Universal Measuring Devices Without Gradations

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Abstract. Measuring cups are everyday instruments used to measure a required amount of liquid for many common tasks such as cooking, . . . etc. A measuring cup usually has gradations marked on its sides. In this paper we study measuring devices ungraded but which nevertheless can measure any integral amount of liquid up to their full capacity. These devices will be called *universal measuring devices*.

1 Introduction

A common device used to measure liquid in many Japanese stores some years back, was a measuring box with a square base of area 6 and height 1. By tilting the box, and using its edges and vertices as markers, it is possible to keep 6, 3, and 1 litters as shown in figure 1. The box would have no extra gradings to measure 2, 4, and 5 liters. A store would keep a container holding large amounts of a certain liquid.

If a customer wanted to buy a certain integral amount of liquid between 1 and 6, the store owner would proceed as follows:

Procedure SERVE

1. The store clerk would immerse his measuring box into the store's container, and fill it.
2. Then he would alternately empty a certain amount of liquid into the store's container and the customer's.

For example to measure 5 liters, the clerk would first fill the container and then pour into the customer's container until 1 liter of liquid was left in the measuring box. The remaining liquid would go back to the store's container. To

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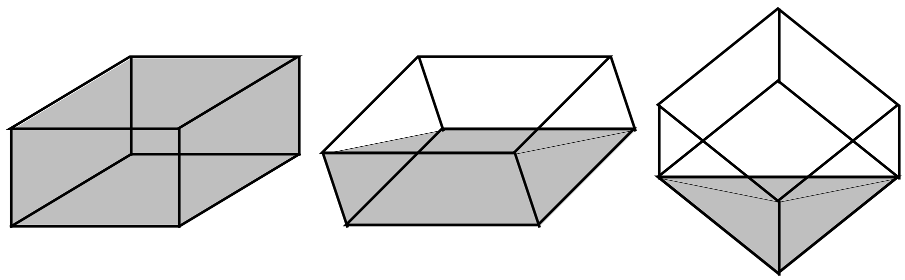

Fig. 1. Measuring 6, 3 and 1 liters

measure 4 liters, he would first fill our box. Next he would pour liquid into the customer's container until 3 liters were left into his box. Then he would pour two liters into the store's container, and put the remaining liter into the customer's container.

In this paper we are interested in studying measuring devices without gradations which nevertheless can be used for measuring any integral amount of liquid up to their full capacity by using the procedure SERVE described above. We call such measuring boxes *universal measuring device*. We determine the largest volume of a universal measuring device *with triangular* and *rectangular* bases. More precisely, we want to determine the dimensions of a universal measuring box of maximum volume obtained by cutting a triangular cylinder perpendicular to the x–y plane by the x–y plane (to produce the base of our container), and another plane that cuts the cylinder above its base. In the case of rectangular cilinders we cut by perhaps two parallel planes above its base to obtain the container. We will assume that the lengths of the edges of our box contained in the original triangular or rectangular cylinder are $h_1 < h_2 < h_3$ or $h_1 < h_2 < h_3 < h_4$ respectively. These will be called the heights of our box. We show that the largest possible volume of such boxes are 41 and 691 respectively, if the areas of the base of these boxes are 3 and 6, and their heights are 12, 13, 16, and $1,\ 32,\ 83,\ 691$ respectively.

2 Universal containers with triangular base

In this and the next sections we prove that the maximum volume of a universal measuring box with triangular base of area 3 is 41. The assumption that the area of the containers base is 3 can be removed easily to study other triangular base boxes. This assumption will simplify our analysis as the reader will soon realize. Under this restriction the heights of a universal measuring box with triangular base are $12, 13$, and 16.

