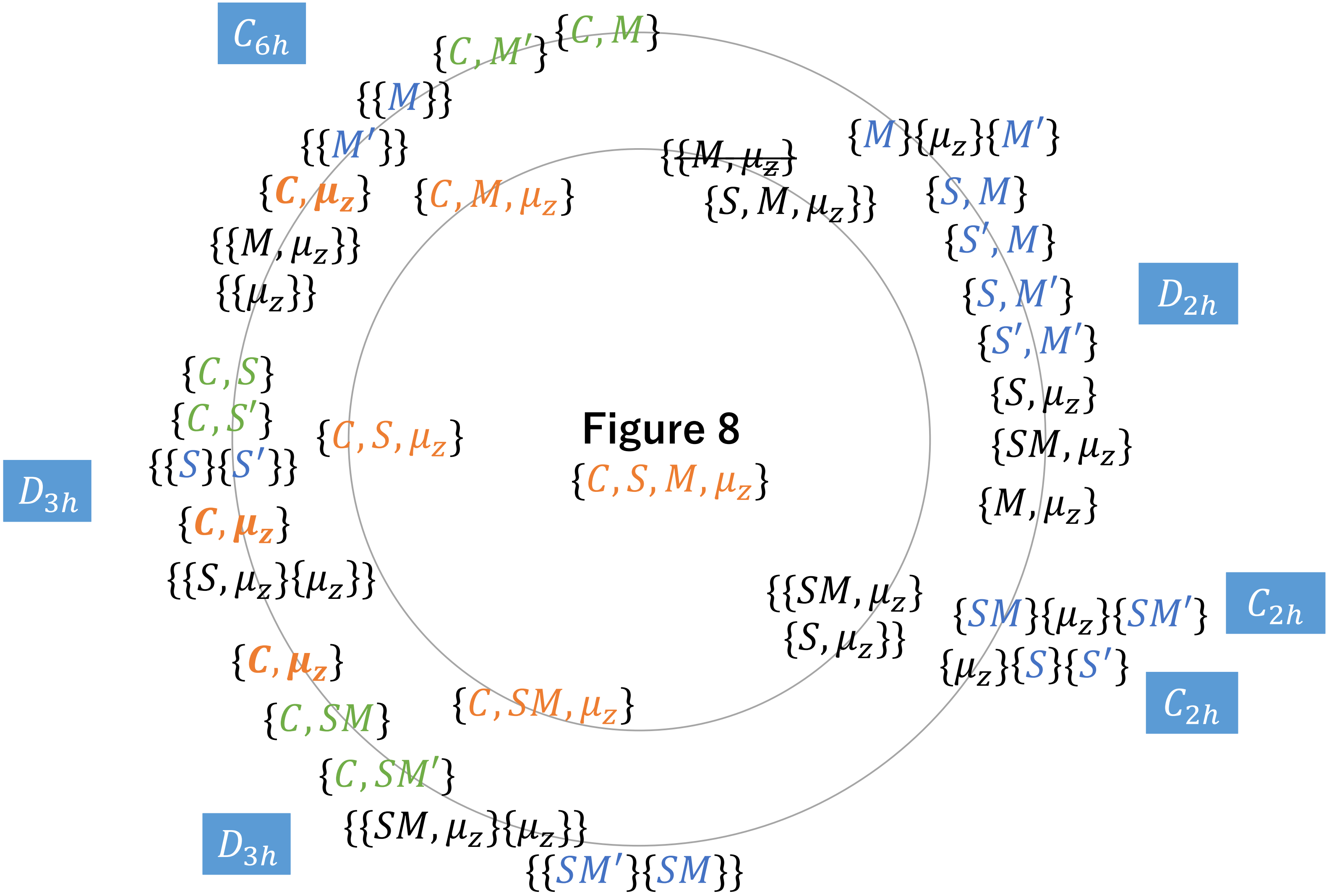
$\{\{M, \mu_z\}$
$\{S, M, \mu_z\}\}$ D_{2h}

D_{3h} $\{C, S, \mu_z\}$

$\{\{SM, \mu_z\}$ C_{2h}
$\{S, \mu_z\}\}$ C_{2h}

D_{3h} $\{C, SM, \mu_z\}$

From their orthogonal irreducible representations,



C_{6h}

$\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

$\{C,M,\mu_z\}$

$\{$~~$\{M,\mu_z\}$~~

$\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{S,M\}$

$\{S',M\}$

$\{S,M'\}$

D_{2h}

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

$\{C,S\}$

$\{C,S'\}$

$\{\{S\}\{S'\}\}$

D_{3h}

$\{C,S,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

C_{2h}

$\{\mu_z\}\{S\}\{S'\}$

C_{2h}

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$

$\{C,SM,\mu_z\}$

$\{C,SM'\}$

D_{3h}

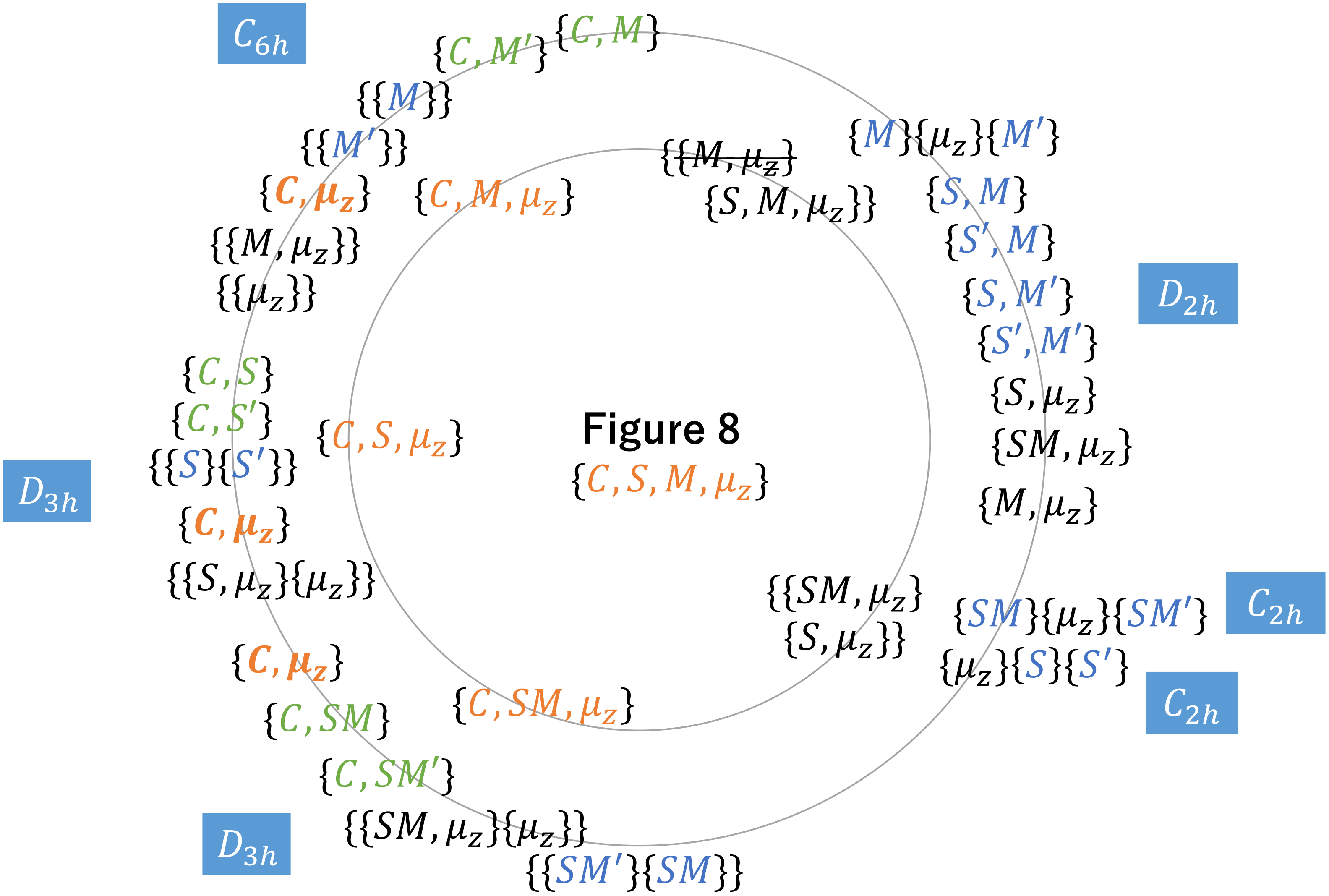
$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

we obtain bifurcation from bifurcations, shown on the second ring.

Figure 8

$\{C, S, M, \mu_z\}$



C_{6h}

$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C, \mu_z\}}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

{~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{S, M\}$

$\{S', M\}$

D_{2h}

$\{S, M'\}$

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

$\{C, S\}$

$\{C, S'\}$

$\{\{S\}\{S'\}\}$

D_{3h}

$\{C, S, \mu_z\}$

$\boldsymbol{\{C, \mu_z\}}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$ C_{2h}

$\{\mu_z\}\{S\}\{S'\}$ C_{2h}

$\boldsymbol{\{C, \mu_z\}}$

$\{C, SM\}$

$\{C, SM, \mu_z\}$

$\{C, SM'\}$

D_{3h}

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

Here, a non-planar choreographies are colored by green, non-planar by blue, and prime is μ_z.

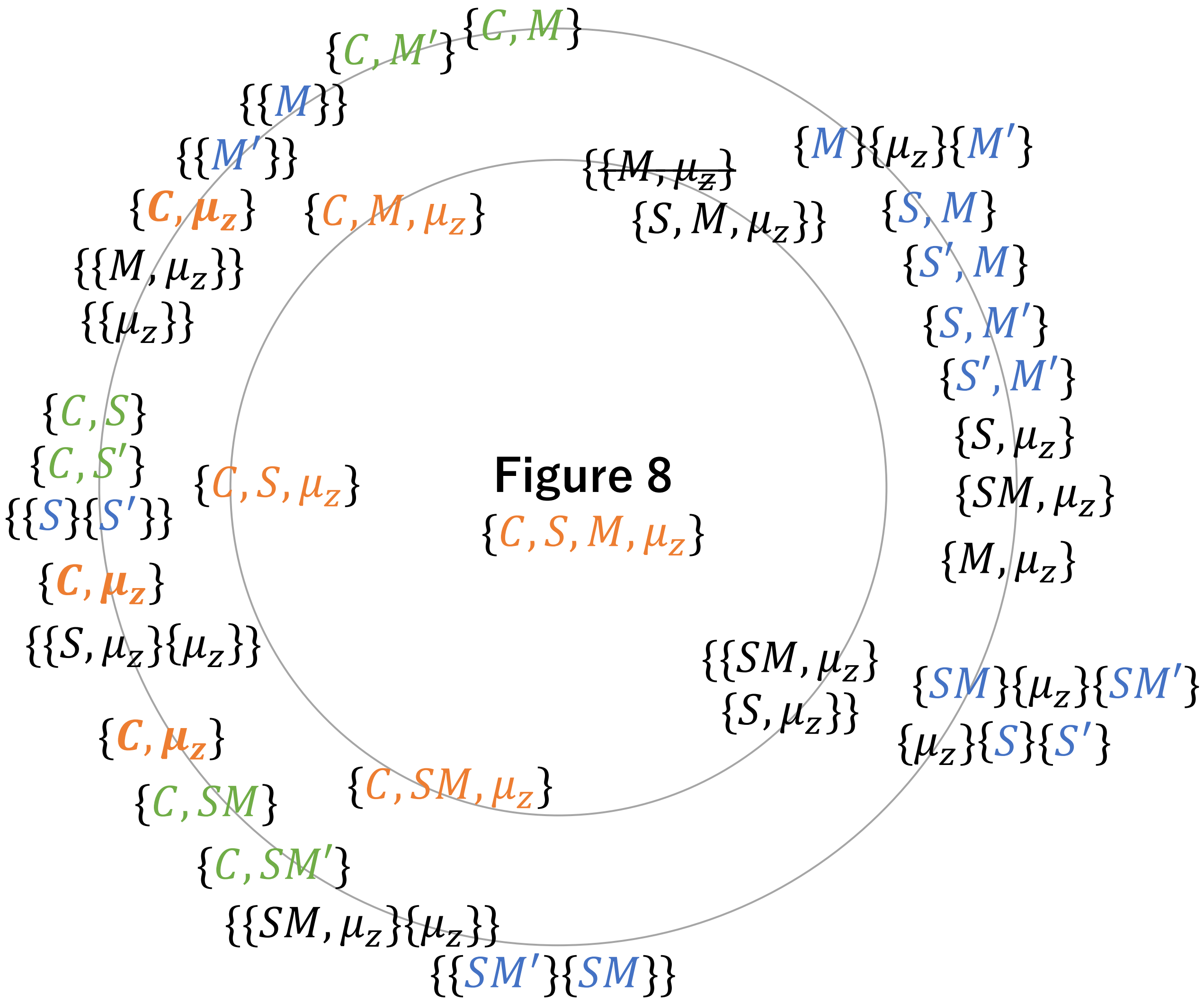


Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M\}$

$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

$\{C,S\}$

$\{C,S'\}$

$\{\{S\}\{S'\}\}$

$\{C,S,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$

$\{C,SM,\mu_z\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}\{S\}\{S'\}$

$\{M,\mu_z\}$

$\{SM,\mu_z\}$

$\{S,\mu_z\}$

$\{S',M'\}$

$\{S,M'\}$

$\{S',M\}$

$\{S,M\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{\cancel{\{M,\mu_z\}}$
$\{S,M,\mu_z\}\}$

However, several of them can not be solutions because of angular momentum restrictions.

