Three-Dimensional Development of a Hyper-Cube in Four Dimensions

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Abstract: We report on three-dimensional development of a hyper-cube in four dimensions. There are 261 distinct development figures and all the figures are shown.

1 Introduction

The Inhabitants of Flatland (Abbott, 1991) can get an idea of solid figures by general methods, projection, section and development of them. Though projection may be a more familiar method for them to recognize solid figures, they can also recognize them by sections and developments of them. We have investigated all possible sections of hyper-cube (Coxeter, 1973) in four and five dimensions (Nakamura, 1980 and Fukuda *et al.*, 1997). In this paper, we report on three-dimensional development of a hyper-cube in four dimensions. In section 2, we describe a method to develop a polyhedron in three dimensions. In section 3, we apply the method to a hyper-cube in four dimensions.

2 Three dimensions

A Cube in three dimensions has eleven distinct figures of development, as shown in

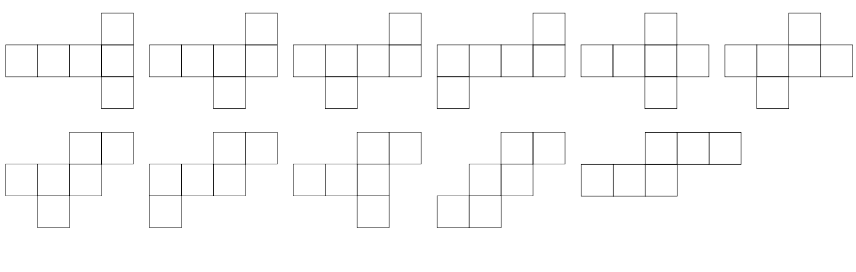


Fig. 1. Eleven figures of development of a cube.

Fig.1. These figures can be obtained by the following steps. For explanation we develop the cube shown in Fig.2, and label the faces from F0 to F5.

(3D.1) Pick up a face, face F0 here, of the cube and place it on a plane.

(3D.2) List the faces, as indicated by the dashed lines in Fig.3, adjacent to the faces placed in previous steps by referring the cube in Fig.2. Select a face from the above faces, and pick it up from the cube then put it in the position chosen on the plane so that the vertices coincide with those of previous face(s).

(3D.3) Continue the procedure (3D.2) until all faces are used as shown in Fig.3, and then obtain one of the figures of development as shown in Fig.1.

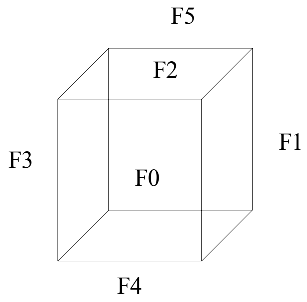


Fig. 2. Cube.

(3D.4) If all possible choices in step (3D.2) are examined, a huge number of figures of development are obtained. Some of them may coincide with each other. Since the cube is an object in three dimensions, figures of their development should be considered identical if they are identical in three dimensions. After reducing the identical figures, we obtain eleven distinct figures as shown in Fig.1.

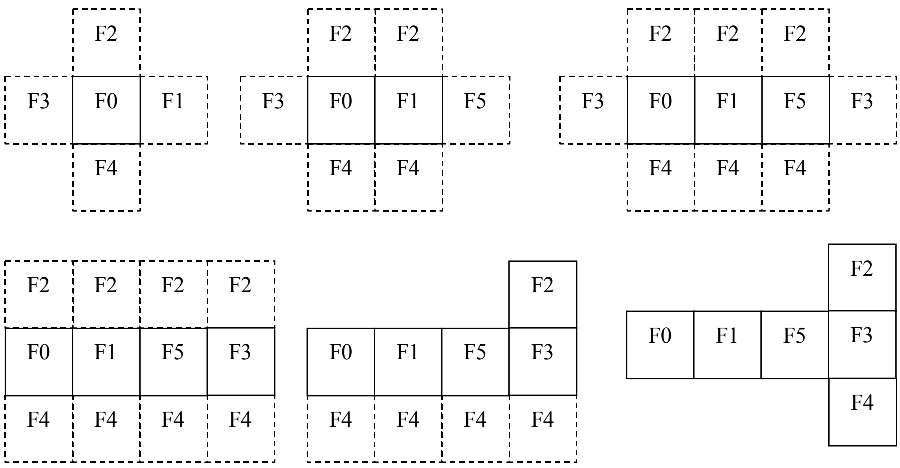


Fig. 3. Intermediate figures of one development of a cube.

This procedure can be used to obtain all the figures of development by computer and is applicable to any polyhedron in three dimensions. We have obtained the total number of distinct figures of development for regular polyhedra, 2 for a regular tetrahedron and 11 for a regular octahedron. The number is larger than 10,000 for a regular dodecahedron and for a regular icosahedron.

