Infinite Series of Generalized Gosper Space Filling Curves

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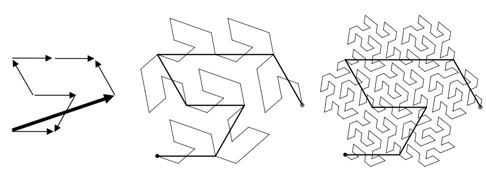
Abstract. We report on computer search for generalized Gosper curve for $37 < N < 61$, where N is the degree of the generalized Gosper curve. From the results of the computer search and some geometrical insight, we conjecture that the degree N satisfies $N = 6n+1$. We investigate the existence of infinite series of generalized Gosper curves. We show how to generate these series and introduce two new methods, the 'decomposition method' and the 'modified layer method'.

1 Generalized Gosper space filling curves

The Gosper curve is a space filling curve discovered by William Gosper, an American computer scientist, in 1973, and was introduced by Martin Gardner in 1976 [1, 2]. The curve is constructed by recursively replacing a bold arrow, called *initiator*, by seven arrows, called *generators*, Fig. 1(a), Fig. 1(b) and Fig. 1(c) illustrate the curves obtained by replacing the initiator by generators once and twice respectively.



(a) (b) (c)

Fig. 1. Gosper curve.

The Gosper curve is said to be a monster curve since it is a path from the root to the tip of the initiator visiting all interior lattice points on a regular triangular lattice. In 2001 [3], we found many such monster curves, called *generalized Gosper curves*, by computer search. Their shapes are similar to the original Gosper curve as shown in Fig. 2.

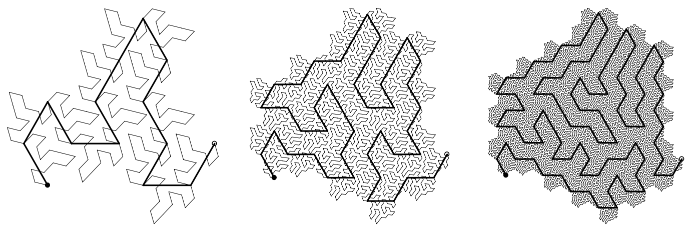
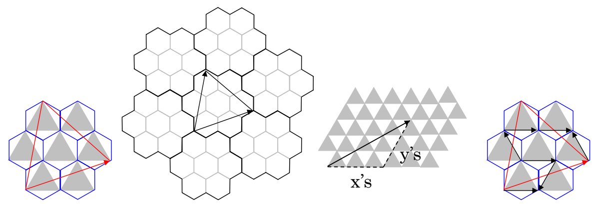


Fig. 2. Generalized Gosper curve with $N = 13, 43, 91$.

The generalized Gosper curves are defined by the following procedure used in the computer search in [3].

1. Choose the degree of the generalized Gosper curve N, the number of arrows included in the generator.
2. Assemble N hexagons having equilateral triangles inside as shown in Fig. 3 (a). We call this assembly of hexagons *B-figure*.
3. Choose two vertices from the triangles in B-figure as root and tip of the initiator.
4. Create a tiling by B-figures as shown in Fig. 3 (b), by using edges of the equilateral triangle having the initiator as its one side, as translation vectors. Then
$$N = x^2 + y^2 + xy \tag{1}$$
where x and y $(= 0, 1, 2, \ldots)$ are the numbers of horizontal and vertical triangles between the end points of the initiator as shown in Fig. 3 (c).
5. The B-figure has to have three-fold rotational axis. Therefore,
$$N = 3m \text{ or } 3m + 1, \ m = 0, 1, 2, \ldots. \tag{2}$$
6. Both the root and tip of the initiator have to be three-fold rotational axes of the tiling pattern of the B-figure.
7. By taking one edge from each equilateral triangle in the B-figure, construct a path from the root to the tip of the initiator. Confirm that the path visits all lattice points in the B-figure, except the points on the boundary, where the points are covered by only one triangle.
8. Convert the edges in the path to the arrow so that the direction of the arrow is counterclockwise in the equilateral triangle. Then obtain the generator arrows as shown in Fig. 3 (d).



(a) (b) (c) (d)

Fig. 3. B-figure and initiator triangle.

In [3], computer search for generalized Gosper curves for $N \leq 37$ was performed and the existence of infinite series of generalized Gosper curves was confirmed. In this paper, we report on two new results. In section 2, results of computer search for $37 < N \leq 61$ are reported. In section 3, many infinite series of generalized Gosper curves are shown.

2 Degree and Number of Curves

Computer search was performed for the generalized Gosper curves for $N \leq 61$. We were able to determine the number of generalized Gosper curves C_N and the number of corresponding B-figures B_N for given N. In Table 1, we tabulate C_N, B_N and also $C_N(i)$, the number of curves on the i'th B-figure, for N given by Eqs. (1) and (2).