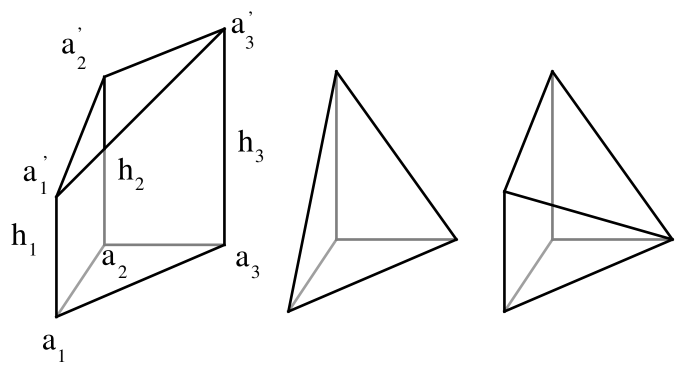

Fig. 2. The volumes of the polyhedron shown here are $h_1 + h_2 + h_3$, h_2 and $h_1 + h_2$ respectively.

2.1 An interesting relationship

We start by determining the volumes that can be measured using corner points of our container. We start by proving the following theorem.

Theorem 1. *Let $\mathcal{B}$ be a box with base area 3, and heights $h_1 < h_2 < h_3$. Then we can measure the following amounts of liquid: $h_1,\ h_2,\ h_3,\ h_1 + h_2,\ h_1 + h_3,\ h_2 + h_3,\ h_1 + h_2 + h_3$.*

Proof: Assume that the vertices of $\mathcal{B}$ are labelled $a_1,\ a_2,\ a_3,\ a_1',\ a_2',\ a_3'$ as shown in figure 2. Assume that the distance between a_i and a_i' is h_i, $i = 1, 2, 3$. It is well known that the volume of a tetrahedron with base of area A and height h is $\frac{Ah}{3}$. Since the area of the base of our measuring box, i.e. the area of the triangle with vertices $a_1,\ a_2,\ a_3$ is 3, it follows immediately that the volume of the tetrahedron with vertices $a_1,\ a_2,\ a_3,\ a_i'$ is h_i, $i = 1, 2, 3$.

We now show that the volume of the polyhedron with vertices a_1, a_2, a_3, a_i', and a_j' is precisely $h_i + h_j$, $i \neq j$, $i,\ j \in \{1, 2, 3\}$. We make now the following observation:

Let $\mathcal{T}$ be a triangular cylinder, and assume that the area of the triangle obtained by cutting $\mathcal{T}$ along any plane perpendicular to its edges is 3. Then if we choose two points $p,\ q$ on one of its edges, and any two points r and s, one in each of the remaining edges of $\mathcal{T}$, then the volume of the tetrahedron with vertices $p,\ q,\ r,\ s$ is equal to the distance between p and q, see figure 3.

Suppose w.l.o.g. that $i = 1,\ j = 2$. Dissect now the polyhedron with vertices $a_1,\ a_2,\ a_3,\ a_1',\ a_2'$ into two tetrahedron with vertices $a_1,\ a_2,\ a_3,\ a_1'$ and $a_2,\ a_3,\ a_1',\ a_2'$ respectively, see figure 4. The volume of our first tetrahedron is h_1, and by our previous observation the area of the second one is the distance from a_2 to a_2' which is h_2.

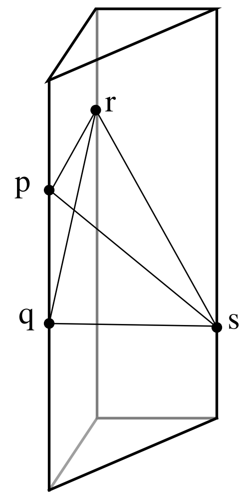

Fig. 3. Once we fix p and q, the volumes of the tetrahedron with vertices $p,\ q,\ r$, and s remains constant regardless of the positions of r and s.

Using similar arguments, we can now show that the volume of our measuring box is $h_1 + h_2 + h_3$, and our result is proved. $\square$

3 The second reduction

Using our previous result, it follows that if $h_1 = 12$, $h_2 = 13$, and $h_3 = 16$ we can measure the following amounts, sorted in increasing order: $l_1 = h_1 = 12$, $l_2 = h_2 = 13$, $l_3 = h_3 = 16$, $l_4 = h_1 + h_2 = 25$, $l_5 = h_1 + h_3 = 28$, $l_6 = h_2 + h_3 = 29$, and $l_7 = h_1 + h_2 + h_3 = 41$.