3 Four dimensions

The procedure of development for three dimensions is also applicable to higher dimensions. We develop the hyper-cube shown in Fig.4, the schematic figures of the

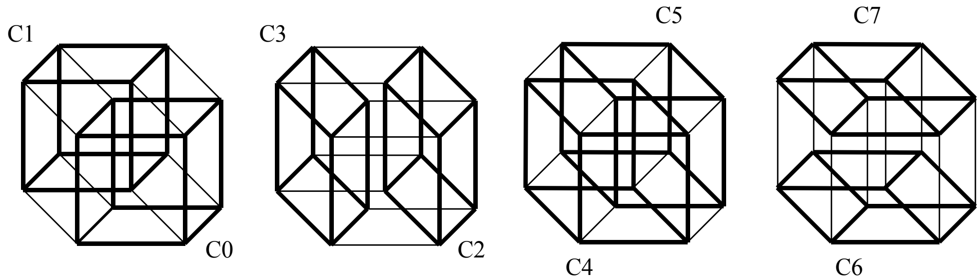


Fig. 4. Schematic figures of the projection of a hyper-cube.

projection of a hyper-cube, and explain each step in four dimensions in detail. We label eight cells from C0 to C7. Four figures in Fig.4 represent the same hyper-cube and each figure indicates two cells by the bold lines.

(4D.1) Pick up a cell of the hyper-cube, C0 here, and put it in three dimensions.

(4D.2) List the cells, as indicated by the dashed lines in Fig.5, adjacent to the cells placed in previous steps by referring the hyper-cube in Fig.4. Select a cell from the above cells, and pick it up from the hyper-cube then put it in the position chosen in three dimensions so that the vertices coincide with those of previous cell(s). The cell chosen can be mirror-inverted when it is placed in three dimensions. This corresponds to the fact that the face of a cube in three dimensions can be placed on a plane in two ways, face up or face down. In Fig.5, cells indicated by prime are the mirror-inversion of corresponding cells shown in Fig.4. Here we choose C2, which should be positioned after mirror-inversion compared with the C2 in Fig.4.

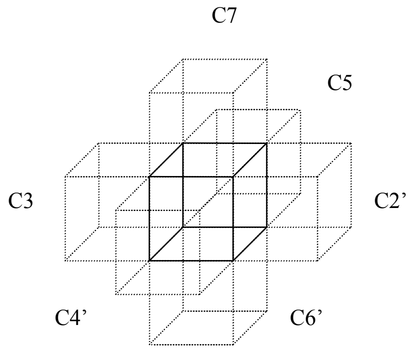


Fig. 5. Intermediate solid figure.

(4D.3) Continue the process above until all cells are used, and then we obtain a solid figure of development as shown in Fig.6.

(4D.4) If all possible choices in step (4D.2) are examined, a huge number of solid figures of development are obtained. Since the hyper-cube is an object in four dimensions, figures of their development should be considered identical if they are identical in four dimensions. This means that two solid figures of development should be considered identical if they or their mirror-inversions are identical in three dimensions, because rotation in four dimensions corresponds to both rotation and mirror inversion in three dimensions. After reducing the identical solid figures in this way, we obtain 261 distinct solid figures as shown in Fig.6. If we distinguish two solid figures of mirror-inversion, the number of distinct figures becomes 455.

In Fig.6, the faces of development are colored according to the 24 faces of the hyper-cube. A developed hyper-cube can be constructed by joining the adjacent faces with the same color. All these solid figures are exhibited on a homepage (Fukuda, 1998) using the Virtual Reality Modeling Language (VRML, 1995).

In computer calculation, we can streamline the procedure (4D.2). If an intermediate solid figure, such as that shown in Fig.5, coincides with one that already appeared, we do not have to complete the process for that figure. This is because identical intermediate solid figures will yield identical results.