Table 1. Number of generalized Gosper curves C_N. N's with $C_N = 0$ are omitted.

N	x	y	$C_N = C_N(1) + C_N(2) + \cdots + C_N(B_N)$
7	1	2	1
13	1	3	1
19	2	3	1
31	1	5	$7 = 1 + 5 + 1$
37	3	4	$8 = 1 + 2 + 3 + 2$
43	1	6	$24 = 3 + 1 + 19 + 1$
49	0	7	$134 = 58 + 5 + 3 + 60 + 4 + 4$
49	3	5	$75 = 41 + 18 + 1 + 6 + 2 + 1 + 5 + 1$
61	4	5	$(486) = 1 + 13 + 14 + 153 + 19 + 166 + 24 + 45 + 10 + 5 + 1 + 34 + 1$

From Table 1, we observe that among $N = x^2 + y^2 + xy$ only $N = 6n + 1$ seem to be degrees allowed for the curves. Therefore, *Conjecture.* The degree N of the generalized Gosper curves has to satisfy

$$N = 6n + 1, \tag{3}$$

where n is positive integers.

We have a rough scheme of the proof of this conjecture, but it is not complete and not simple. We hope to produce the proof in a future paper.

From Table 1 we observe that the number of B-figures, B_N, increases gradually with N. On the other hand, the computational time for B_N increases exponentially with N and this is the reason why we stopped calculation at $N = 61$. We show all B-figures with $43 \leq N < 61$ in Fig. 4–6. The B-figures for $N < 43$ are given in [3]. We note that the calculation for $N = 61$ is not completed, and that the number C_N given in parenthesis in Table 1 is a lower bound.

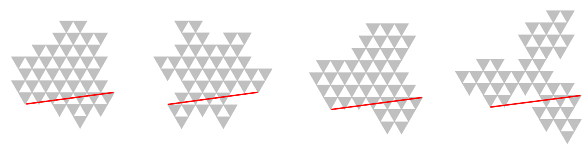


Fig. 4. B-figures for $N = 43$. $i = 1, 2, 3, 4$ from the left.

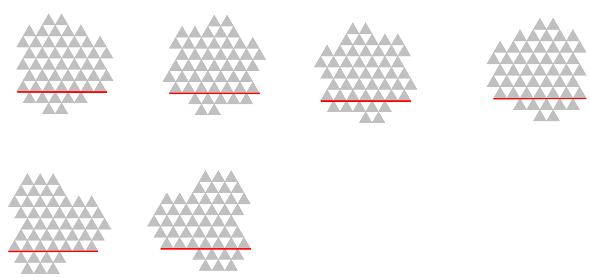


Fig. 5. B-figures for $N = 49, (x, y) = (7, 0)$. $i = 1, 2, \ldots, 6$ from the left top to right bottom.

3 Infinite Series of Curves

We discussed the natural extension of the Gosper curve in [3]. It is a simple procedure to generate infinitely many curves with

$$N = 3k^2 + 3k + 1, \;\; k = 1, 2, 3, \ldots \tag{4}$$

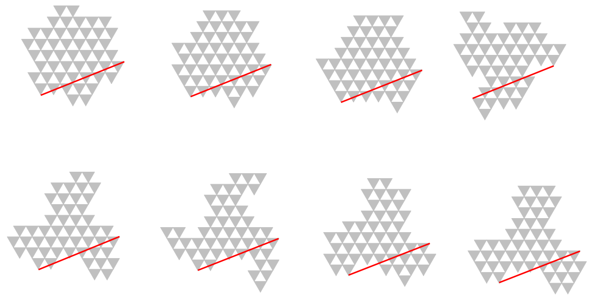


Fig. 6. B-figures for $N = 49, (x, y) = (5, 3)$. $i = 1, 2, \ldots, 8$ from the left top to right bottom.

from the Gosper curve as shown in Fig. 7, where we can find a simple relation

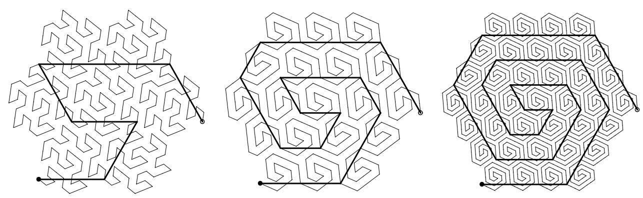


Fig. 7. Infinite series of generalized Gosper curves with $N = 3k^2+3k+1$. $N = 7, 19, 37$.

between successive generators in k. Since the B-figure for $k + 1$ is obtained by adding a layer to the B-figure for k as shown in Fig. 8, we call this procedure *layer method*. In this section, we introduce two new methods to generate infinite series and show that there exist several other infinite series of curves.