Consider now any subsequence $S = (l_{i_1} < \cdots < l_{i_k})$ of the sequence

$$(12, 13, 16, 25, 28, 29, 41).$$

We can now associate to it an amount of liters V_S that can be measured with a container with heights $12, 13$, and 16 as follows:

$$V_S = (l_{i_k} - l_{i_{k-1}}) + \cdots + l_{i_1}, \quad \text{for } k \text{ odd}$$

or

$$V_S = (l_{i_k} - l_{i_{k-1}}) + \cdots + (l_{i_2} + l_{i_1}), \quad \text{for } k \text{ even}.$$

For example if $S = (16,\ 29,\ 41)$, $V_S = (41-29)+16 = 28$. If $S = (12,\ 16,\ 25,\ 28)$ then $V_S = (28-25) + (16-12) = 7$. From this, it is straightforward to develop

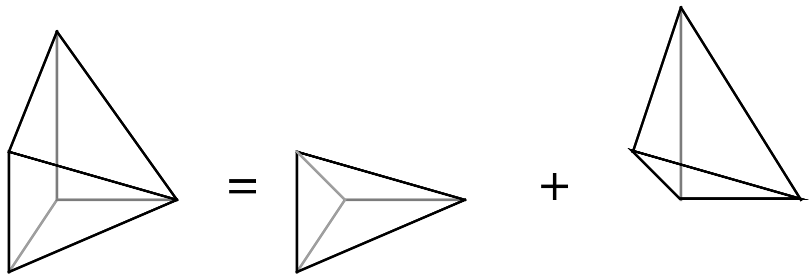

Fig. 4. Dissecting the polyhedron with vertices $\{a_1,\ a_2,\ a_3,\ a_1',\ a_2'\}$ into two tetrahedra with volumes h_1 and h_2 respectively.

a method to measure V_S liters using our container. Two examples will suffice to illustrate this:

If $S = (12,\ 16,\ 25,\ 28)$, to measure $V_S = 7$ liters we proceed as follows: First fill our measuring box, and pour back $41 - 28$ liters into the store's container. Now proceed as follows: empty $3 = 28 - 25$ liters into the customer's container until 25 liters are left in our measuring box. Then pour $9 = (25 - 16)$ liters into the store's container. Next pour $4 = 16 - 12$ liters into the customer's container and finally empty the remaining 12 liters into the store's container.

If $S = (16,\ 29,\ 41)$, to measure $V_S = 28$ liters, we fill our container, then empty $12 = 41 - 29$ liters into the customer's container. Then pour $13 = 29 - 16$ liters into the store's container, and finally pour the remaining 16 liters into the customer's container. The reader may now easily verify that any integer i, $1 \leq i \leq 41$ there is a subset S of $(12, 13, 16, 25, 28, 29, 41)$ such that $i = V_S$. In the next section we develop an easy test to verify this fact.

Observe that since there are exactly 2^7 subsets of $\{h_1, h_2, h_3, h_1 + h_2, h_1 + h_3, h_2 + h_3, h_1 + h_2 + h_3\}$ it follows that the maximum amount of liters a universal container with triangular base can measure is $2^7 - 1 = 128 - 1$, the empty set corresponds to 0 liters.

In the next section we show that the largest volume a universal measuring box with triangular base can have is 41, thus proving that $12, 13, 16$ are the heights of an optimal universal measuring box with triangular base of area 3.