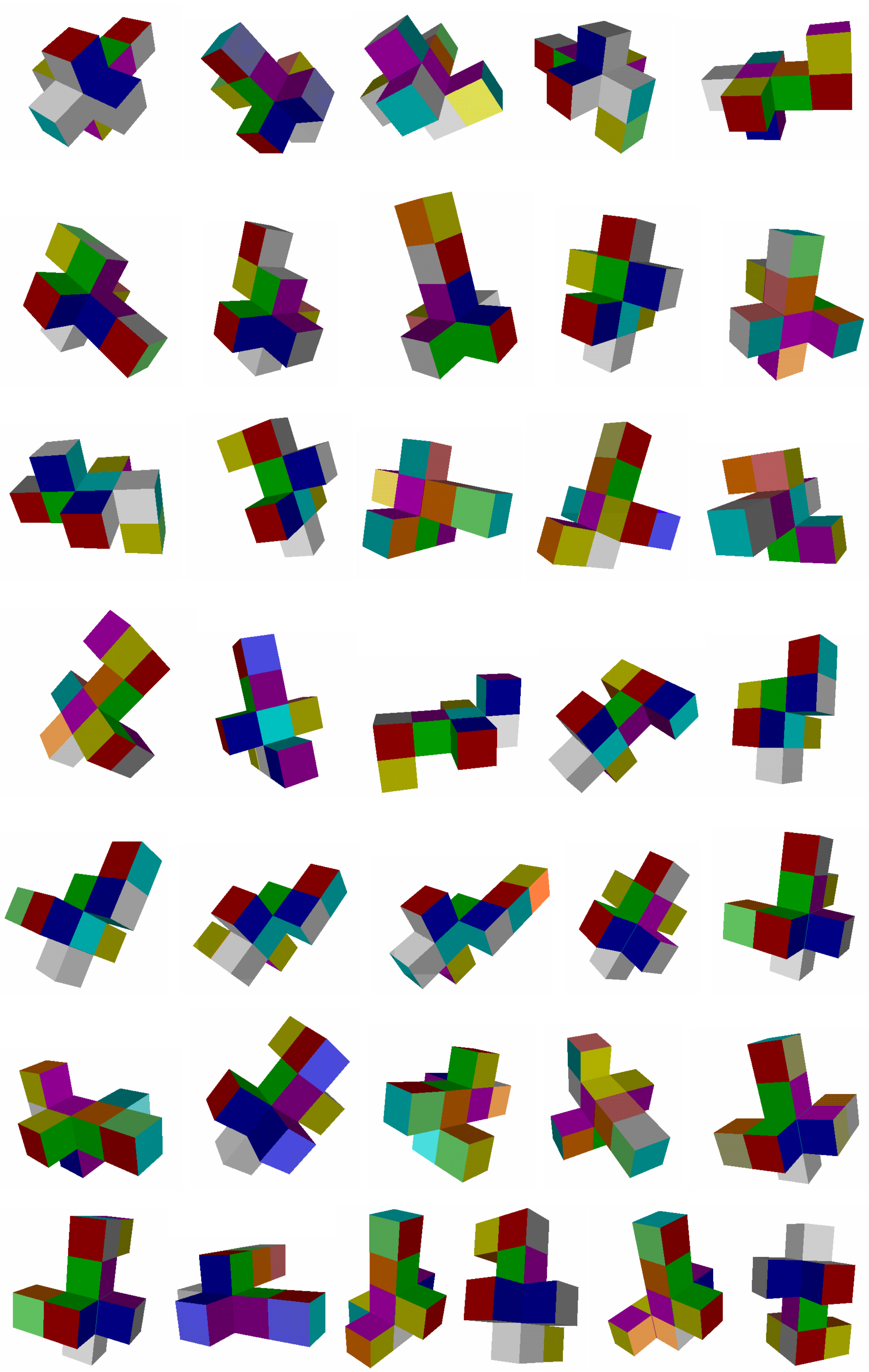


Fig. 6. 261 figures of development of a hyper-cube.