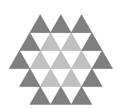


Fig. 8. Layer method.

3.1 Decomposition method

We consider the infinite series from $N = 13$. From the case $N = 7$, if we wrap the B-figure by the layer method, we obtain 31 triangles shown in Fig. 9 (a). By

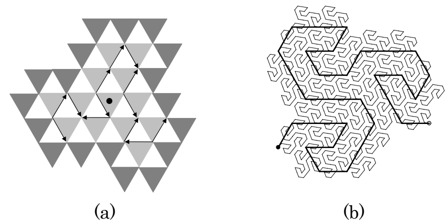


Fig. 9. Wrapped B-figures for $N = 13$ and generalized Gosper curve on the B-figure.

comparing the results for $N = 31$ in [3], we find that this is a B-figure having one generalized Gosper curve shown in Fig. 9 (b). However, it is difficult to find any simple relation between the curve with $N = 13$ and Fig. 9 (b).

We introduce an assembly of hexagons which are obtained by replacing the nodes of the generator by hexagons except two end points as shown in Fig. 10. We call this assembly of $N - 1$ hexagons *T-figure*. From the definitions of the

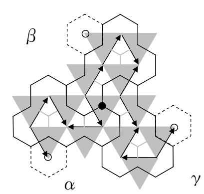


Fig. 10. T-figure for $N = 13$. The filled circle is the three fold rotational axis.

B-figure, the T-figure has three-fold rotational axis. Therefore, the T-figure can be decomposed into three congruent pieces α, β and γ as shown in Fig. 10.

Then we find that we can obtain the B-figure by adding layers of hexagons onto each decomposed T-figure, as shown in Fig. 11, and can construct a path from α to β then γ. In Fig. 11, the dashed hexagon with a circle inside is an endpoint of the generator, and the filled circle is a three-fold rotational axis of the T-figure.

As can be seen from Fig. 10, for $N = 13$, the path in α and γ starts from the dashed hexagon with a circle inside and goes toward the hexagon with three-fold rotational axis. The path in β starts from the dashed hexagon without a

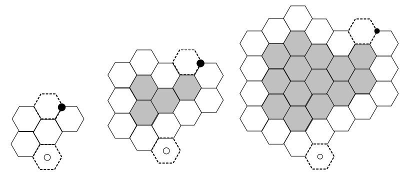


Fig. 11. Decomposed T-figure for $N = 13$ and wrapping process.

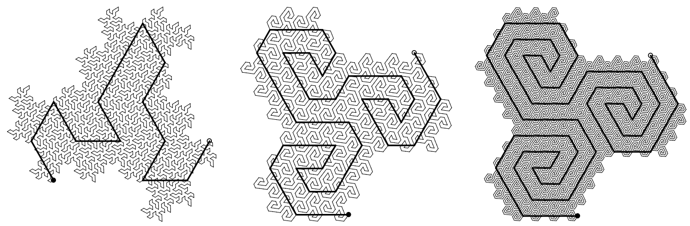


Fig. 12. Infinite series of generalized Gosper curves from $N = 13$ by decomposition method. $N = 13, 43, 91$.

circle and goes toward the hexagon with three-fold rotational axis. The path in successive bigger pieces are obtained similarly. In this way, we get an infinite series from $N = 13$ shown in Fig. 12 with

$$N = 9k^2 + 3k + 1, k = 1, 2, 3, \ldots. \tag{5}$$

We call this method *decomposition method*.

Up to now, we have not found another series generated by the layer method other than the series found in [3]. However, the decomposition method generates several infinite series. For example, the decomposition method works for a curve with $N = 31$ and yields an infinite series shown in Fig. 13 with

$$N = 9k^2 + 15k + 7, \; k = 1, 2, 3, \ldots. \tag{6}$$

The decomposed piece in this case is shown in Fig. 14.

For the generalized Gosper curve with $N = 31$ shown in Fig. 15 (a), a layer is added to the decomposed T-figure in different a way as shown in Fig. 16. In this case, the starting points and terminal points of the path in the decomposed T-figures are also different from the previous two examples.

The path in α and β starts from the dashed hexagon with a circle inside and goes toward the hexagon with a three-fold rotational axis. The path in γ starts from the dashed hexagon without a circle and goes toward the hexagon with a circle. This procedure yields the infinite series shown in Fig. 15 with

$$N = 9k^2 + 15k + 7, \; k = 1, 2, 3, \ldots \tag{7}$$

from $N = 31$.

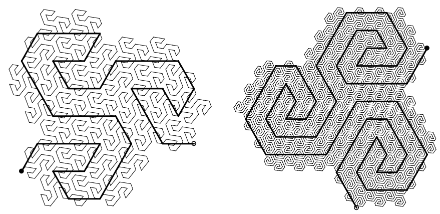


Fig. 13. Infinite series of generalized Gosper curves from $N = 31$ by decomposition method. $N = 31, 73$.

