Social Common Capital, Congestion Tax and Non-tradable Goods in a Small Open Economy

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Abstract
We construct a static general equilibrium model with social common capital or highway-type congested public capital. Non-tradable goods are also introduced into this small open economy. We obtain the effects of an improvement in the terms of trade, as well as an operation in congestion tax, on the degree of congestion and social welfare. We also find that the non-tradable goods become crucial to the results of comparative statics when they are used as the numeraire.

Introduction
When considering social common (overhead) capital or public goods of a kind being employed as a factor of production, there are usually three types of formulations, as summarized by Barro and Sala-i-Martin (1992): (i) rival and excludable public goods; (ii) non-rival and non-excludable public goods; and (iii) rival and non-excludable public goods.

The first type formulates a class of social common capital, such as a social security system, that may indeed be private goods but owned publicly and allocated to each user by, for example, the government instead of by certain invisible market mechanisms. The second type, e.g., Barro (1990), formulates Samuelson’s (1954) pure public goods, such as free radio broadcasting, which provides each user the same amount of service, without being affected by the intensity of economic activities. The third type, e.g., Futagami and Mino (1995), Rioja (1999), and Fisher and Turnovsky (1998), formulates a
class of public goods subject to congestion, such as standard roads and airports. In these models, the same amount of service derived from the public goods is enjoyed by all users or in an amount proportional to the user’s share of private capital. The congestion phenomenon is modelled by assuming that the quality of public goods is adversely affected by the intensity of economic activities, usually the aggregate stock of private capital.

Although the three types of formulations cover a wide range of productive public goods, there is yet an important class of public goods—that of highway-type public capital including toll highways, water and sewer systems and postal systems—to which none of the three formulations can apply. First, these formulations assume that the service derived from the public goods is fixed or proportional to the private capital. However, users of highway-type public capital usually have a certain level of control over how much of the service to use. For example, drivers can determine by themselves how far and in which direction to travel. Second, also as a consequence of the first, the degree of congestion should be determined endogenously by users’ optimal behavior, rather than being treated as an exogenous parameter.

Uzawa (1974) provides, to our knowledge, the first model that incorporates all the characterizations of highway-type public capital, and he called this class of public capital “social overhead capital.” That is, given the stock of public capital, more use of its services results in a greater degree of congestion and lower productivity, and the use depends on both the relative price of factors of production and the degree of substitution between them. Uzawa’s model is a one-good model and focuses on the growth problem within a closed economy. Asako (2009) sets up a two-good model and uses it to study, above all, the effect of trade on the degree of congestion and social welfare, in order to obtain the optimal congestion tax. One of his results is that free trade does not necessarily improve social welfare with the presence of highway-type public capital.

This paper introduces non-tradable goods into Asako’s model, to see what difference it will bring about. Changes in the terms of trade and congestion tax are considered: effects on the degree of congestion, the price of non-tradable goods, and social welfare are solved for. The optimum pattern for a congestion tax is provided as well. Finally, we identify the special role of the price numeraire in our model.
1. The Model

1.1 Utility and Technology

Assume that three goods are produced. Good 1 and good 2 are tradable while good 3 is non-tradable. There are a large number of well-behaved households, whose average features are collectively integrated as the representative household. The utility of the representative household, which is deemed to be social welfare within the representative agent framework, is denoted by

\[ U = U(C_1, C_2, C_3, X_h; s), \]

where \( C_i (i=1,2,3) \) is the consumption on the \( i \)-th good; \( X_h \) is the amount of social common capital service used by the household; and \( s \) denotes the degree of congestion whose definition will be explained below. The utility function is assumed to be: (i) strictly quasi-concave on \((C_1, C_2, C_3, X_h)\); (ii) strictly increasing in \( C_1, C_2, C_3, X_h \) and decreasing in \( s \); and (iii) twice continuously differentiable. To measure the sensitivity of the household over the degree of congestion, define by \( \varepsilon_h \) the elasticity of utility with respect to congestion

\[ \varepsilon_h = -\frac{\partial \ln U}{\partial \ln s}. \]

There are two production factors: the service of social common capital, and that of private capital. Private capital is simply capital in the normal meaning with stock \( K \). While social common capital is rather “special” in the sense that no user can exclusively possess total or partial stock of social common capital, each user can derive service from it. Note the total amount of service derived from social common capital, denoted by \( X \), could be any amount and does not have to be correlated with its stock, denoted by \( V \).