3.1 Proof of optimality

We now prove that the maximum volume of a universal measuring box with triangular base is 41. Note first that there are only two possible orderings for the measurable quantities of such a box. These are:

$$0 < h_1 < h_2 < h_3 < h_1 + h_2 < h_1 + h_3 < h_2 + h_3 < h_1 + h_2 + h_3, \qquad (1)$$

and

$$0 < h_1 < h_2 < h_1 + h_2 < h_3 < h_1 + h_3 < h_2 + h_3 < h_1 + h_2 + h_3. \tag{2}$$

Consider the second ordering now, and take the sequence formed by the difference of consecutive elements for this ordering: $s_1 = h_1 - 0$, $s_2 = h_2 - h_1$, $s_3 = h_1$, $s_4 = h_3 - (h_1 + h_2)$, $s_5 = h_1$, $s_6 = h_2 - h_1$, and $s_7 = h_1$.

Observe next that for any subsequence S of

$$(h_1, h_2, h_1 + h_2, h_3, h_1 + h_3, h_2 + h_3, h_1 + h_2 + h_3) \tag{3}$$

we can express V_S as a sum of some subset of $(s_1, \ldots, s_7)$, e.g. if

$$S = (h_1, h_1 + h_2, h_1 + h_3, h_1 + h_2 + h_3)$$

then

$$V_S = ((h_1 + h_2 + h_3) - (h_1 + h_3)) + ((h_1 + h_2) - h_1)$$

equals

$$s_7 + s_6 + s_3 + s_2.$$

It is also straightforward to see that to any subset of $\{s_i;\ i = 1, \ldots, 7\}$ we can associate a unique subsequence of

$$(h_1, h_2, h_1 + h_2, h_3, h_1 + h_3, h_2 + h_3, h_1 + h_2 + h_3).$$

Thus the number of integers V_S that can be formed with subsequences of

$$(h_1, h_2, h_1 + h_2, h_3, h_1 + h_3, h_2 + h_3, h_1 + h_2 + h_3)$$

equals the number of elements that can be obtained as sums of subsets of $\{s_i;\ i = 1, \ldots, 7\}$.

Let us denote by $a = h_1$, $b = h_2 - h_1$ and $c = h_3 - (h_1 + h_2)$. Note that any number obtained as the sum of a subset of $\{s_i; i = 1, \ldots, 7\}$ must be of the form $ai + bj + ck$, where $i \in \{0, 1, 2, 3, 4\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1\}$. This is because $h_1, h_2 - h_1$ and $h_3 - (h_1 + h_2)$ repeat themselves in the set of differences 4, 2 and 1 times respectively. It follows now that the maximum capacity of a box in this case is $5 \times 3 \times 2 - 1 = 29$.

Now we consider the sequence

$$h_1, h_2 - h_1, h_3 - h_2, h_1 + h_2 - h_3, h_3 - h_2, h_2 - h_1, h_1,$$

of differences of consecutive elements for (1).

Similarly to the previous paragraph, denote by $a = h_1$, $b = h_2 - h_1$, $c = h_3 - h_2$ and $d = (h_1 + h_2) - h_3$. In this case we have that any number generated by (1) is one of the form $ai + bj + ck + d\ell$, where $i, j, k \in \{0, 1, 2\}$ and $\ell \in \{0, 1\}$. Again this is because h_1, $h_2 - h_1$, $h_3 - h_2$ and $(h_1 + h_2) - h_3$ repeat themselves in the sequence of differences 2, 2, 2 and 1 times respectively. Thus a first upper bound for the number of generated numbers when using a sequence like (1) is $3 \times 3 \times 3 \times 2 - 1 = 53$.

We now prove that 12 of these quantities are repeated. To this end note that $a - c = d$. Thus it is true that

$$ai + bj + ck + d\ell = a(i+1) + bj + c(k-1) + d(\ell - 1),$$

whenever $i \in \{0,1\}, k \in \{1,2\}, \ell = 1$ and $j \in \{0,1,2\}$. Therefore the number of repetitions is 12, and therefore using this case we can generate at most $53 - 12 = 41$ numbers. Our result now follows since using the box with triangular base of area 3 and heights $B = (12, 13, 16)$ we can measure from 1 to 41 liters.