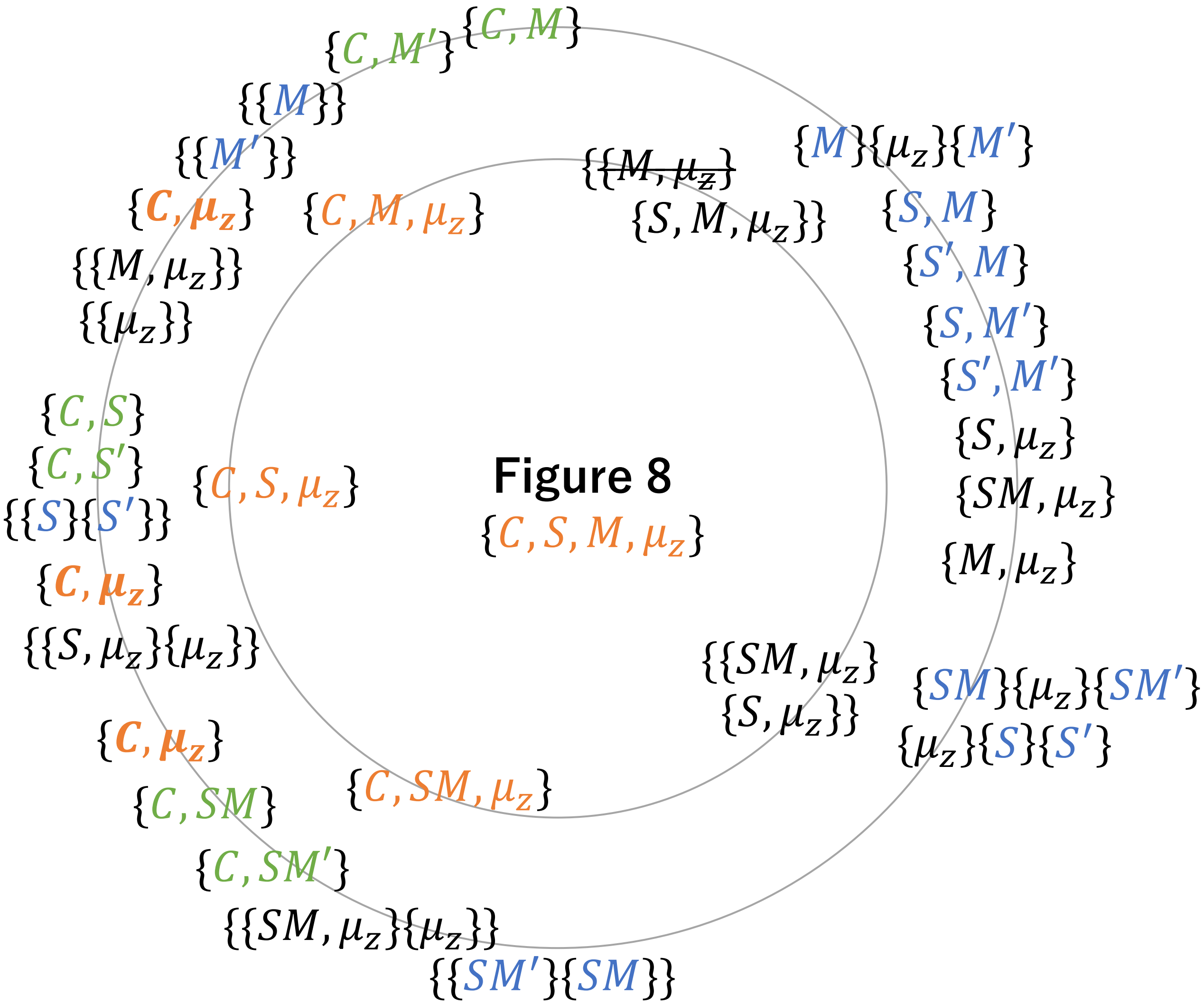


Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

$\{C,S\}$

$\{C,S'\}$

$\{\{S\}\{S'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{C,SM\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{C,M,\mu_z\}$

$\{$~~$\{M,\mu_z\}$~~

$\{S,M,\mu_z\}\}$

$\{C,S,\mu_z\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$

$\{C,SM,\mu_z\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{S,M\}$

$\{S',M\}$

$\{S,M'\}$

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

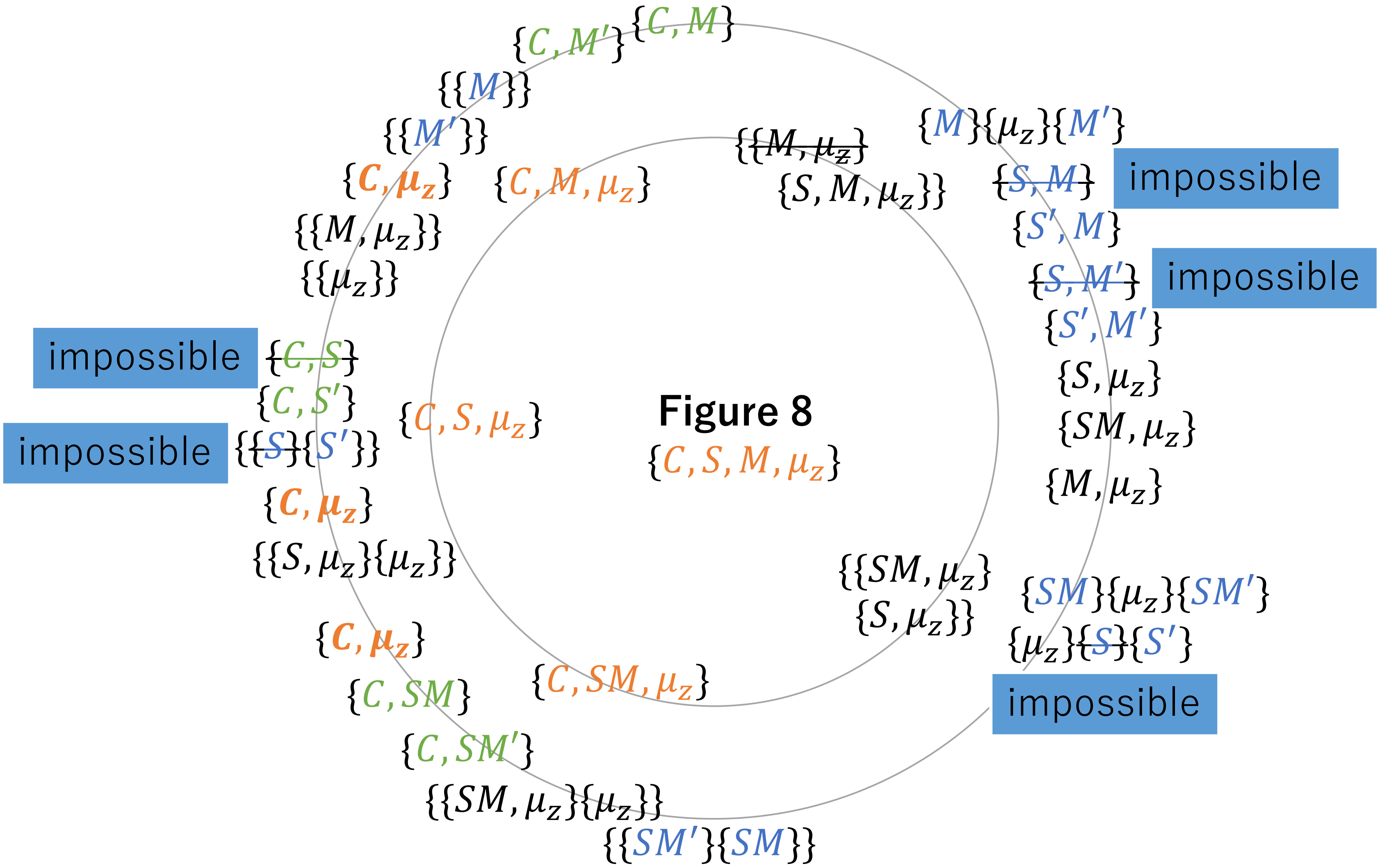
$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}\{S\}\{S'\}$

If angular momentum is zero, three body motion is planar. On the other hand, operation S changes sign of the angular momentum.

Figure 8

$\{C,S,M,\mu_z\}$



$\{C,M\}$

$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,M,\mu_z\}$

$\{$~~$\{M,\mu_z\}$~~

$\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S,M\}$~~ impossible

$\{S',M\}$

~~$\{S,M'\}$~~ impossible

$\{S',M'\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

impossible ~~$\{C,S\}$~~

$\{C,S'\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$

$\{C,SM,\mu_z\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

Thus, bifurcation with S but without μ_z is impossible. (Fujiwara 2020)

$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

$\{$~~$\{M, \mu_z\}$~~
$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~ impossible

$\{S', M\}$

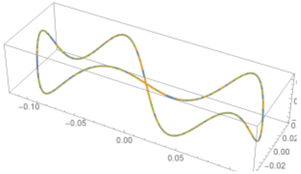
~~$\{S, M'\}$~~ impossible

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$



impossible ~~$\{C, S\}$~~

$\{C, S'\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, S, \mu_z\}$

Figure 8

$\{C, S, M, \mu_z\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, SM\}$

$\{C, SM, \mu_z\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

For example, this choreography is impossible.

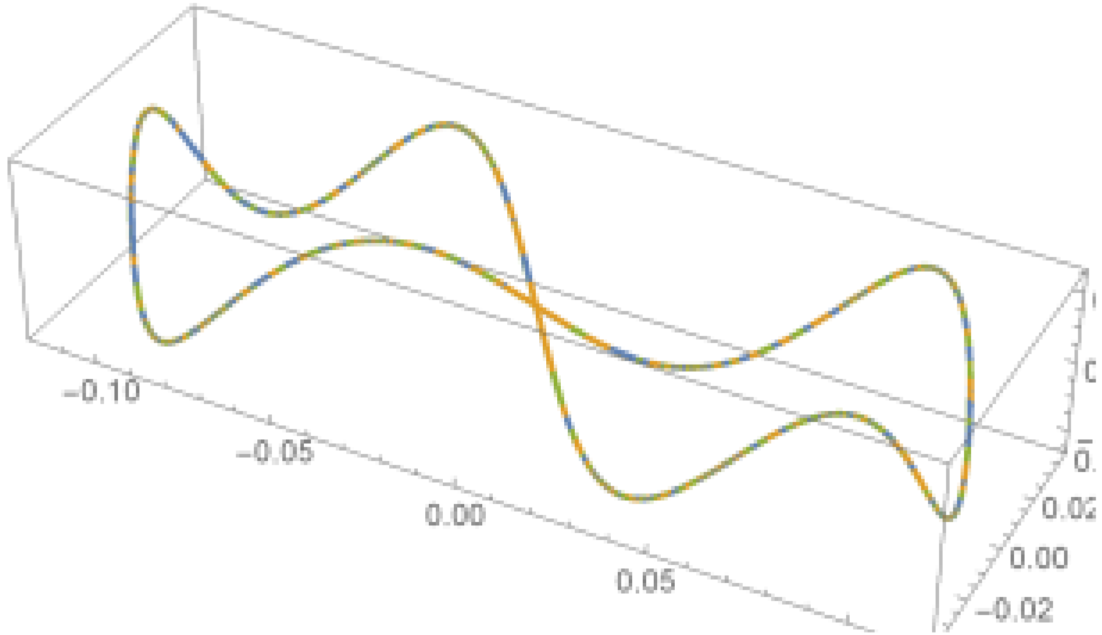
$\{C, M'\}$ $\{C, M\}$

impossible

ssible

possible

impossib



$M'\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M'\}$ $\{C,M\}$

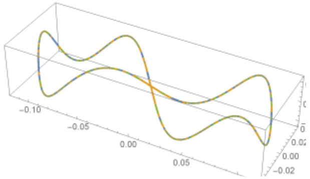
$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$ $\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$



impossible ~~$\{C,S\}$~~

$\{C,S'\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$ $\{C,SM,\mu_z\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{$~~$\{M,\mu_z\}$~~ $\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S,M\}$~~ impossible

$\{S',M\}$

~~$\{S,M'\}$~~ impossible

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

$\{\{SM,\mu_z\}$ $\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible

$\{C, M\}$

$\{C, M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

impossible ~~$\{C, S\}$~~

$\{C, S'\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, S, \mu_z\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

possible $\{C, SM\}$

$\{C, SM'\}$

$\{C, SM, \mu_z\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

Figure 8

$\{C, S, M, \mu_z\}$

$\{$~~$\{M, \mu_z\}$~~
$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~ impossible

$\{S', M\}$

~~$\{S, M'\}$~~ impossible

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

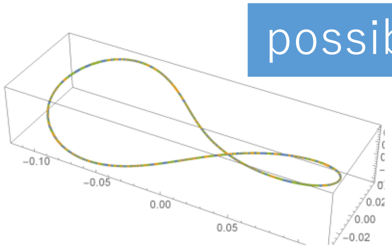
$\{M, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible



While, for example, this is a possible non-planar choreography.