There are a large number of firms producing good \( i \) with the same technology, so that firm \( i \) is referred to as the representative firm of those firms. Firm \( i \) has technology

\[ Q_i = g_i(s)F_i(X_i, K_i), \]

where \( Q_i \) is the output; \( g_i(s) \in [0,1] \) measures the firm-specific impact of congestion; \( F_i(X_i, K_i) \) measures the positive contribution of productive factors, i.e., \( X_i \) and \( K_i \) which are, respectively, the service of social common capital and private capital employed by firm \( i \). The congestion part \( g_i(\cdot) \) and the positive contribution part \( F_i(\cdot) \) are assumed to be separable for analytical simplicity.
Assume that $F_i(X_i, K_i)$ is: (i) strictly quasi-concave on $(X_i, K_i)$; (ii) first-order homogeneous on $(X_i, K_i)$; (iii) strictly increasing in $X_i$ and $K_i$; and (iv) twice continuously differentiable. By the homogeneity, the output can be rewritten as

$$Q_i = g_i(s) f_i(x_i) K_i,$$

where the intensive variable $x_i \equiv X_i / K_i$ is the ratio of social common capital to private capital and $f_i(x_i) \equiv F_i(x_i, 1)$.

1.2 Impact of Congestion

A natural way of defining the degree of congestion $s$ may as well be

$$s = \frac{X}{V}.$$  \hspace{1cm} (2)

The relation between $g_i(s)$ and $s$ is similar to that between vehicle speed and traffic density. In the literature on road congestion, e.g., Inman (1978) and Verhoef (1999), vehicle speed and traffic density are usually understood as being adversely related. Assume that $g_i(s)$ is: (i) non-increasing; (ii) continuous; and (iii) piecewise continuously differentiable. Other possible properties of $g_i(s)$, such as increasing/decreasing marginal congestion or jam degree of congestion resulting in $g_i(\cdot) = 0$, are left open in this paper.\(^7\) To measure the sensitivity of firm $i$ over the degree of congestion, define by $\epsilon_i$ the elasticity of production damage with respect to congestion

$$\epsilon_i = -\frac{\partial \ln Q_i}{\partial \ln s} = -\frac{d \ln g_i}{d \ln s}.$$

Note that there might be some values of $s$ with which or over a certain range where $\epsilon_i$ does not exist. However, because of the piecewise continuously differentiable assumption on $g_i(s)$, that should not hinder further analysis. At any rate, this problem can be treated as if there were no singular points, and all that is needed is some additional discussion on the singular points when faced with specific problems. According to the non-increasing assumption, $\epsilon_i \geq 0$. Firm $i$ is said to be more (or less) congestion sensitive with respect to firm $j$ when $\epsilon_i > \epsilon_j$ (or $\epsilon_i < \epsilon_j$).

1.3 Market Failure

The markets for goods and private capital are perfectly competitive. Denote the price of good $i$ by $p_i$ and the interest rate (rental price for private capital)
by \( r \). The service of social common capital is charged at \( \theta \) for per unit use. This fee \( \theta \), called congestion tax, is assumed to be levied by the government with literally no managing cost.

The representative household, taking the prices of goods and social common capital service as given, maximizes the utility \( U(C_1, C_2, C_3, X_h; s) \) under an income budget constraint. Similarly, each representative firm maximizes profit \( (p_i Q_i - \theta X_i - r K_i) \), taking output and factor prices as given. Then for the representative household, it follows that the optimality condition implies

\[
p_i = \frac{U_i}{\lambda},
\]

where \( U_i = \partial U / \partial C_i \) is the marginal utility of consumption for each good \( i \) and \( \lambda \) is the Lagrange multiplier attached to the budget constraint of the optimization problem. It is well known that \( \lambda \) is interpreted as the shadow price of income and is equal to the marginal utility of income. Also for the representative firm \( i \), we should have

\[
r = p_i g_i(s) \frac{\partial F_i}{\partial K_i} = p_i g_i(s) \left( f_i - x_i f'_i \right),
\]

for profit maximization, where \( f'_i = df_i(x_i) / dx_i \).