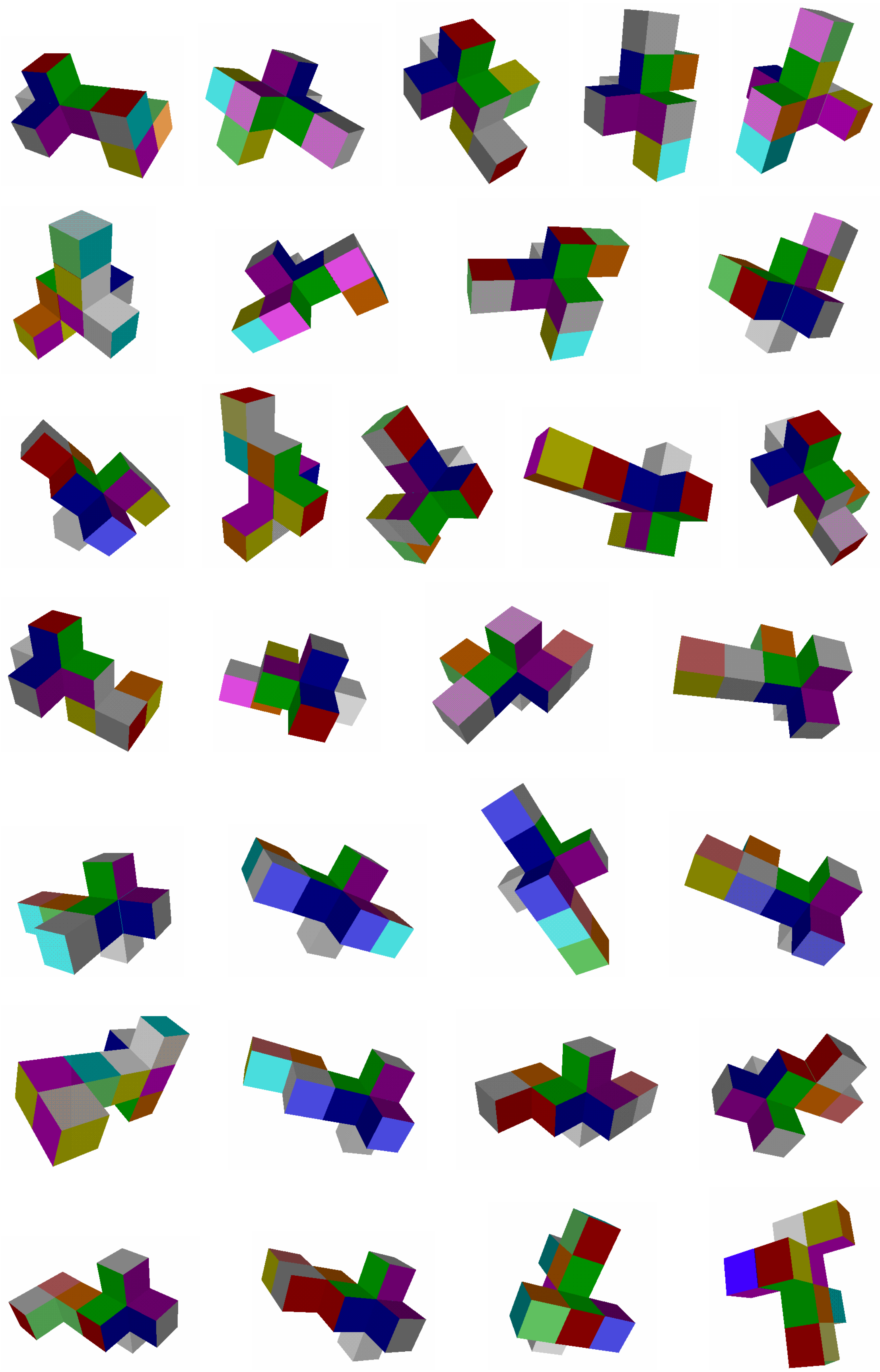
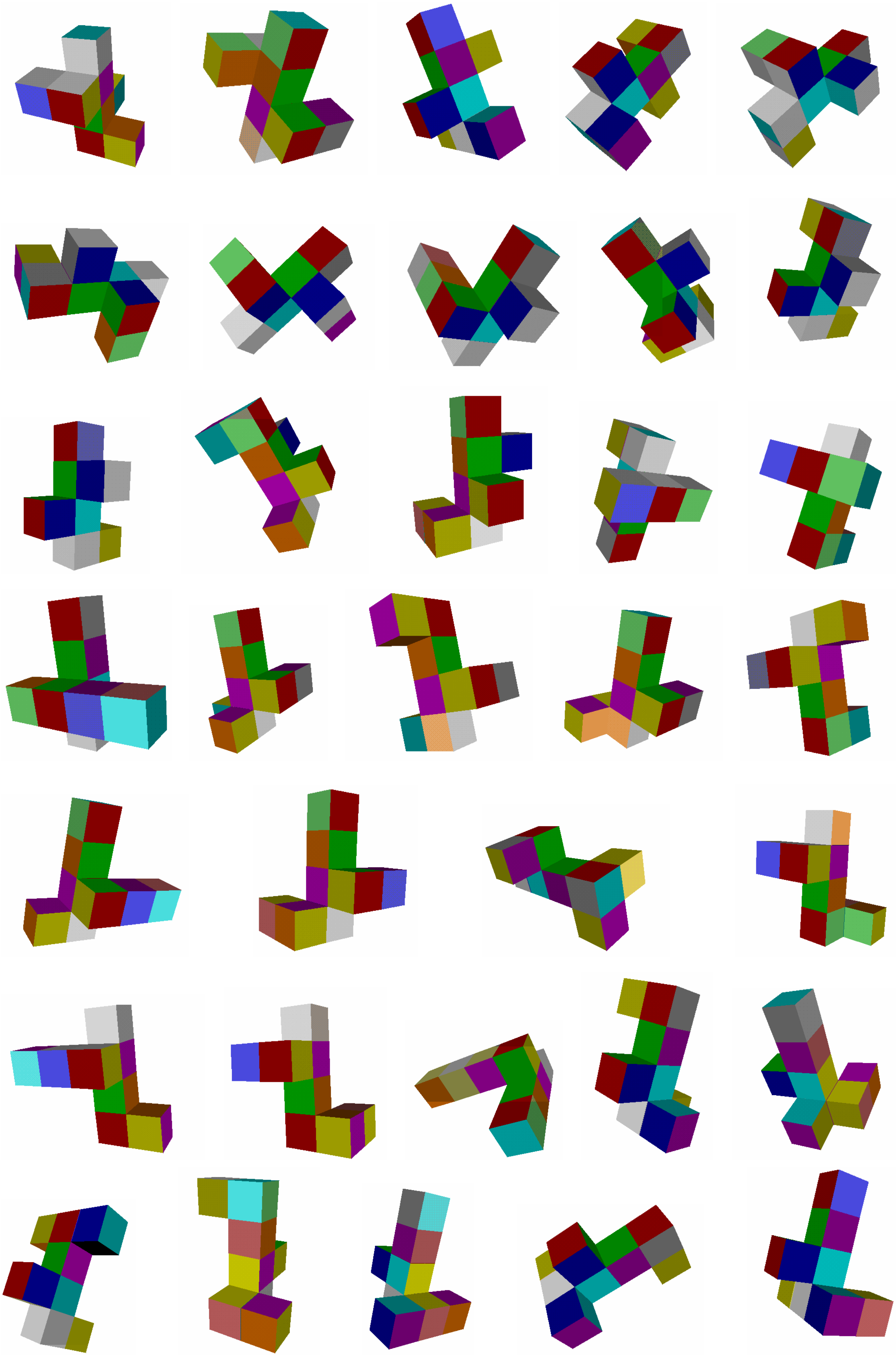