$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

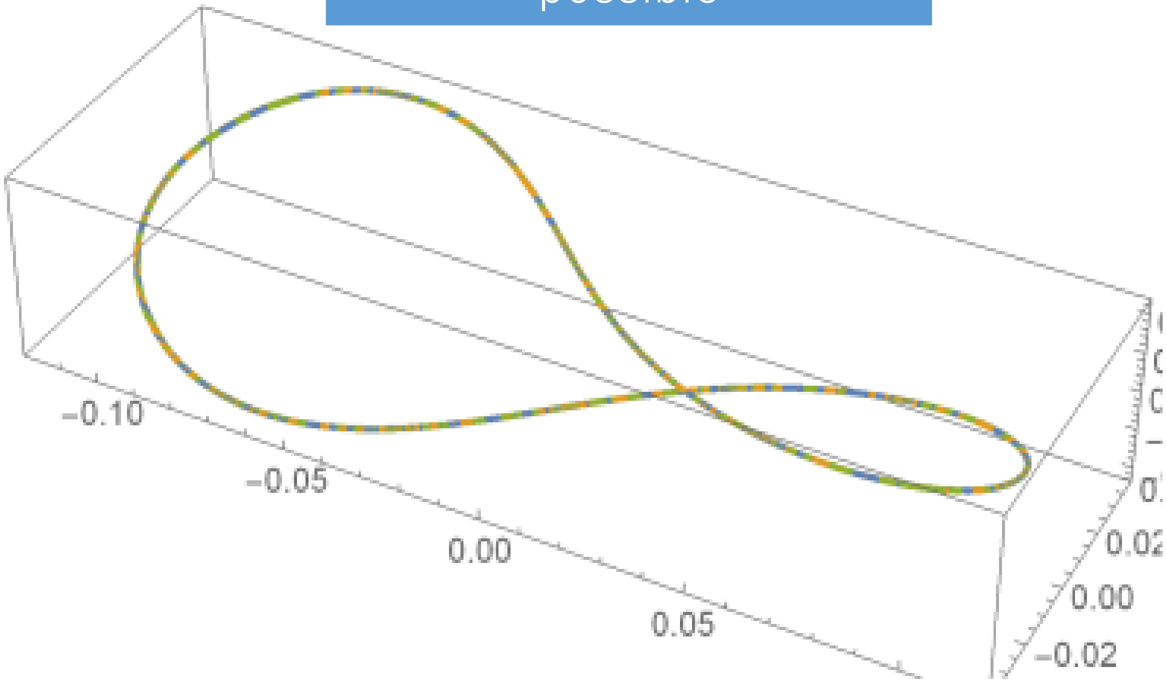
possible

ssible

ossible

impossib

impossible



$1'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

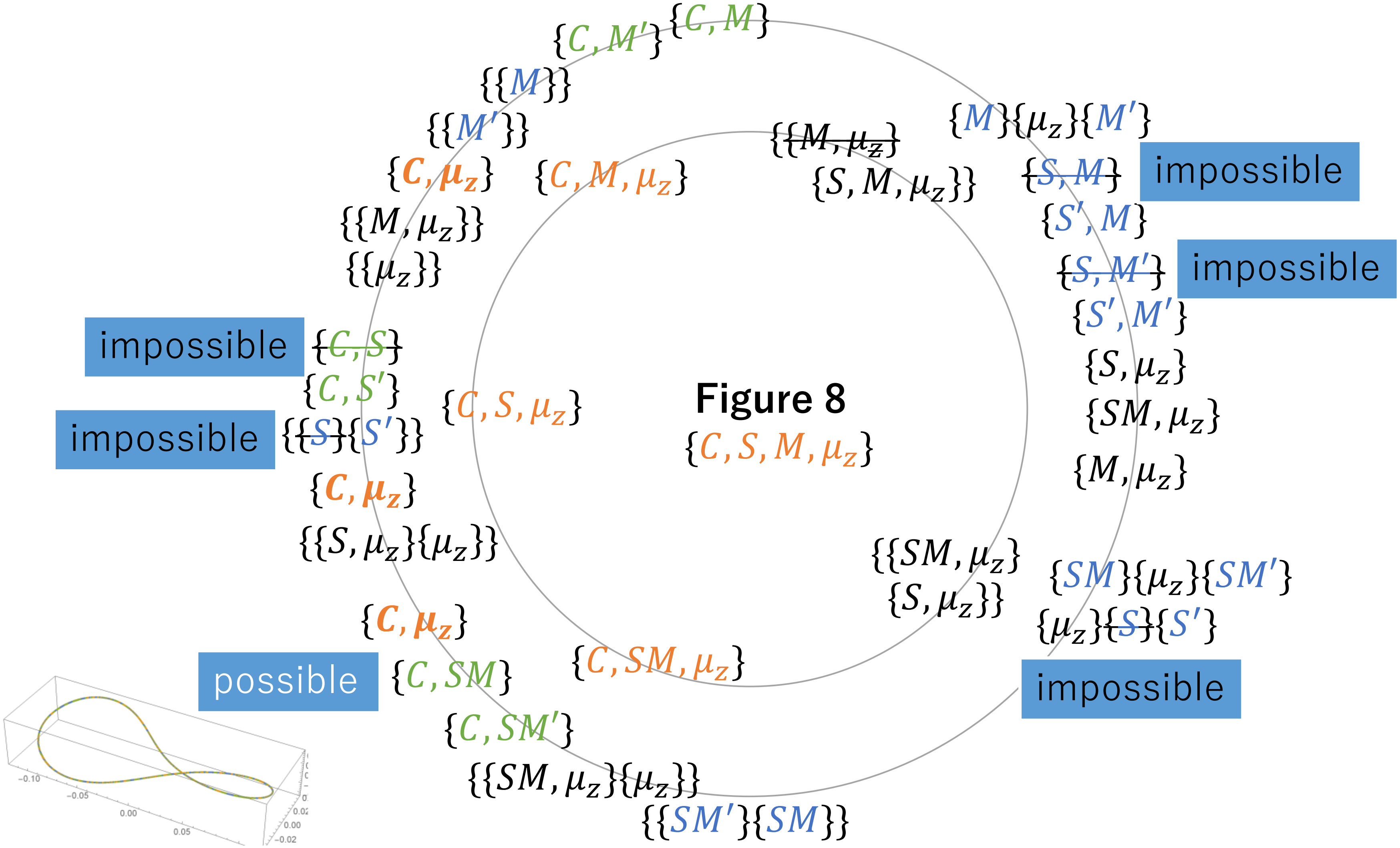


Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

impossible ~~$\{C,S\}$~~

$\{C,S'\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

possible $\{C,SM\}$

$\{C,SM'\}$

$\{C,SM,\mu_z\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{$~~$\{M,\mu_z\}$~~

$\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S,M\}$~~ impossible

$\{S',M\}$

~~$\{S,M'\}$~~ impossible

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

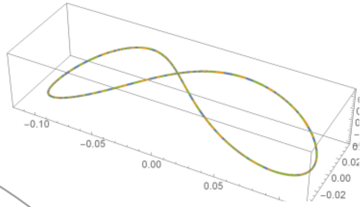
$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible

Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M\}$ possible



$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

impossible $\cancel{\{C,S\}}$

$\{C,S'\}$

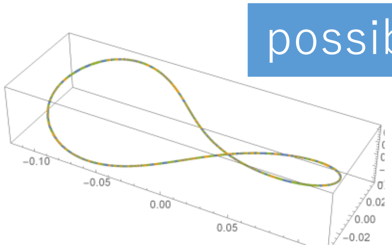
impossible $\{\cancel{\{S\}}\{S'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\boldsymbol{\{C,\mu_z\}}$

possible $\{C,SM\}$



$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{M\}\{\mu_z\}\{M'\}$

$\cancel{\{S,M\}}$ impossible

$\{S',M\}$

$\cancel{\{S,M'\}}$ impossible

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}\cancel{\{S\}}\{S'\}$

impossible

$\{C,M,\mu_z\}$

$\{\cancel{\{M,\mu_z\}}$
$\{S,M,\mu_z\}\}$

$\{C,S,\mu_z\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{C,SM,\mu_z\}$

Also this is a possible non-planar choreography.

$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

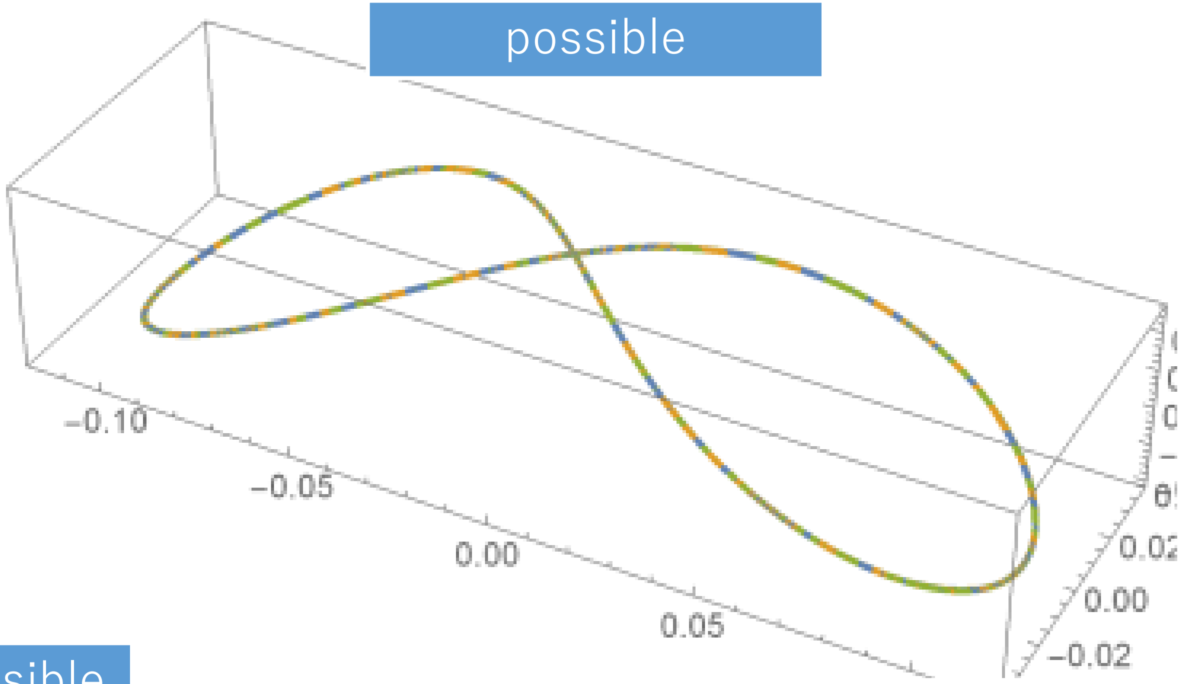
possible

ssible

ossible

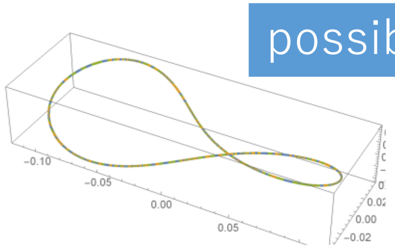
imposs

impossib



$M'\}$

possible



$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

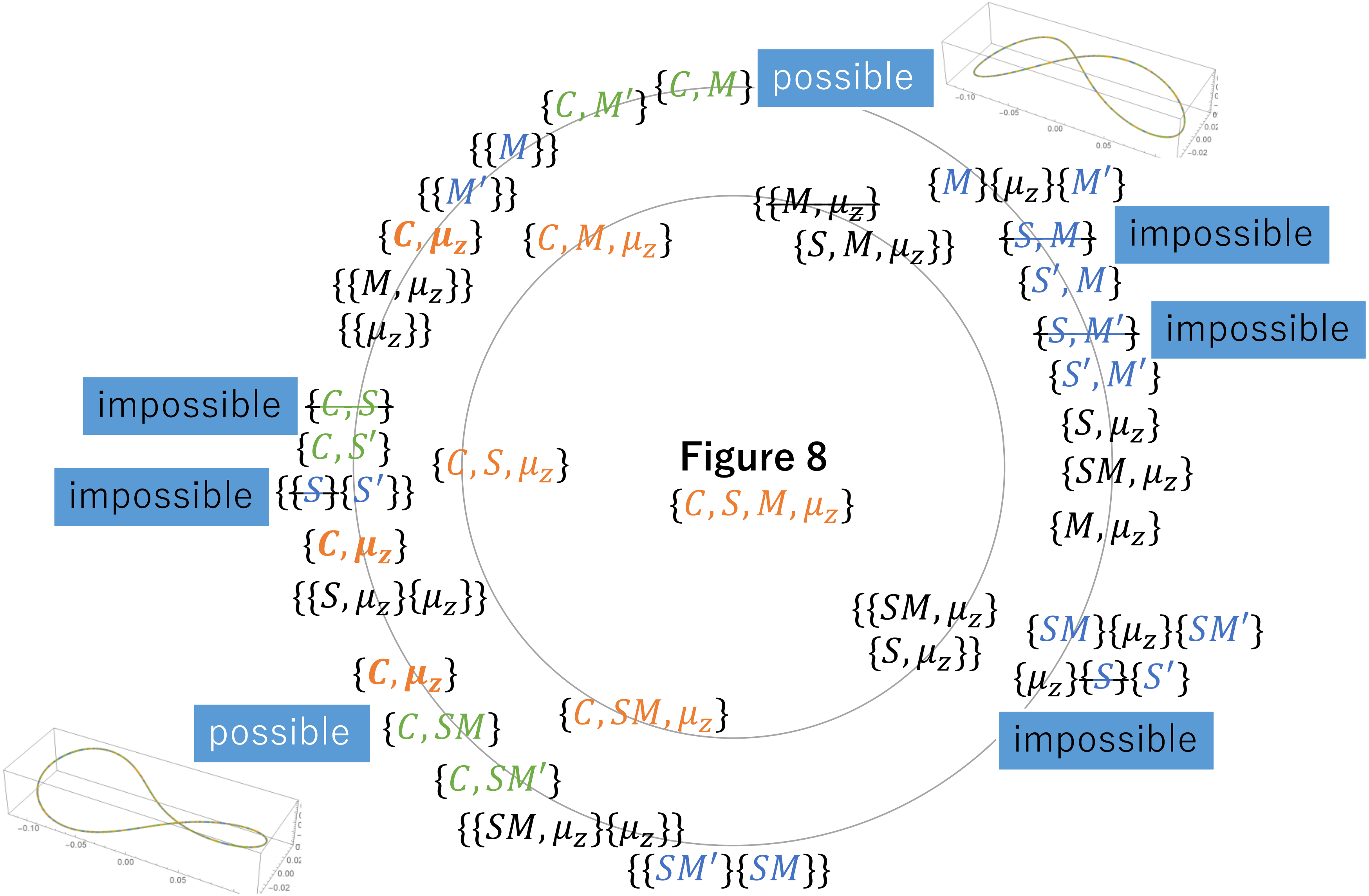


Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M\}$ possible

$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

impossible ~~$\{C,S\}$~~

$\{C,S'\}$

$\{C,S,\mu_z\}$

impossible $\{$~~$\{S\}$~~$\{S'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