4 Generating sequences

Let $\Sigma = (n_1, \ldots, n_k)$ be a sequence of numbers such that $n_i < n_{i+1}$, $i = 1, \ldots, k-1$. For each subsequence $S = (n_{i_1}, \ldots, n_{i_j})$ of Σ we associate the number $V_S = (n_{i_j} - n_{i_{j-1}}) + (n_{i_{j-2}} - n_{i_{j-3}}) + \ldots + n_{i_1}$, j odd, or $n_S = (n_{i_j} - n_{i_{j-1}}) + (n_{i_{j-3}} - n_{i_{j-4}}) + \ldots + (n_{i_2} - n_{i_1})$, j even.

For example if $\Sigma = (4,\ 12,\ 23,\ 24,\ 69,\ 71,\ 81,\ 213,\ 225,\ 282\ )$ and $S = (12,\ 69,\ 81, 282)$ then $V_S = (282 - 81) + (69 - 12)$

We call Σ a *generating sequence* if for each integer i, $1 \leq i \leq n_k$ there is a subsequence S of Σ such that $i = V_S$. The best known generating sequences are the ones corresponding to powers of 2, i.e. $(1, 2, 4, \ldots, 2^k)$.

In this section we develop a linear time test to decide if a sequence of numbers is generating.

Let $\Sigma' = (0, n_1, \ldots, n_k)$, and if $s_0 = 0$, let $\Psi = (s_0, s_1, \ldots, s_k)$ be the sequence obtained by sorting the sequence $(n_i - n_{i-1}, i = 1, \ldots, k)$, this is considering $n_0 = 0$. For example if Σ is the sequence

$(4,\ 12,\ 23,\ 24,\ 69,\ 71,\ 81,\ 213,\ 225,\ 282\ )$

we have

$\Psi = (0\ ,1\ ,\ 2,\ 4,\ 8,\ 10,\ 11,\ 12,\ 45,\ 57,\ 132).$

We now prove:

Theorem 2. *The sequence Σ is generating if and only if $s_i - 1 \leq s_0 + \cdots + s_{i-1}$, $i = 1, \ldots, k$.*

Using our previous result we can now easily verify that

$$\Sigma = (4,\ 12,\ 23,\ 24,\ 69,\ 71,\ 81,\ 213,\ 225,\ 282\ )$$

is generating since

$2 - 1 = 1$
$4 - 1 = 1 + 2$

$8-1 = 1+2+4$
$10-1 < 1+2+4+8$
$11-1 < 1+2+4+8+10$
$12-1 < 1+2+4+8+10+11$
$45-1 < 1+2+4+8+10+11+12$
$57-1 < 1+2+4+8+10+11+12+45$
$132-1 < 1+2+4+8+10+12+45+57$

We proceed now to prove our result. Let $\Sigma = (n_1, \ldots, n_k)$. Observe that $n_i - n_j = (n_i - n_{i-1}) + (n_{i-1} - n_{i-2}) + \cdots + (n_{j+1} - n_j)$. Thus for any subsequence $S = (n_{i_1}, \ldots, n_{i_j})$ of Σ, V_S can be written as a sum of elements of Ψ, e.g. in our previous example if

$$S = (23, 24, 69, 81, 282),$$

$$N_S = (282 - 81) + (69 - 24) + 23,$$

which equals,

$$(282-225) + (225-213) + (213-81) + (69-24) + (23-12) + (12-4) + (4-0),$$

i.e.,

$$V_S = 57 + 12 + 132 + 45 + 11 + 8 + 4.$$

This implies that Σ is generating if any integer from 1 to n_k can be written as the sum of some elements of Ψ.

Suppose next that $s_i - 1 \leq s_1 + \cdots + s_{i-1}$, $i = 1, \ldots, k$. We now prove that Σ is generating. Notice that $s_1 = 1$. By induction assume that any integer between 1 and $s_1 + \ldots + s_{i-1}$ can be expressed as the sum of some subset of $\{s_1, \ldots, s_{i-1}\}$. Observe now that any integer of the form $m + s_i$, $0 \leq m \leq s_1 + \cdots + s_{i-1}$ can now be generated the sum of the elements of a subset of $\{s_1, \ldots, s_{i-1}, s_i\}$. Since by hypothesis $s_i - 1 \leq s_1 + \ldots + s_{i-1}$. It follows that any integer less than equal to $s_1 + \ldots + s_i$ can be expressed as the sum of the elements of some subset of $\{s_1, \ldots, s_i\}$.