Fig. 6. (continued).



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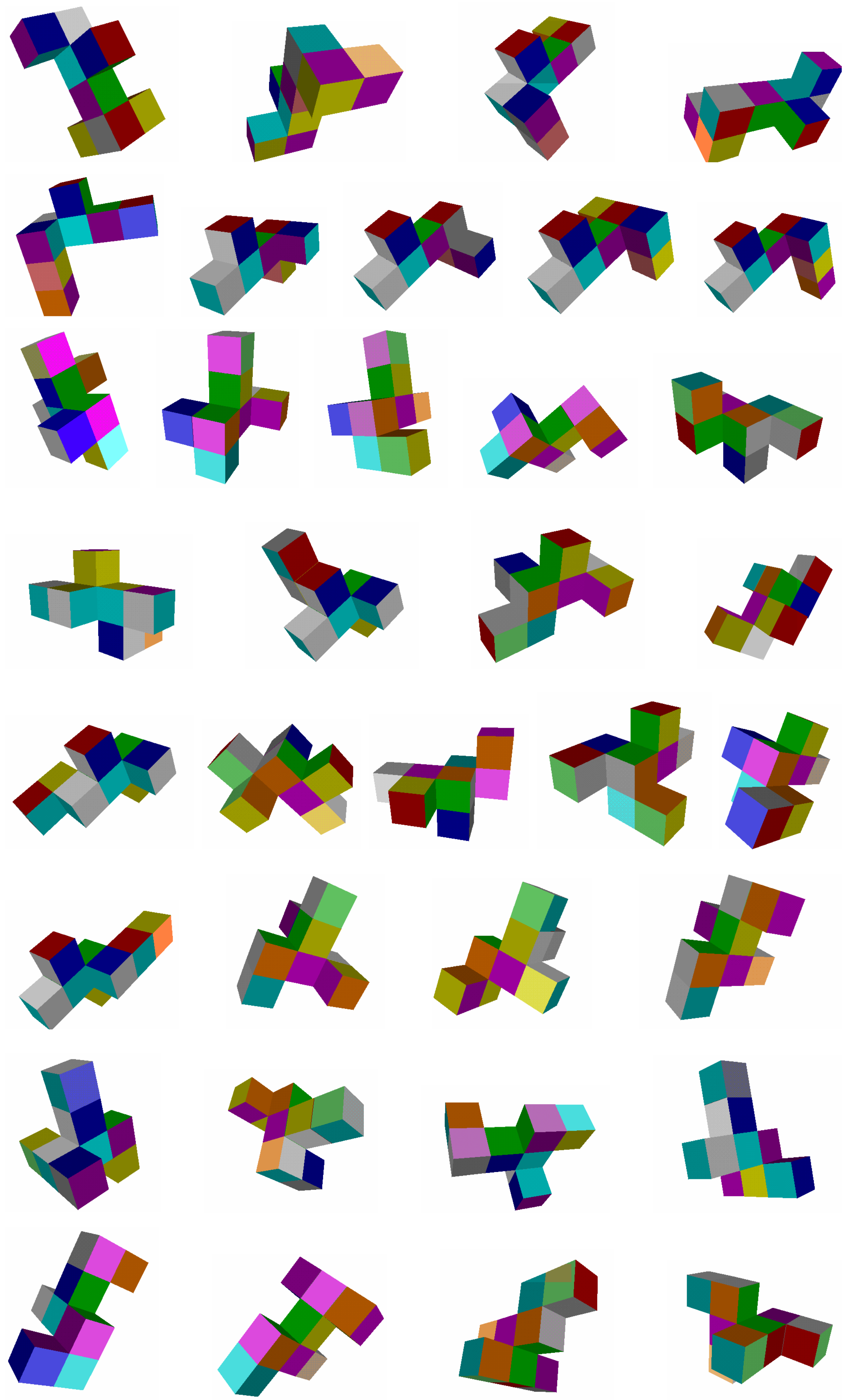