possible $\{C,SM\}$

$\{C,SM,\mu_z\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

impossible

$\{M,\mu_z\}$

$\{SM,\mu_z\}$

$\{S,\mu_z\}$

$\{S',M'\}$

~~$\{S,M'\}$~~ impossible

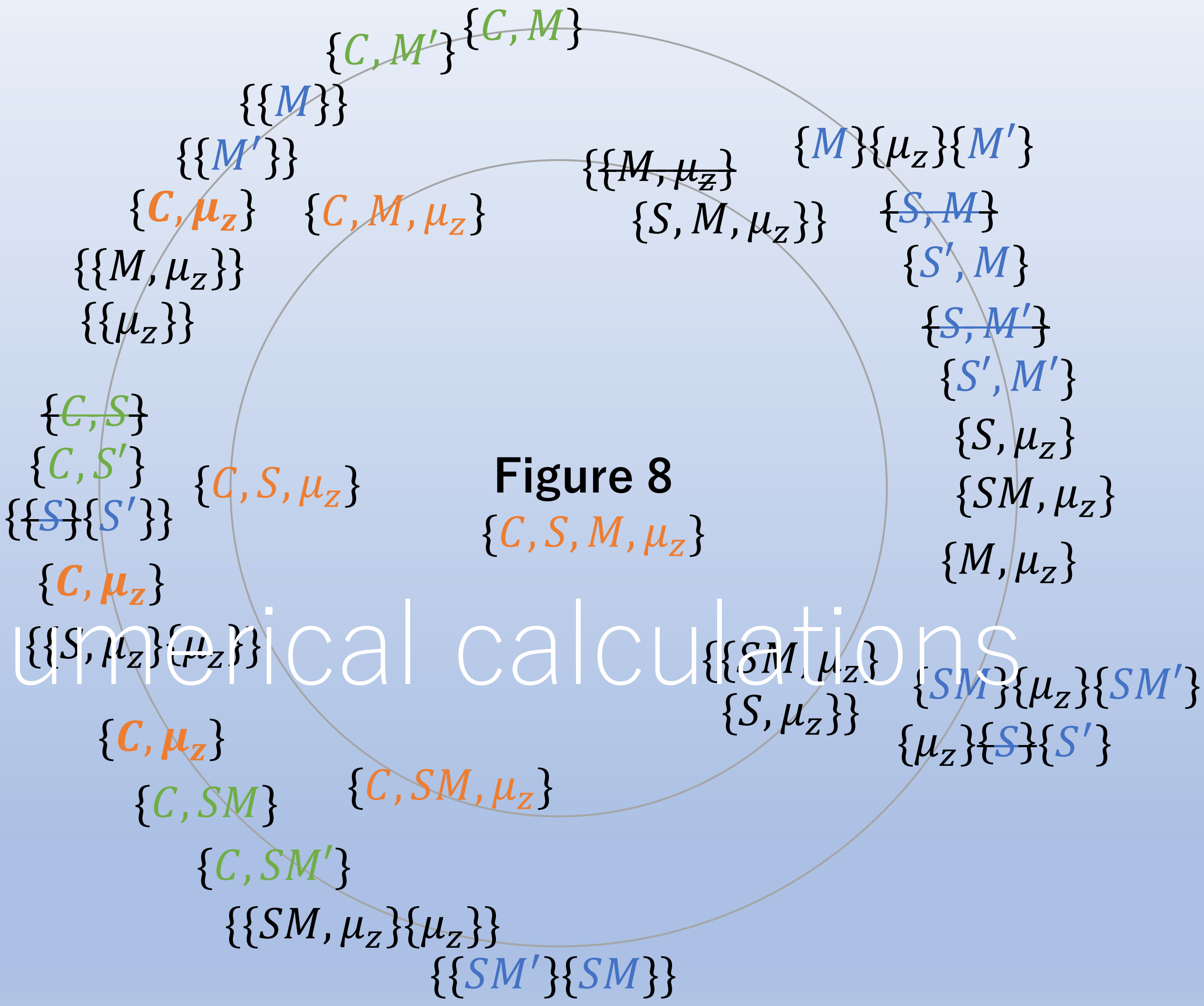
$\{S',M\}$

~~$\{S,M\}$~~ impossible

$\{M\}\{\mu_z\}\{M'\}$

$\{$~~$\{M,\mu_z\}$~~
$\{S,M,\mu_z\}\}$

6. Numerical calculations



$\{C,M\}$

$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{C,\mu_z\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C,S\}$~~

$\{C,S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

$\{C,\mu_z\}$

$\{\{S,\mu_z\}$~~$\{\mu_z\}$~~$\}$

$\{C,\mu_z\}$

$\{C,SM\}$

$\{C,SM,\mu_z\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{$~~$\{M,\mu_z\}$~~
$\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S,M\}$~~

$\{S',M\}$

~~$\{S,M'\}$~~

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

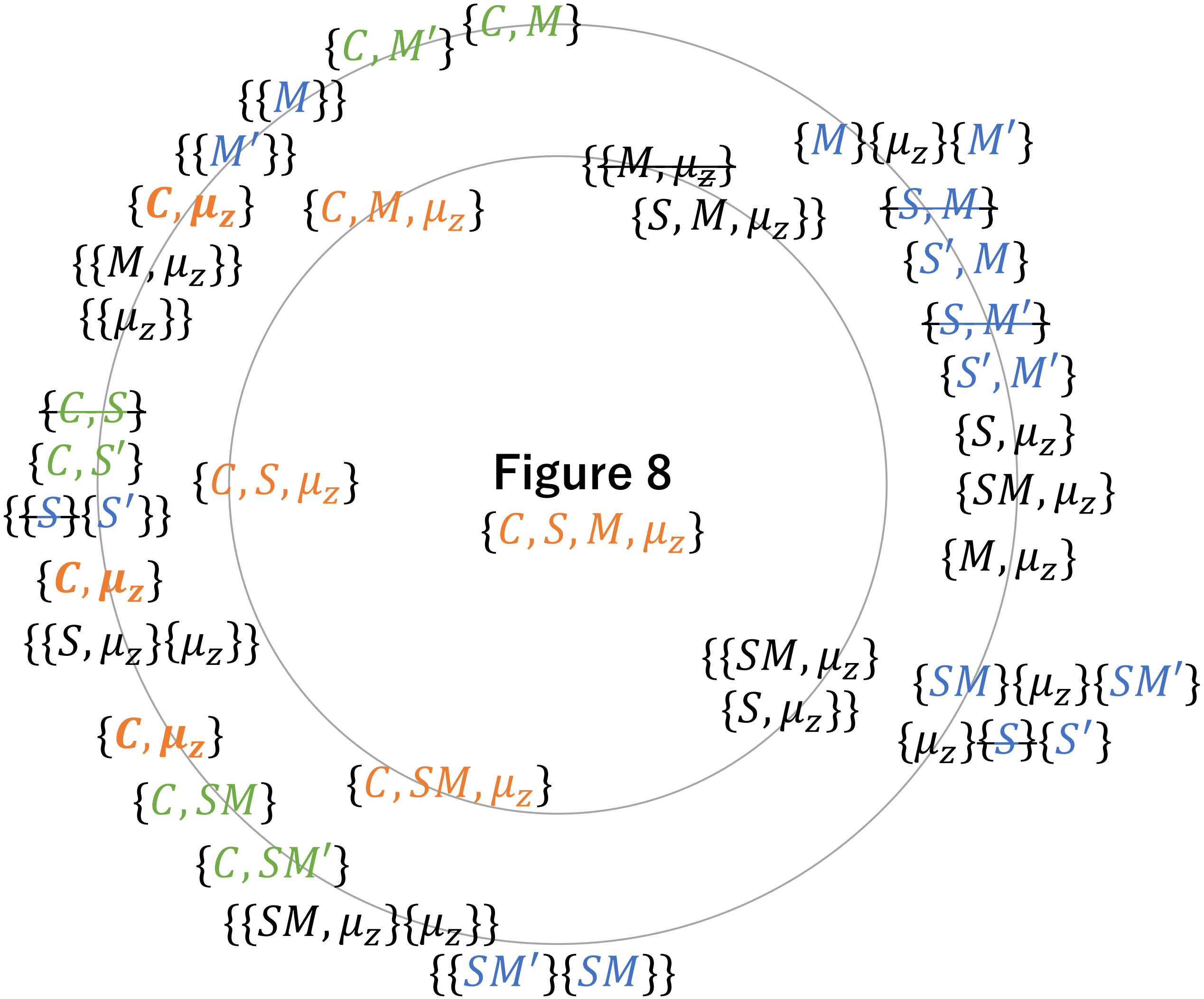
$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

Figure 8

$\{C, S, M, \mu_z\}$



$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{C, \mu_z\}$

$\{C, M, \mu_z\}$

$\{$~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~

$\{S', M\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C, S\}$~~

$\{C, S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, S, \mu_z\}$

$\{C, \mu_z\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{C, \mu_z\}$

$\{C, SM\}$

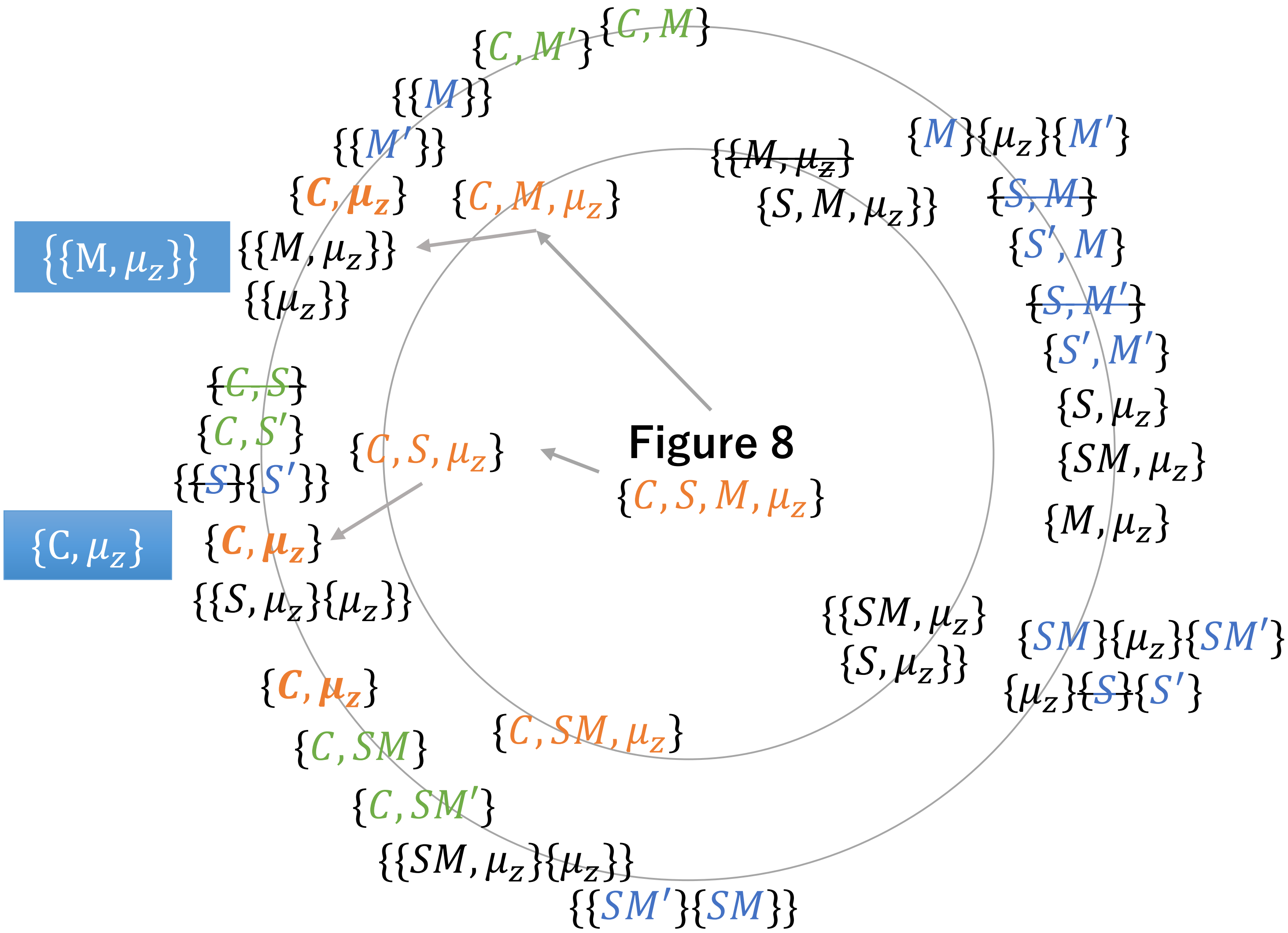
$\{C, SM, \mu_z\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

From 2020, we are calculating numerically bifurcations on the second ring.