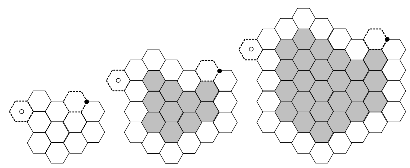


Fig. 14. Decomposed T-figure for $N = 31$ and wrapping process.

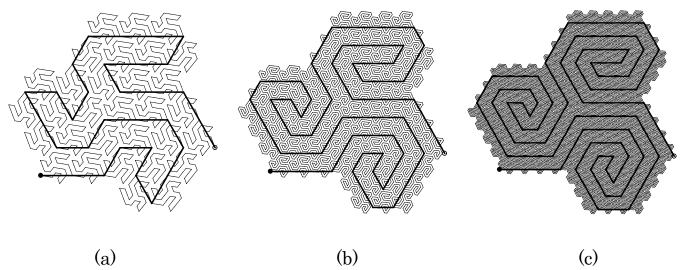


Fig. 15. Infinite series of generalized Gosper curves from $N = 31$ by decomposition method. $N = 31, 73, 133$.

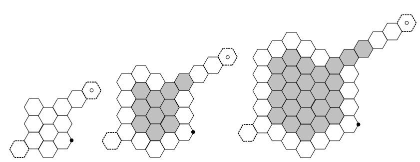


Fig. 16. Decomposed T-figure for $N = 31$ and wrapping process.

3.2 Modified layer method

In Fig. 17, a generalized Gosper curve for $N = 37$ and the corresponding B-figure

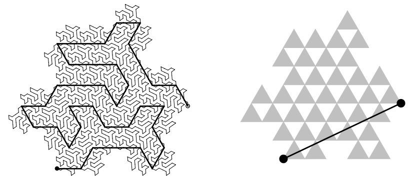


Fig. 17. A generalized Gosper curve for $N = 37$ and its B-figure.

are shown. From this curve, we can generate an infinite series by modifying the layer method as follows.

1. It is impossible to wrap the B-figure in Fig. 17 by a single layer as shown in Fig. 18 (a).

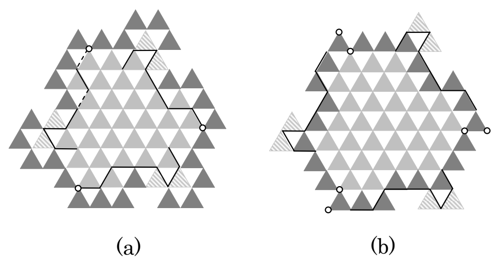


Fig. 18. Modified layer for $N = 37$.

2. We modify by removing the striped triangles from the B-figure and wrap it by a single layer as shown in Fig. 18 (b).
3. After wrapping the modified B-figure, we add the removed triangles (striped triangles) as shown in Fig. 18 (b).
4. Then we can connect a starting point of the new initiator and a terminal point of the old initiator by a path, and a terminal point of the new initiator and a starting point of the old initiator by a path.
5. It is possible to continue this procedure to obtain a new infinite series with

$$N = 3k^2 + 15k + 19,\ k = 1, 2, 3, ... \quad (8)$$

starting from $N = 37$ as shown in Fig. 19.

4 Summary

In this paper, we reported on new results of computer search for generalized Gosper curves for $37 < N < 61$. We then observed that the empirically allowed

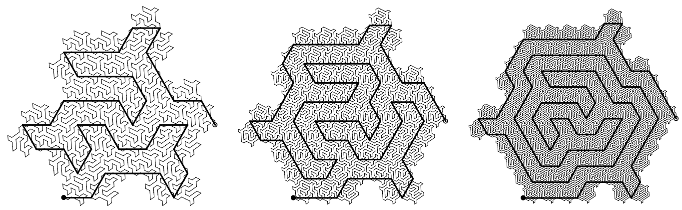


Fig. 19. Infinite series of generalized Gosper curve from $N = 37$ by modified layer method. $N = 37, 61, 91$.

degree N for generalized Gosper curves seems to be $N = 6n + 1$. The proof of this conjecture looks complex and lengthy. It is left for a future work.

Finally, we pointed out that there exist several infinite series of generalized Gosper curves generated by three methods we introduced: the layer method, the decomposition method, and the modified layer method. However, so far as our investigation is concerned, several other series and methods exist and more than three quarters of generalized Gosper curves can generate infinitely many curves. For example, an infinite series with

$$N = 9k^2 + 3k + 1 \tag{9}$$

shown in Fig. 2 can not be generated by the above three method. Thus more systematic investigation of infinite series for every generalized Gosper curve is desired.

References

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