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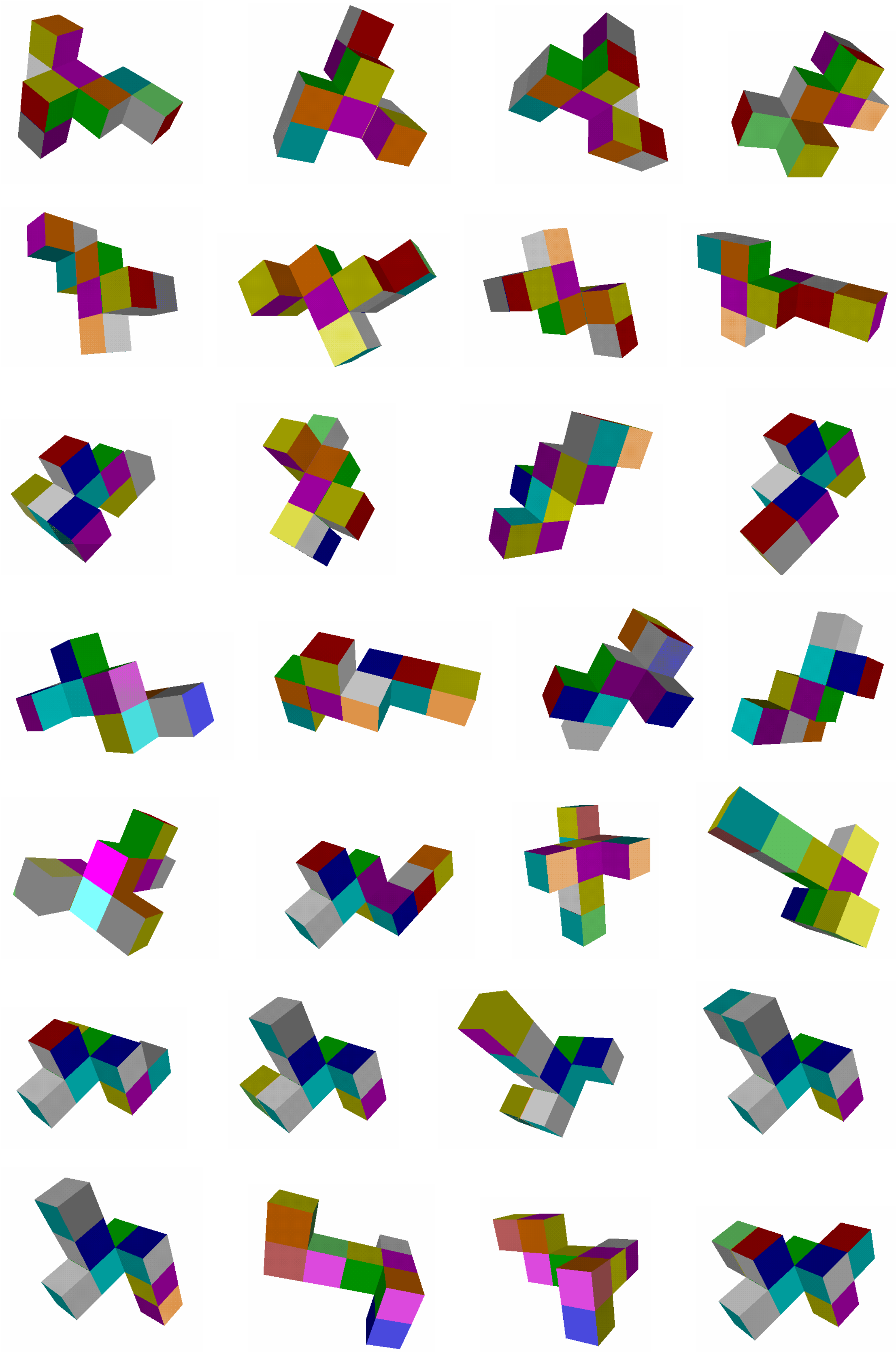