$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{\{M'\}\}$

$\{$~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$ $\{C, M, \mu_z\}$

~~$\{S, M\}$~~

$\{\{\mathrm{M}, \mu_z\}\}$

$\{\{M, \mu_z\}\}$

$\{S', M\}$

$\{\{\mu_z\}\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

~~$\{C, S\}$~~

$\{S, \mu_z\}$

$\{C, S'\}$

$\{C, S, \mu_z\}$

Figure 8

$\{C, S, M, \mu_z\}$

$\{SM, \mu_z\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{M, \mu_z\}$

$\{\mathrm{C}, \mu_z\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, SM\}$

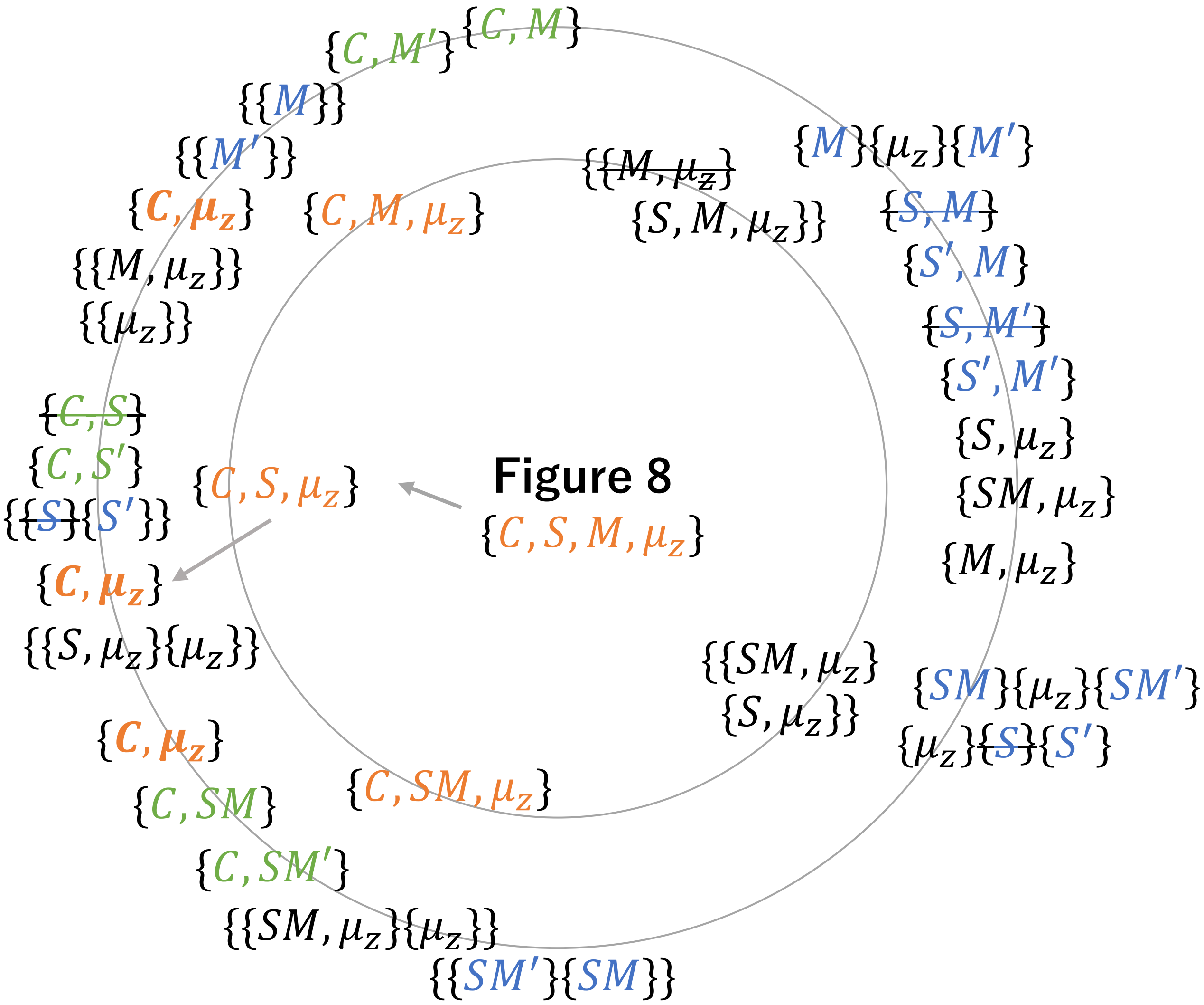
$\{C, SM, \mu_z\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

In LJ system, two bifurcations are interesting.



$\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C,S\}$~~

$\{C,S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$

$\{C,SM'\}$

$\{C,SM,\mu_z\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{M\}\{\mu_z\}\{M'\}$

$\{$~~$\{M,\mu_z\}$~~ $\{S,M,\mu_z\}\}$

~~$\{S,M\}$~~

$\{S',M\}$

~~$\{S,M'\}$~~

$\{S',M'\}$

$\{S,\mu_z\}$

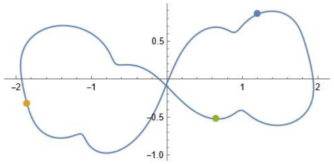
$\{SM,\mu_z\}$

$\{M,\mu_z\}$

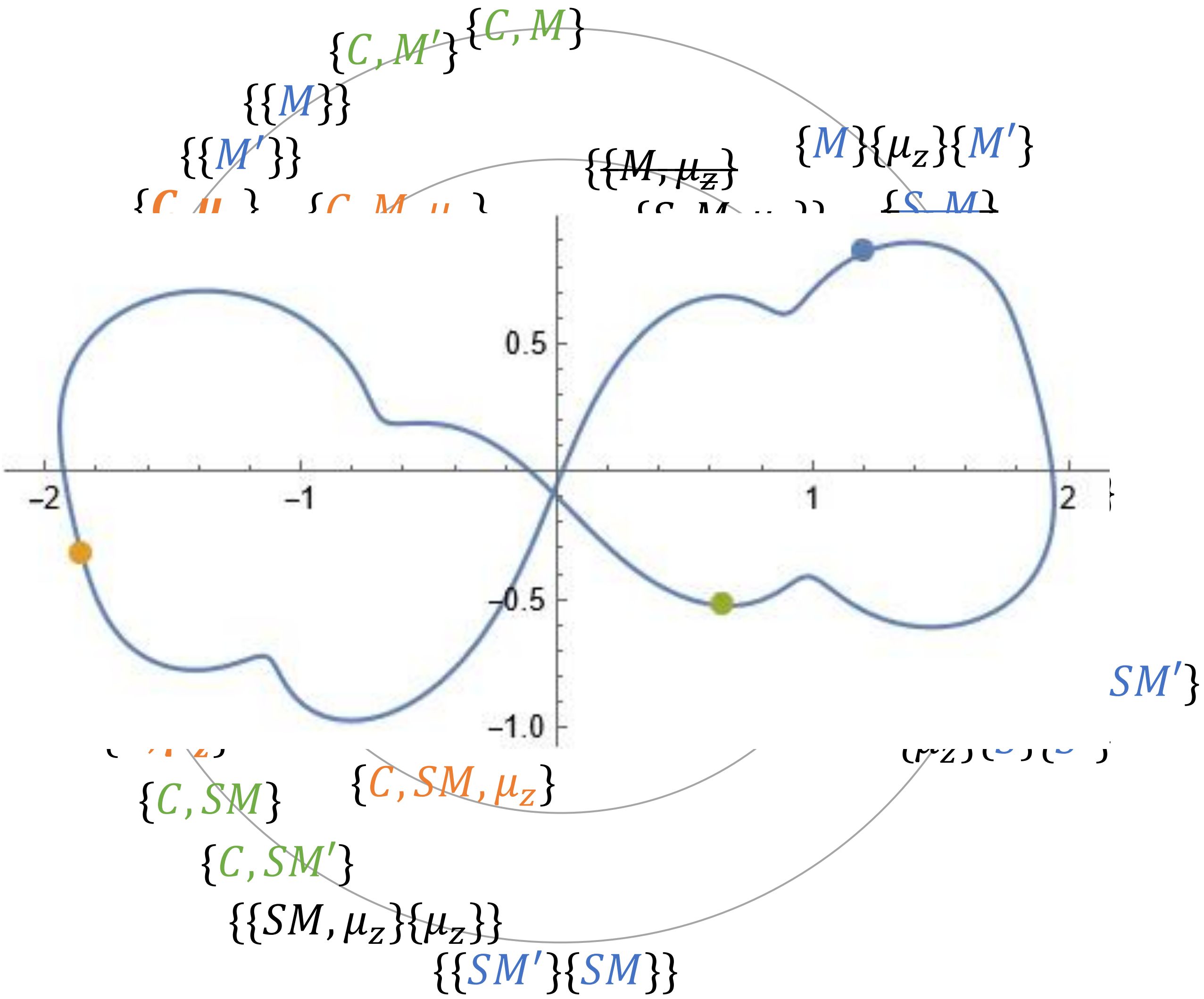
$\{\{SM,\mu_z\}$ $\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$



This is choreographic C without any spatial symmetry in the plane.


$\{C,M\}$
$\{C,M'\}$
$\{\{M\}\}$
$\{\{M'\}\}$
$\{\{M,\mu_z\}$
$\{M\}\{\mu_z\}\{M'\}$
0.5
−0.5
−1.0
−2
−1
1
2
$SM'\}$
$\{C,SM,\mu_z\}$
$\{C,SM\}$
$\{C,SM'\}$
$\{\{SM,\mu_z\}\{\mu_z\}\}$
$\{\{SM'\}\{SM\}\}$

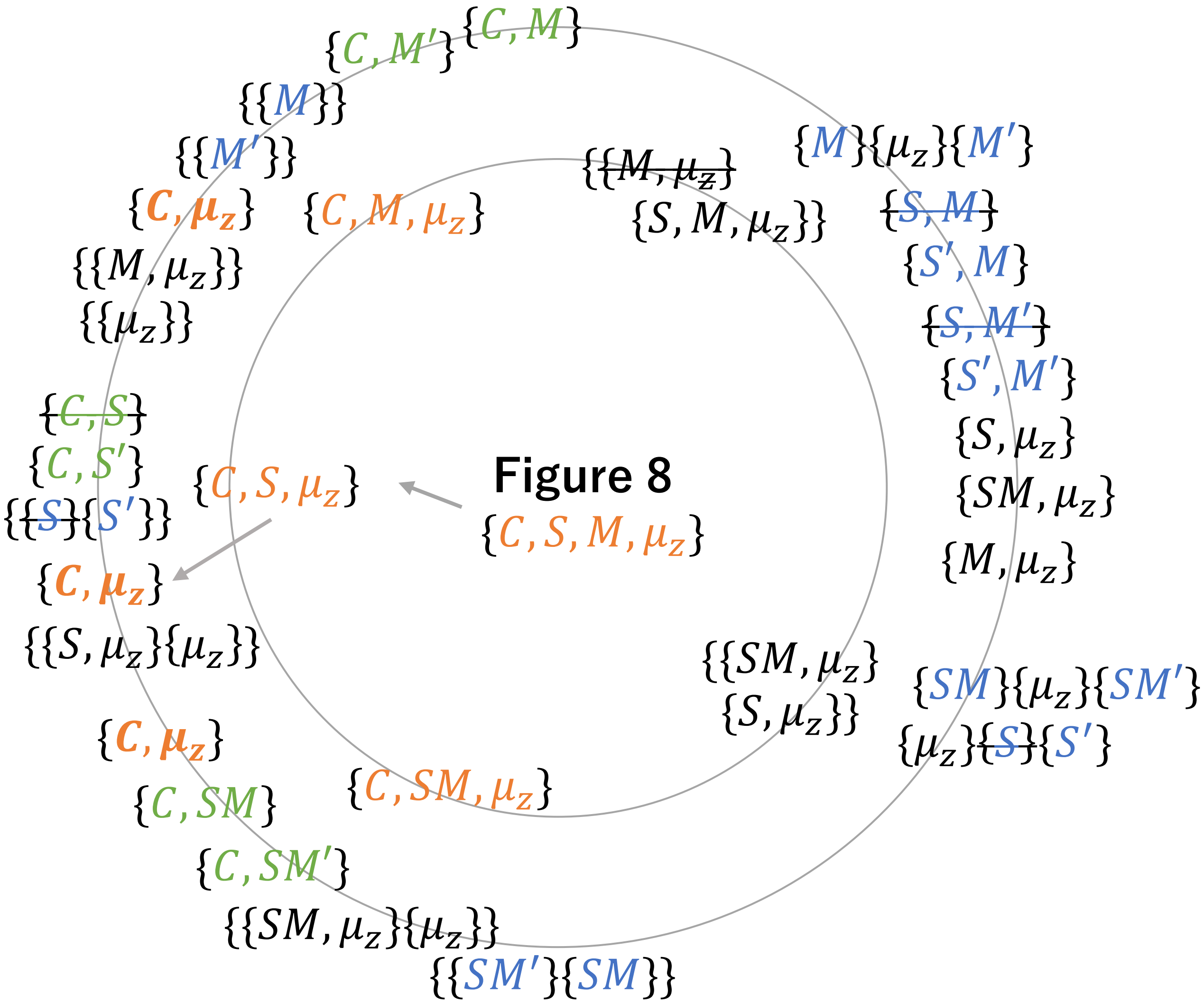


Figure 8

$\{C,S,M,\mu_z\}$

$\{C,M\}$

$\{C,M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

$\cancel{\{C,S\}}$

$\{C,S'\}$

$\{\cancel{\{S\}}\{S'\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{C,SM\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}\cancel{\{S\}}\{S'\}$