Conversely if for some i we have $s_i - 1 > s_1 + \ldots + s_{i-1}$ then clearly Σ' is not a generating sequence since $s_1 + \ldots + s_{i-1} + 1$ can not be generated. Our result follows.

5 Rectangular base boxes

In this section show that an optimal universal measuring box of square base of area 6 is a box of heights 1, 32, 83, 691. Because the ideas are very similar to all the ones used previously, we will only sketch the main arguments.

In a similar way to the case of triangular base boxes, we first find all the amounts of liquid that can be directly measured with a box of heights $h_1 < h_2 < h_3 < h_4$. Additional problems arise since there are a number of different shapes of boxes having themselves different sets of measurable quantities. Similarly to the triangular case, assume that the vertices of these boxes are labeled a_i, a'_i

for $i = 1, 2, 3, 4$, where the distance between a_i and a'_i is h_i. We must note that the set of measurable quantities of a given rectangular base box depends on the relative positions of the heights h_1, h_2, h_3, and h_4. If we start looking at the heights at h_1, then there are at most 6 different shapes that the boxes can have, namely: (h_1, h_4, h_3, h_2), (h_1, h_4, h_2, h_3), (h_1, h_3, h_4, h_2), (h_1, h_3, h_2, h_4), (h_1, h_2, h_4, h_3), and (h_1, h_2, h_3, h_4). Notice that first and sixth, second and fourth, and third and fifth boxes have the same set of measurable quantities. It is easy, but tedious work to check that the box $\mathcal{B}$ with heights (h_1, h_4, h_3, h_2) can measure more quantities that any other box. For $\mathcal{B}$, we find that we can measure the following amounts: $h_1, 3h_1, 6h_1, h_2, 3h_2, 3(h_1+h_2), h_3, 3h_3, 3(h_1+h_3), 3(h_2+h_3)$, and h_4. For example, h_1 can be measured by the points a_2, a_3 and a'_1, $3h_1$ by a_2, a_4 and a'_1, $6h_1$ by the plane parallel to the base passing through a'_1, and so on. It is possible to check, that for any of the other box, its set of measurable quantities is a proper subset of the set of measurable quantities of the box $\mathcal{B}$. It follows that a universal measuring box of optimal dimensions cannot measure more than 2^{11} litters. We can summarize all previous observations in the next theorem.

Theorem 3. *A universal measuring box $\mathcal{B}$ of rectangular base with area 6 of maximum capacity can measure* $\{h_1,\ 3h_1,\ 6h_1,\ h_2,\ 3h_2,\ 3(h_1+h_2),\ h_3,\ 3h_3,\ 3(h_1+h_3),\ 3(h_2+h_3),\ h_4\}$ *liters.*

Having these results at hand, we can find a box of maximum capacity by brute force. We did this by writing a C program that did the following: for any set of heights for any given set of heights $1 \leq h_1 < h_2 < h_3 < h_4 \leq 2^{11}$ we computed its set $\{h_1,\ 3h_1,\ 6h_1,\ h_2,\ 3h_2,\ 3(h_1+h_2),\ h_3,\ 3h_3,\ 3(h_1+h_3),\ 3(h_2+h_3),\ h_4\}$ of measurable quantities. We then applied to them the criterion from theorem 2, and found two universal measuring boxes with heights 1, 32, 83, 691 and 2, 64, 166, 691, that can measure all integers quantities from 1 to 691. For any other set of heights, the obtained sets of measurable amounts were not generating, or had smaller capacities.

Thus we have:

Theorem 4. *The largest capacity of a universal measuring box with square base is 691.*

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