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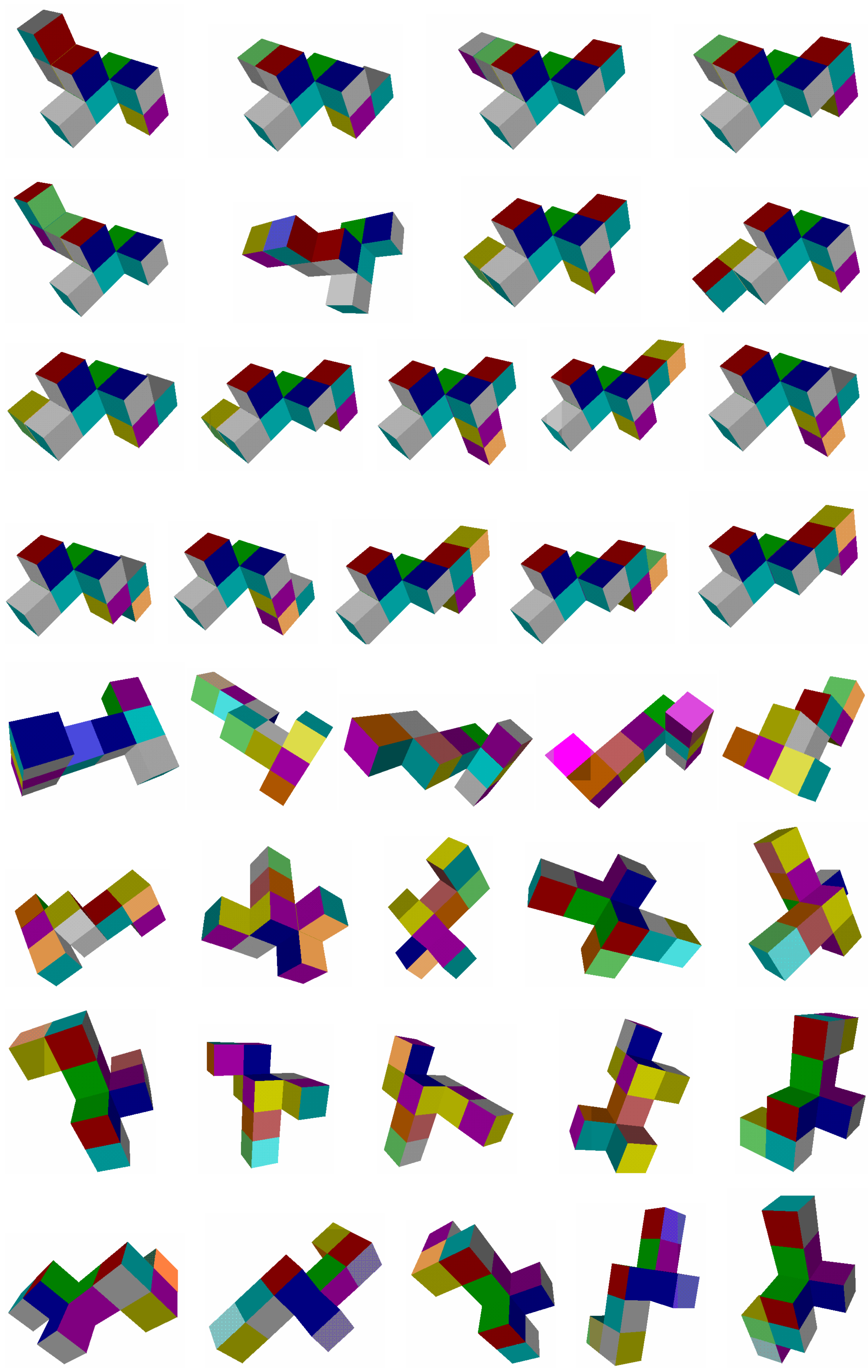

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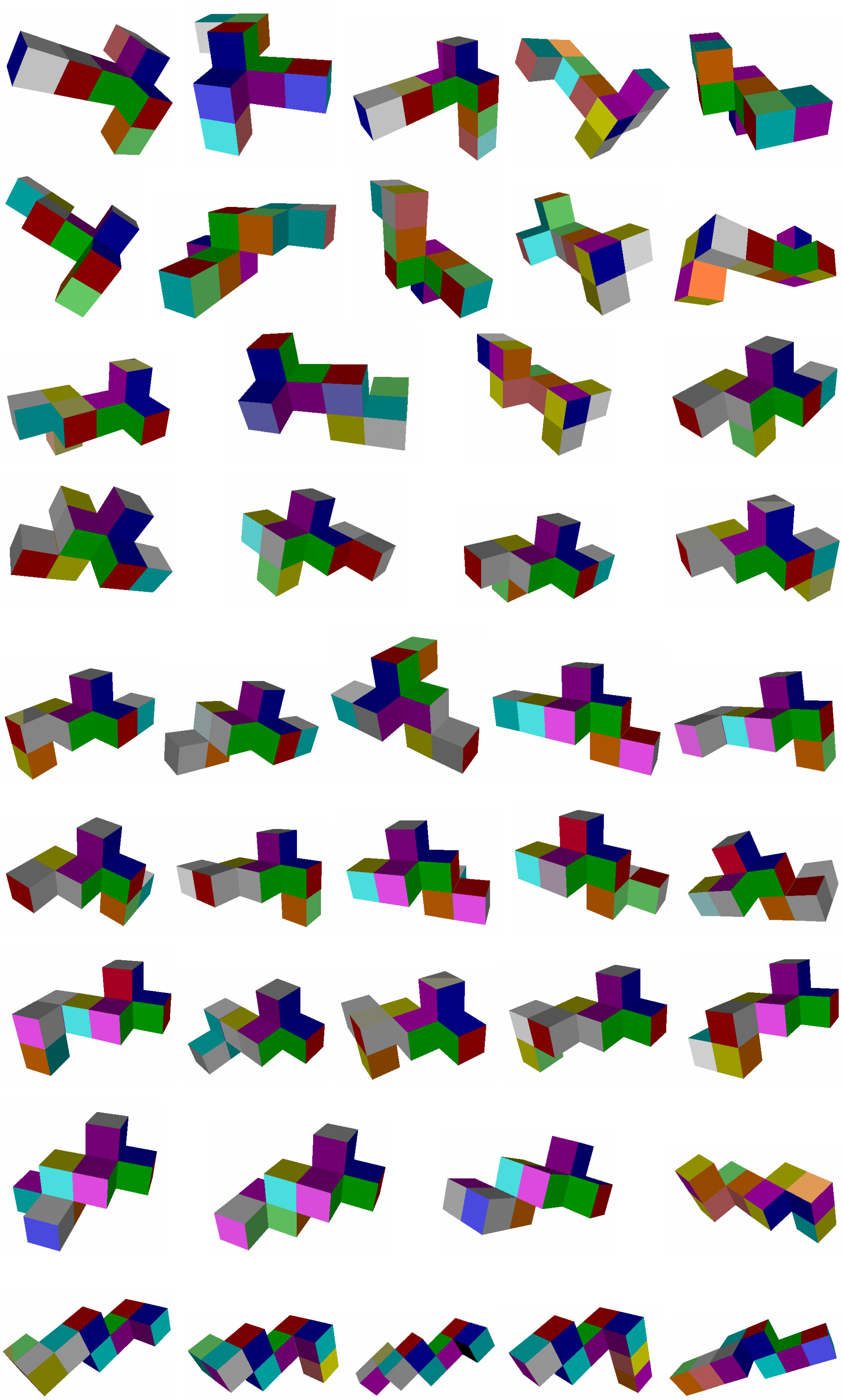