$\{M,\mu_z\}$

$\{SM,\mu_z\}$

$\{S,\mu_z\}$

$\{S',M'\}$

$\cancel{\{S,M'\}}$

$\{S',M\}$

$\cancel{\{S,M\}}$

$\{M\}\{\mu_z\}\{M'\}$

$\{C,M,\mu_z\}$

$\{\cancel{\{M,\mu_z\}}$

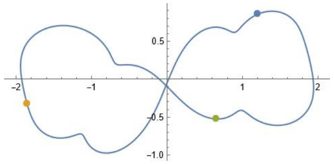
$\{S,M,\mu_z\}\}$

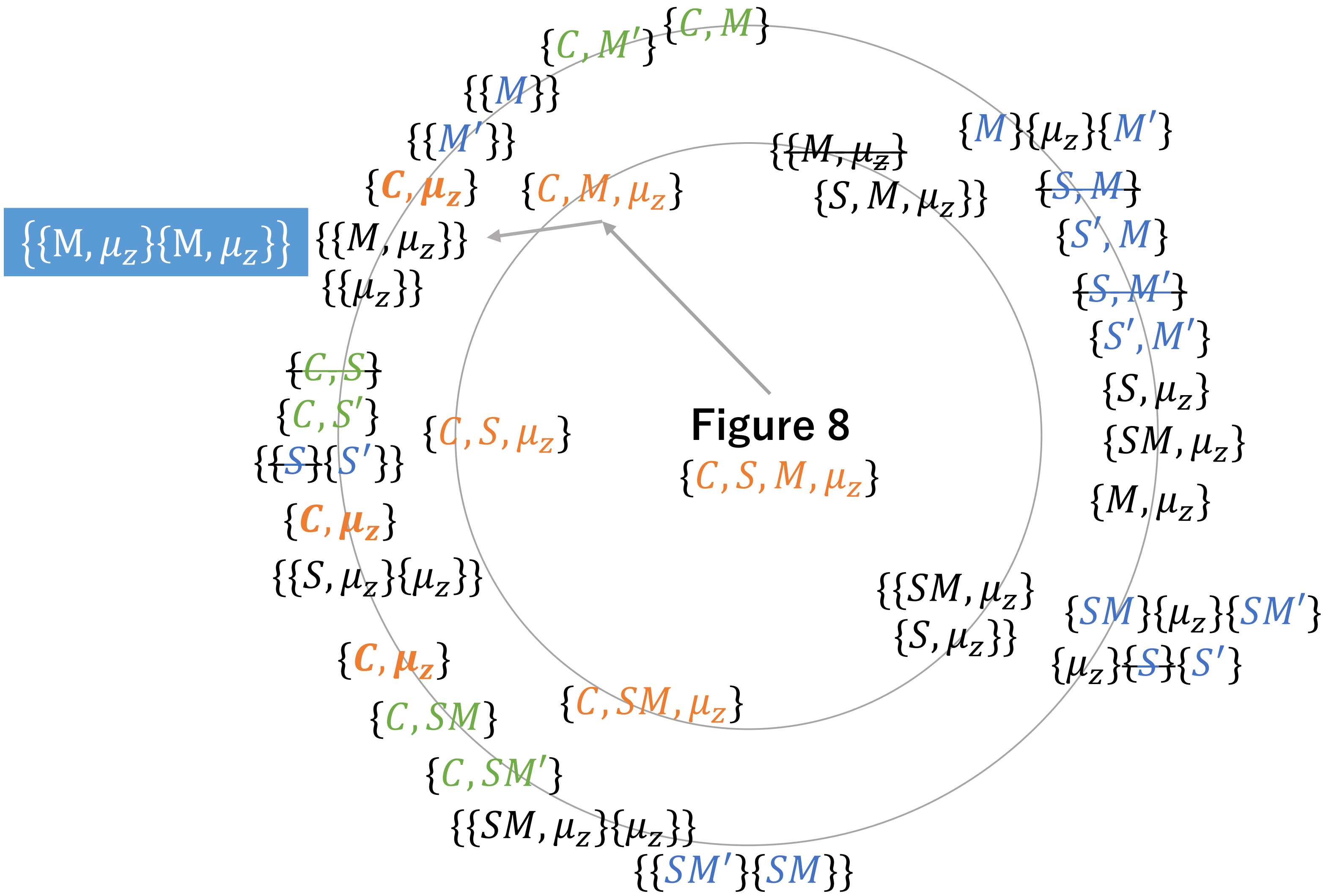
$\{C,S,\mu_z\}$

$\{C,SM,\mu_z\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$





$\{C, M\}$

$\{C, M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}\{M, \mu_z\}\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C, S\}$~~

$\{C, S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, S, \mu_z\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C}, \boldsymbol{\mu_z}\}$

$\{C, SM\}$

$\{C, SM'\}$

$\{C, SM, \mu_z\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{$~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

Figure 8

$\{C, S, M, \mu_z\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~

$\{S', M\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

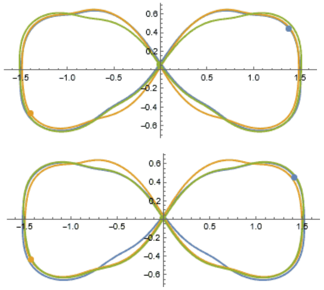
$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

This, abbreviation of repetition, has two eigenfunctions with the same symmetry $\{M, \mu_z\}$ in two-dimensional representation, and



$\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C,\mu_z\}}$ $\{C,M,\mu_z\}$

$\{\{M,\mu_z\}\}$

$\{\{\mu_z\}\}$

$\{$~~$\{M,\mu_z\}$~~

$\{S,M,\mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S,M\}$~~

$\{S',M\}$

~~$\{S,M'\}$~~

$\{S',M'\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

~~$\{C,S\}$~~

$\{C,S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C,S,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\{SM,\mu_z\}$

$\{S,\mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\boldsymbol{\{C,\mu_z\}}$

$\{C,SM\}$

$\{C,SM,\mu_z\}$

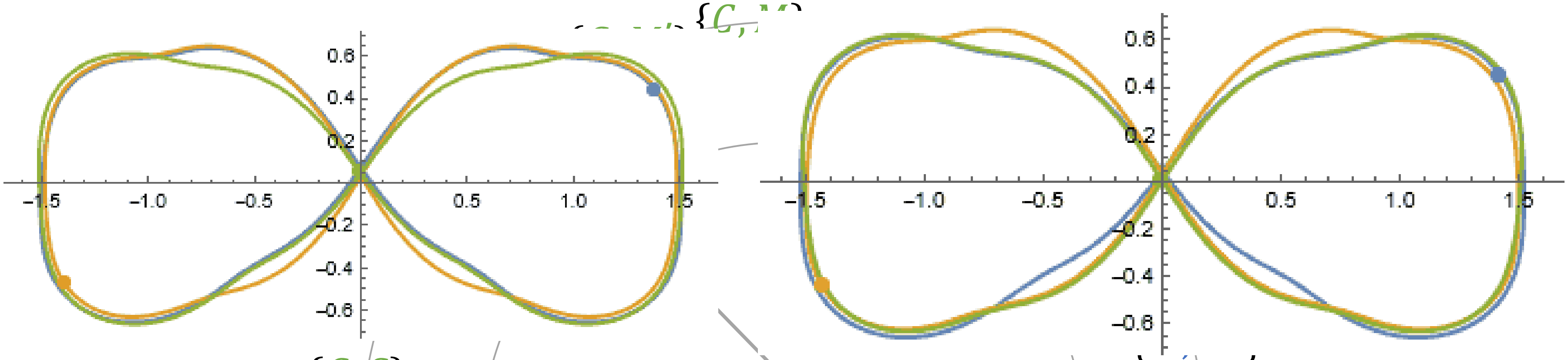
$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

yields two solutions with the same symmetry.

$\{C,$



$\cancel{\{C,S\}}$

$\{C,S'\}$

$\{\cancel{\{S\}}\{S'\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\boldsymbol{C},\boldsymbol{\mu_z}\}$

$\{C,SM\}$

$\{C,SM'\}$

$\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{C,S,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\{C,SM,\mu_z\}$

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$\{M,\mu_z\}$

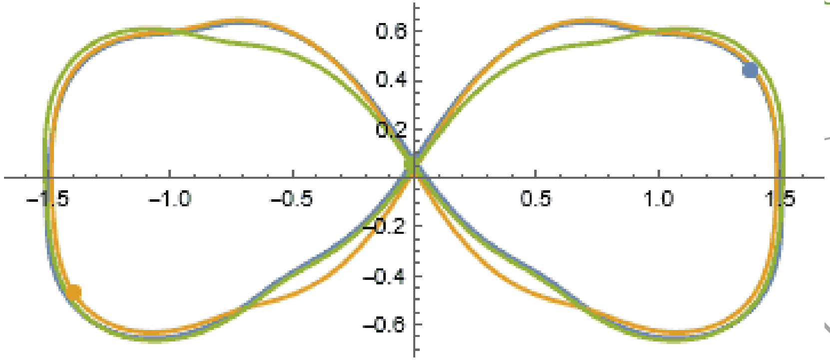
$\{\{SM,\mu_z\}$
$\{S,\mu_z\}\}$

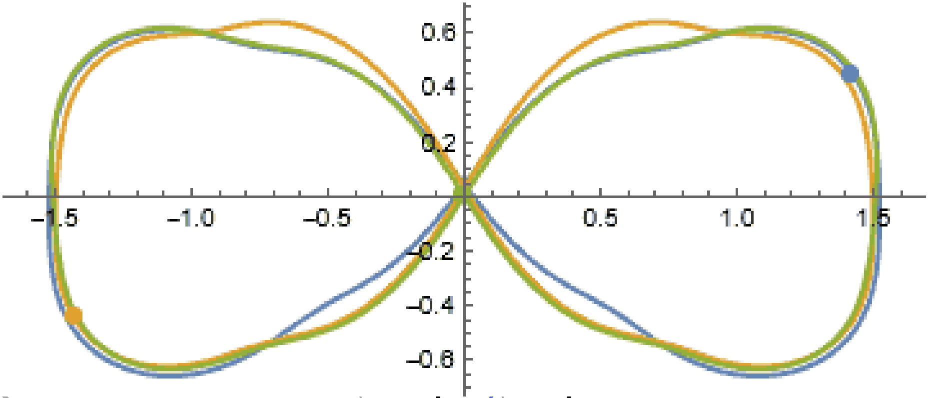
$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}\cancel{\{S\}}\{S'\}$

But until now, we do not know what chooses these solution

$\{C,$





~~$\{C,S\}$~~

$\{C,S'\}$

$\{\{S\}\{S'\}\}$

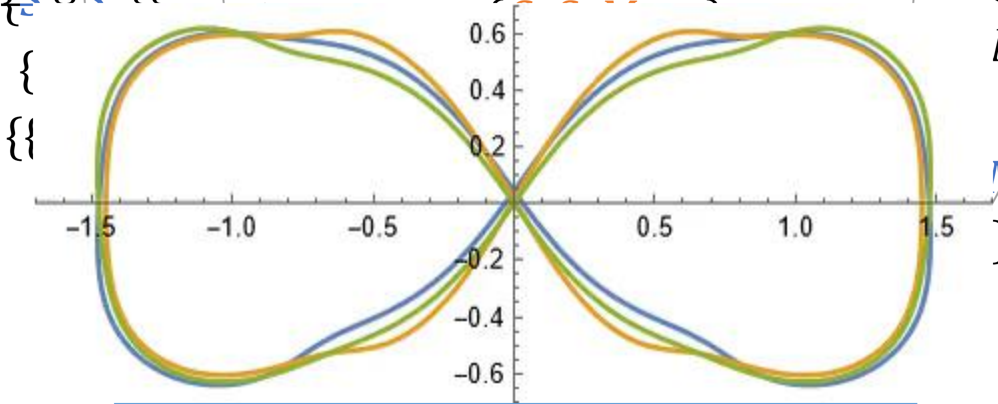
$\{C,S,\mu_z\}$

Figure 8

$\{S,\mu_z\}$

$\{SM,\mu_z\}$

$M,\mu_z\}$



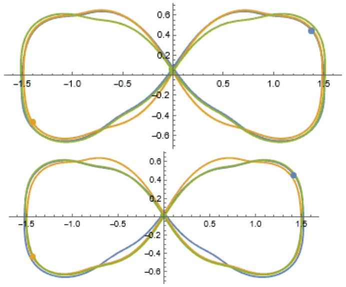
$M\}\{\mu_z\}\{SM'\}$

$\}$~~$\{S\}$~~$\{S'\}$

change of $q+\cos\theta\phi_1+\sin\theta\phi_2$ in θ

$\{\{SM'\}\{SM\}\}$

from the liner combinations of eigenfunctions. The movie is showing change of representation in θ.



$\{C, M\}$

$\{C, M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\boldsymbol{\{C, \mu_z\}}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C, S\}$~~

$\{C, S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\boldsymbol{\{C, \mu_z\}}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\boldsymbol{\{C, \mu_z\}}$

$\{C, SM\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~

$\{S', M\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{C, M, \mu_z\}$

$\{$~~$\{M, \mu_z\}$~~
$\{S, M, \mu_z\}\}$

$\{C, S, \mu_z\}$

$\{\{SM, \mu_z\}$
$\{S, \mu_z\}\}$

$\{C, SM, \mu_z\}$

Figure 8

$\{C, S, M, \mu_z\}$

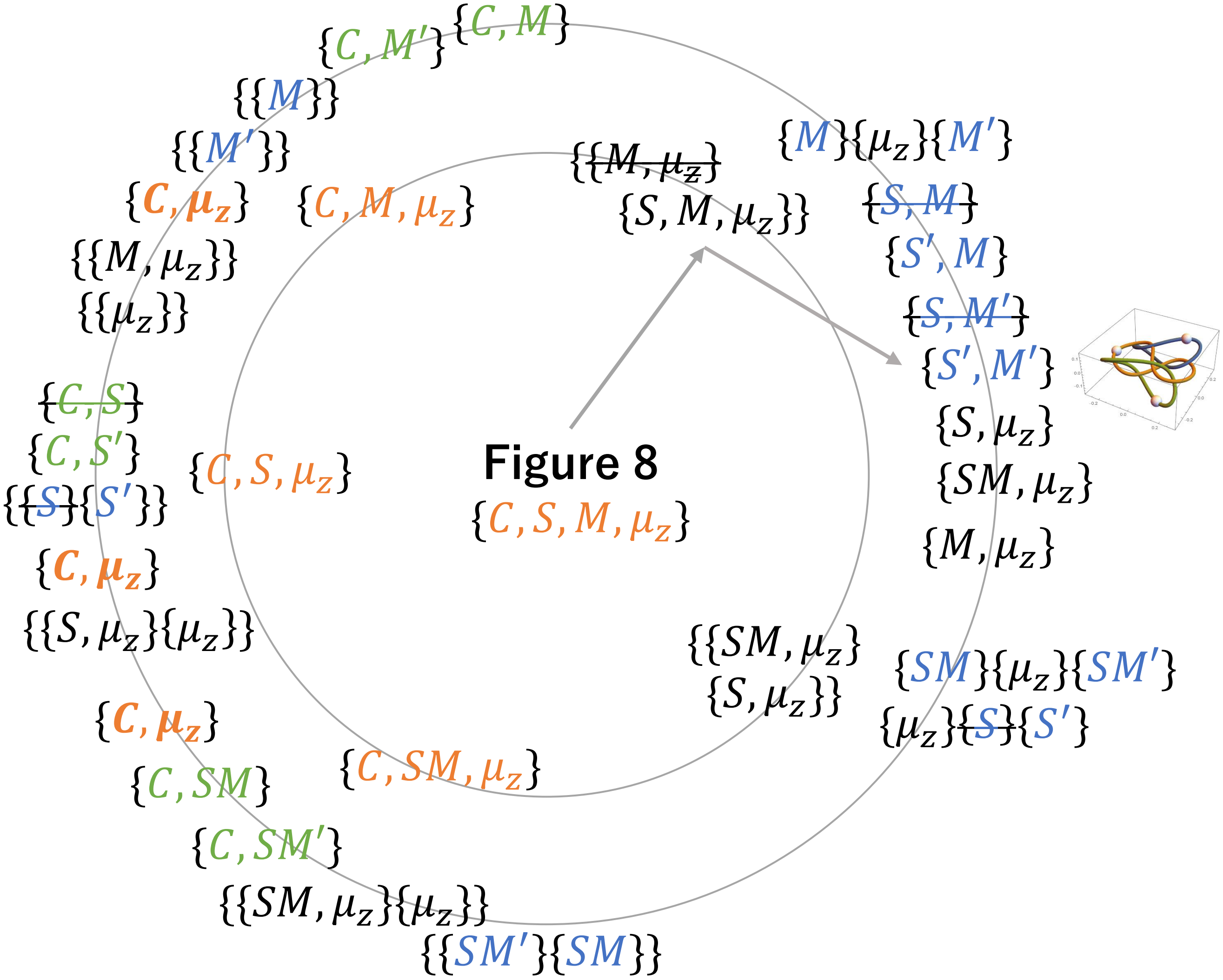


Figure 8

$\{C, S, M, \mu_z\}$

$\{C, M\}$

$\{C, M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{C, \mu_z\}$

$\{C, M, \mu_z\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C, S\}$~~

$\{C, S'\}$

$\{C, S, \mu_z\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, \mu_z\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{C, \mu_z\}$