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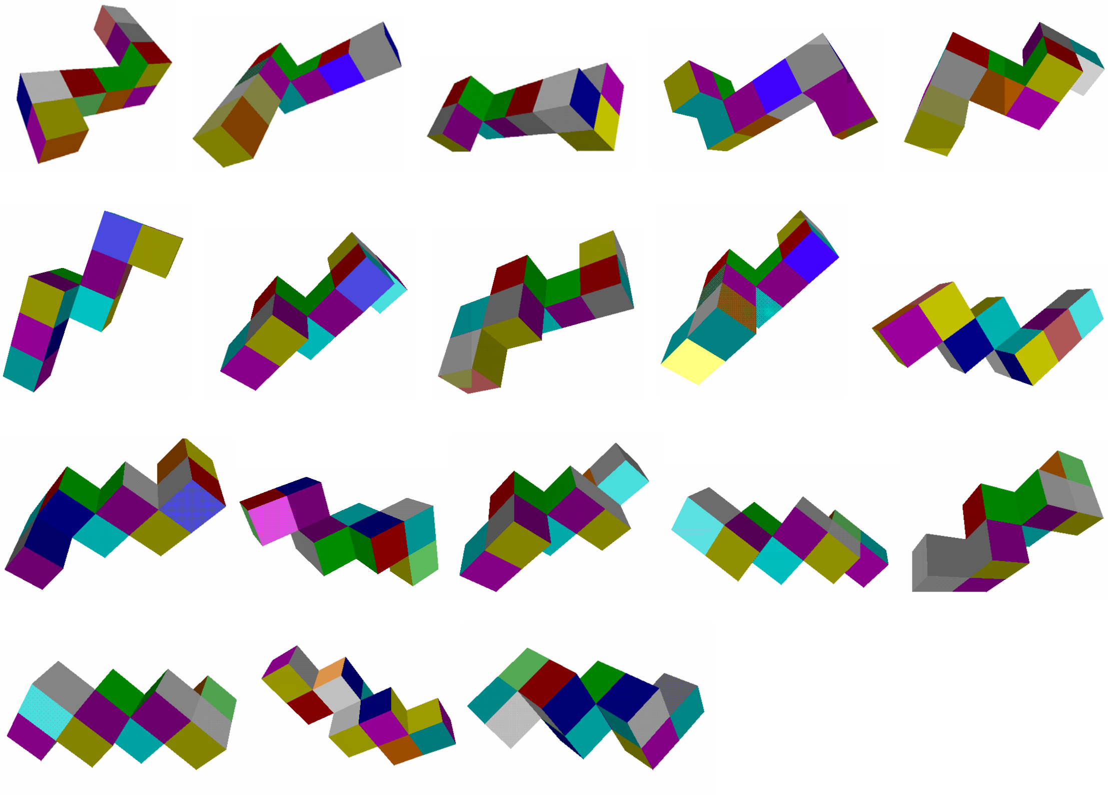

Fig. 6. (continued).

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