$\{C, SM\}$

$\{C, SM, \mu_z\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

$\{$~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~

$\{S', M\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

In homogeneous system, we also found interesting bifurcation solutions, a non-planar solution.

$\{C, M'\}$ $\{C, M\}$

$\{\{M\}\}$

$\{\{$

$\{\{$

$\{C,$

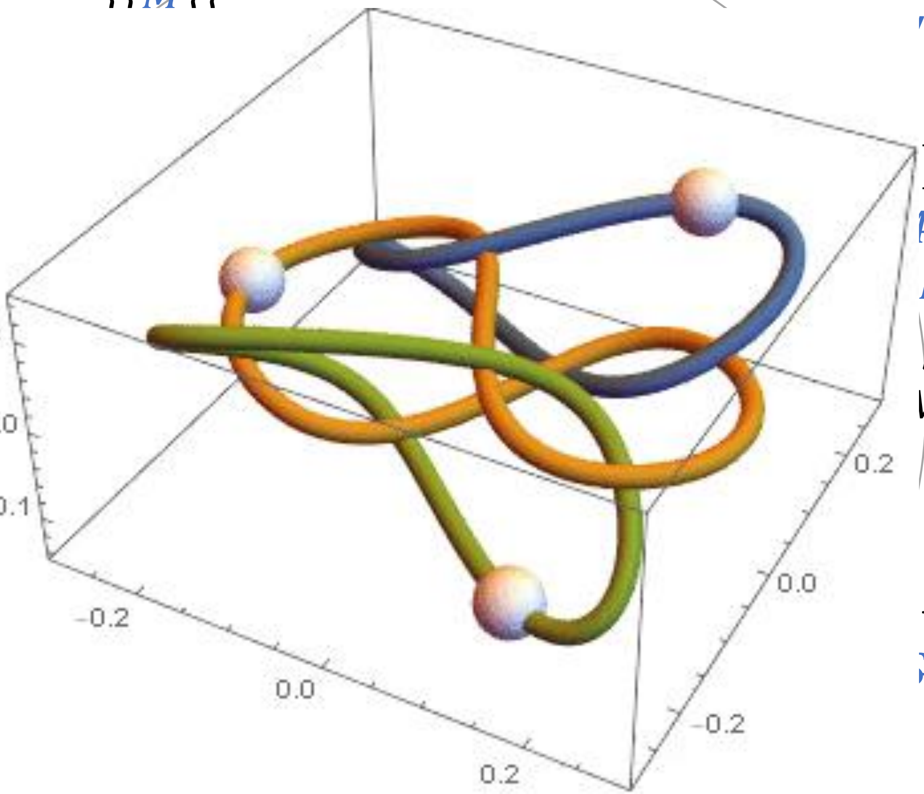
$\{C,$

$\{\{S\}\{$

$\{C,$

$\{\{S,$

$\{$



$'\}$

$\}$

$'\}$

$M'\}$

$\mu_z\}$

$M, \mu_z\}$

$\mu_z\}$

$\{\mu_z\}\{SM'\}$

$\}\{S'\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

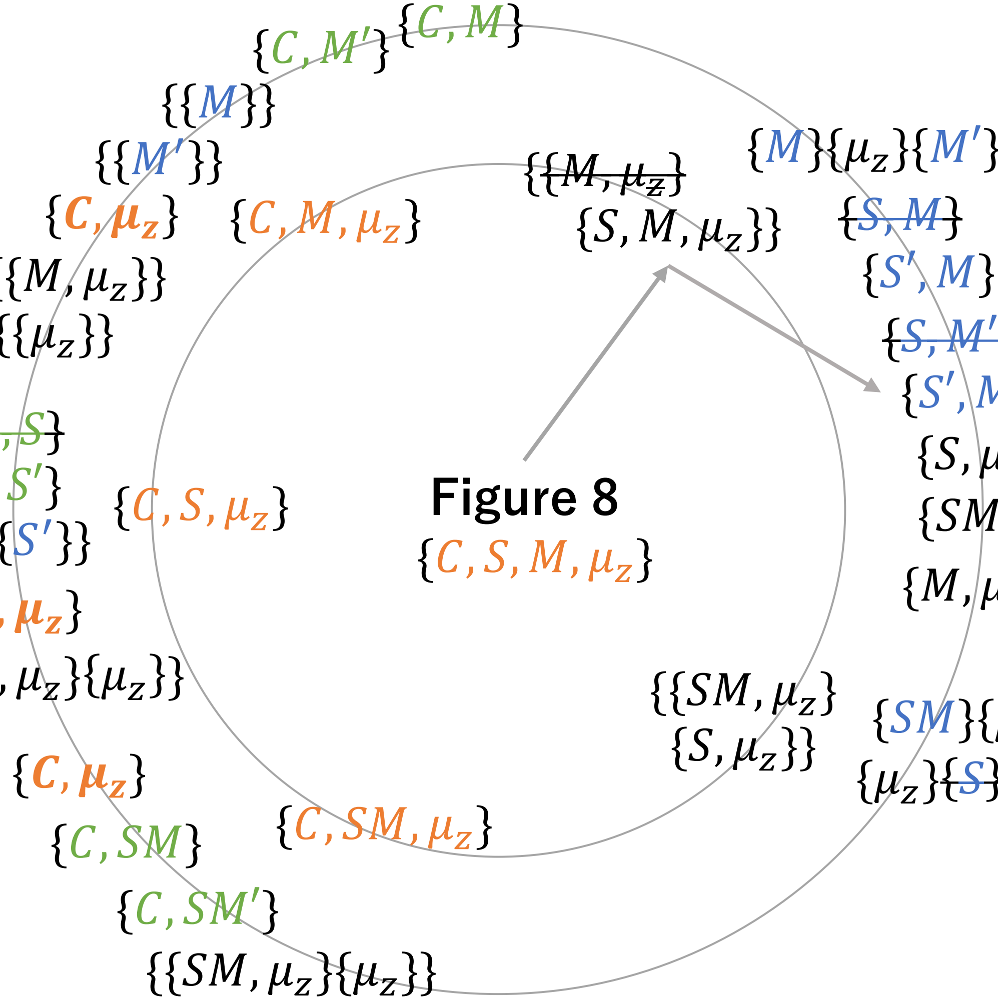


Figure 8

$\{C, S, M, \mu_z\}$

$\{C, M\}$

$\{C, M'\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{C, \mu_z\}$

$\{C, M, \mu_z\}$

$\{$~~$\{M, \mu_z\}$~~

$\{S, M, \mu_z\}\}$

$\{M\}\{\mu_z\}\{M'\}$

~~$\{S, M\}$~~

$\{S', M\}$

~~$\{S, M'\}$~~

$\{S', M'\}$

$\{\{M, \mu_z\}\}$

$\{\{\mu_z\}\}$

~~$\{C, S\}$~~

$\{C, S'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{C, S, \mu_z\}$

$\{C, \mu_z\}$

$\{\{S, \mu_z\}\{\mu_z\}\}$

$\{S, \mu_z\}$

$\{SM, \mu_z\}$

$\{M, \mu_z\}$

$\{\{SM, \mu_z\}$

$\{S, \mu_z\}\}$

$\{SM\}\{\mu_z\}\{SM'\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{C, \mu_z\}$

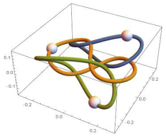
$\{C, SM\}$

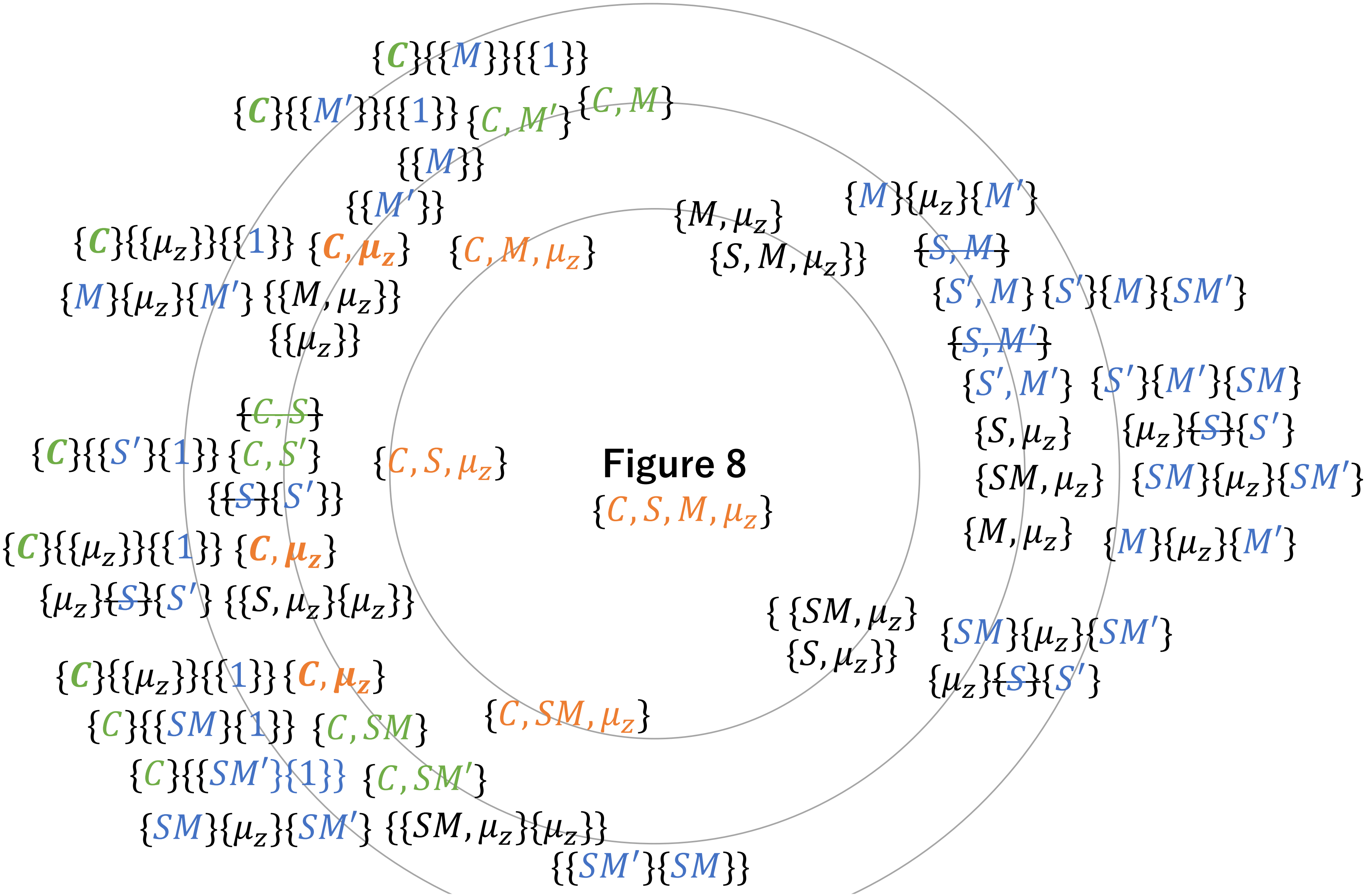
$\{C, SM, \mu_z\}$

$\{C, SM'\}$

$\{\{SM, \mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$





$\{C\}\{\{M\}\}\{\{1\}\}$

$\{C\}\{\{M'\}\}\{\{1\}\}$ $\{C,M'\}$ $\{C,M\}$

$\{\{M\}\}$

$\{\{M'\}\}$

$\{M,\mu_z\}$ $\{M\}\{\mu_z\}\{M'\}$

$\{C\}\{\{\mu_z\}\}\{\{1\}\}$ $\{C,\mu_z\}$ $\{C,M,\mu_z\}$ $\{S,M,\mu_z\}\}$ ~~$\{S,M\}$~~

$\{M\}\{\mu_z\}\{M'\}$ $\{\{M,\mu_z\}\}$ $\{S',M\}$ $\{S'\}\{M\}\{SM'\}$

$\{\{\mu_z\}\}$ ~~$\{S,M'\}$~~

$\{S',M'\}$ $\{S'\}\{M'\}\{SM\}$

~~$\{C,S\}$~~ $\{S,\mu_z\}$ $\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{C\}\{\{S'\}\{1\}\}$ $\{C,S'\}$ $\{C,S,\mu_z\}$

Figure 8

$\{C,S,M,\mu_z\}$

$\{SM,\mu_z\}$ $\{SM\}\{\mu_z\}\{SM'\}$

$\{$~~$\{S\}$~~$\{S'\}\}$

$\{M,\mu_z\}$ $\{M\}\{\mu_z\}\{M'\}$

$\{C\}\{\{\mu_z\}\}\{\{1\}\}$ $\{C,\mu_z\}$

$\{\mu_z\}$~~$\{S\}$~~$\{S'\}$ $\{\{S,\mu_z\}\{\mu_z\}\}$

$\{\ \{SM,\mu_z\}$ $\{S,\mu_z\}\}$ $\{SM\}\{\mu_z\}\{SM'\}$

$\{C\}\{\{\mu_z\}\}\{\{1\}\}\{C,\mu_z\}$ $\{\mu_z\}$~~$\{S\}$~~$\{S'\}$

$\{C\}\{\{SM\}\{1\}\}$ $\{C,SM\}$ $\{C,SM,\mu_z\}$

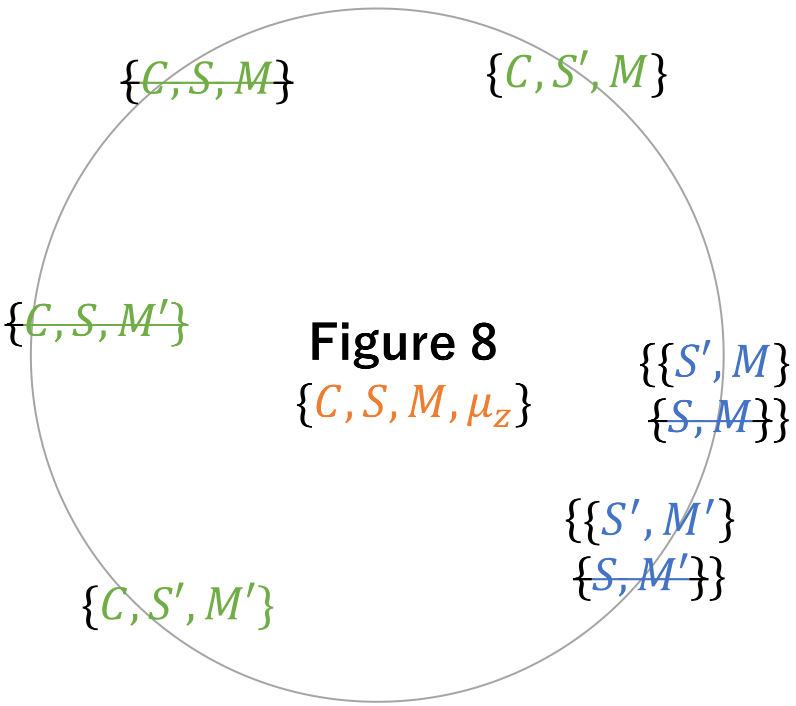
$\{C\}\{\{SM'\}\{1\}\}$ $\{C,SM'\}$

$\{SM\}\{\mu_z\}\{SM'\}$ $\{\{SM,\mu_z\}\{\mu_z\}\}$

$\{\{SM'\}\{SM\}\}$

Finally, this is a full diagram for planar bifurcations from figure 8 choreography. On the third ring, the third bifurcations are added.

7. Bifurcation of figure eight to the non-planar motion


{~~C, S, M~~}

$\{C, S', M\}$

{~~C, S, M'~~}

Figure 8

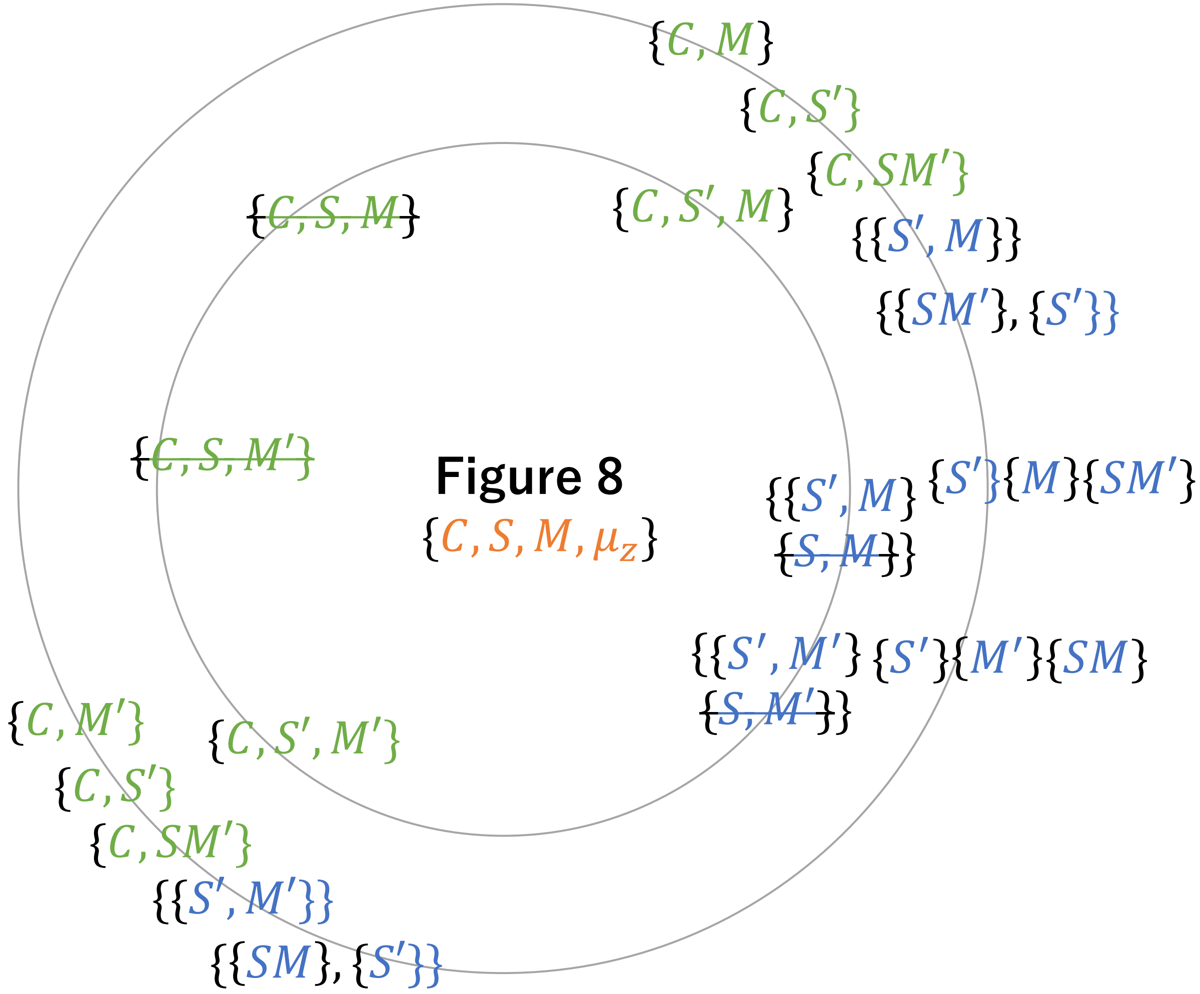
$\{C, S, M, \mu_z\}$

$\{\{S', M\}$
{~~S, M~~}}

$\{\{S', M'\}$
{~~S, M'~~}}

$\{C, S', M'\}$

This is a diagram for direct bifurcations from figure eight choreography to non-planar solutions. Until now we found no this type of solution numerically.



$\{C, M\}$

$\{C, S'\}$

$\{C, SM'\}$

$\{\{S', M\}\}$

$\{\{SM'\}, \{S'\}\}$

~~$\{C, S, M\}$~~

$\{C, S', M\}$

~~$\{C, S, M'\}$~~

Figure 8

$\{C, S, M, \mu_z\}$

$\{\{S', M\}$ $\{S'\}\{M\}\{SM'\}$

~~$\{S, M$~~$\}\}$

$\{\{S', M'\}$ $\{S'\}\{M'\}\{SM\}$

~~$\{S, M'$~~$\}\}$

$\{C, M'\}$

$\{C, S', M'\}$

$\{C, S'\}$

$\{C, SM'\}$

$\{\{S', M'\}\}$

$\{\{SM\}, \{S'\}\}$

However, they can bifurcate,

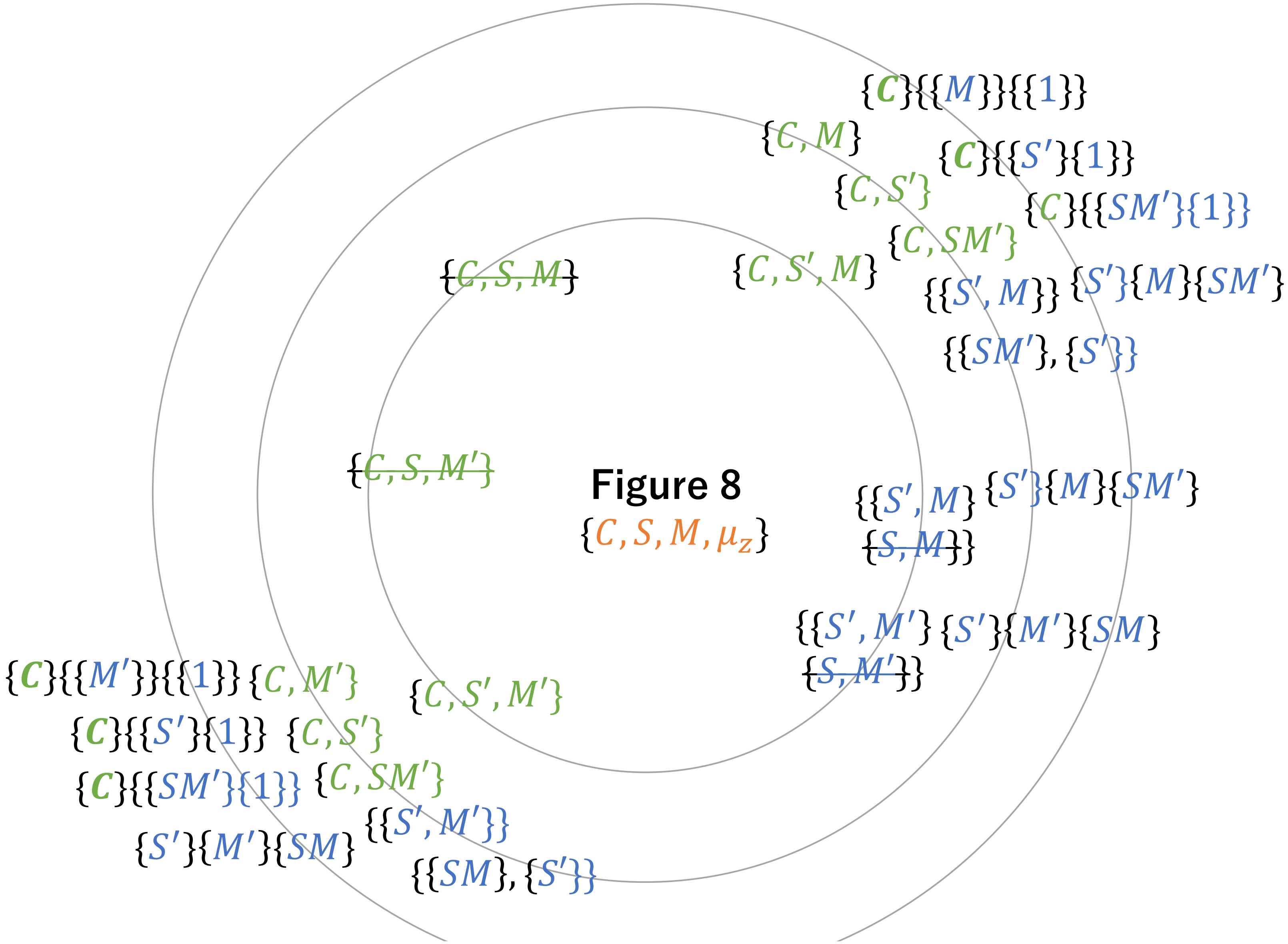


Figure 8

$\{C,S,M,\mu_z\}$

$\{\boldsymbol{C}\}\{\{M\}\}\{\{1\}\}$

$\{C,M\}$

$\{\boldsymbol{C}\}\{\{S'\}\{1\}\}$

$\{C,S'\}$

$\{C\}\{\{SM'\}\{1\}\}$

$\{C,SM'\}$

$\{C,S',M\}$

$\{\{S',M\}\}$ $\{S'\}\{M\}\{SM'\}$

$\{\{SM'\},\{S'\}\}$

~~$\{C,S,M\}$~~

~~$\{C,S,M'\}$~~

$\{\{S',M\}$ $\{S'\}\{M\}\{SM'\}$

~~$\{S,M$~~$\}\}$

$\{\{S',M'\}$ $\{S'\}\{M'\}\{SM\}$

~~$\{S,M'$~~$\}\}$

$\{\boldsymbol{C}\}\{\{M'\}\}\{\{1\}\}$ $\{C,M'\}$

$\{C,S',M'\}$

$\{\boldsymbol{C}\}\{\{S'\}\{1\}\}$ $\{C,S'\}$

$\{\boldsymbol{C}\}\{\{SM'\}\{1\}\}$ $\{C,SM'\}$

$\{\{S',M'\}\}$

$\{S'\}\{M'\}\{SM\}$

$\{\{SM\},\{S'\}\}$

and bifurcate.

8. Future work

We want to

- complete numerical search for bifurcation both for homogeneous and LJ system and
- understand bifurcation for two-dimensional representation, especially for degenerate symmetry case.

Thank you.