Market failure stems from the key assumption about how the decision of utilizing social common capital service is made. Namely, each economic agent takes no account of the influence of its own behavior on the degree of congestion since there are a large number of households and firms. Specifically, the representative household uses the service of social common capital in the following manner:

\[
\theta = \frac{U_h}{\lambda},
\]

where \( U_h = \partial U / \partial X_h \) denotes the marginal utility of services derived from social common capital. Similarly, representative firms determine how much social common capital service to use by the condition that

\[
\theta = p_i g_i(s) \frac{\partial F_i}{\partial X_i} = p_i g_i(s) f'_i.
\]

Note that an external economy consideration \( \partial s / \partial X_h \) is neglected in (5) as is the case \( \partial s / \partial X_i \) is neglected in (6).
1.4 Market Equilibrium and Trade Balance

Without loss of generality, assume that good 1 is imported and good 2 is exported. The government transfers all revenue to the household, then the market-clearing condition for goods and the introduction of trade require that we have

\[ C_1 = Q_1 + pE, \]  
\[ C_2 = Q_2 - E, \]  
\[ C_3 = Q_3, \]  

where \( E \) denotes the export of good 2 and \( p \equiv p_2/p_1 \) denotes the relative price of exportable to importable or the terms of trade. We assume that the economy is a small open economy, implying that \( p \) is determined exogenously to this country. The market-clearing condition for private capital and the definition of \( X \) require, respectively, that

\[ K = K_1 + K_2 + K_3, \]  
\[ X = X_1 + X_2 + X_3 + X_h. \]  

Counting equations (1) to (10), the income constraint, the definition of intensive capital ratio variables \( x_i (i=1,2,3) \), and terms of trade \( p \), there are in total 24 equations. According to Walras’s Law, the number of independent equations then is 23. Excluding the exogenous parameters \( K, V, \theta \) and \( p \), we have the list of endogenous variables \( Q_i, C_i, K_i, X_i, x_i, X_h, X, s, r, \lambda, E, p_i \) to determine, which add up to 24 variables. But, by setting the price numeraire, the number of variables is reduced to 23, which is equal to the number of equations. This is merely an informal, naive check of the solution.

The characterizations of the solution depend to large extent on the properties of the congestion damage function \( g_i(\cdot) \). Therefore, given a specific form of \( g_i(\cdot) \), there may be unique equilibrium solution or multiple solutions, or even the case of no solution.

2. Change in the Terms of Trade

2.1 Effects on Congestion

Let good 1 be the numeraire, i.e. \( p_1 = 1 \) to mean \( p_2 = p \) for given terms of trade \( p \). Suppose there occurs a small change in \( p \), denoted by \( dp \), while congestion tax \( \theta \) remains unchanged, i.e. \( d\theta = 0 \). Using (4) and (6), we obtain\(^9\)
\[-\varepsilon_1 \hat{s} + \beta_1 \hat{\varepsilon} = 0, \quad (11)\]
\[\hat{p} - \varepsilon_2 \hat{s} + \beta_2 \hat{\varepsilon} = 0, \quad (12)\]
\[\hat{p}_3 - \varepsilon_3 \hat{s} + \beta_3 \hat{\varepsilon} = 0, \quad (13)\]
where a hat operator "\(^\hat{\quad}\)" denotes a proportional change (e.g. \(\hat{p}_2 = dp_2/p_2\)) and the newly introduced variables \(w = \theta/r\) and \(\beta_i = rK_i/p_iQ_i\) are, respectively, the relative factor price ratio and income share of private capital in firm \(i\).

Note that \(\hat{w} = -\hat{\varepsilon}\) since \(d\theta = 0\) here.

From (11) and (12), given that \(\varepsilon_1/\beta_1 \neq \varepsilon_2/\beta_2\), we can solve for \(\hat{s}\) and \(\hat{\varepsilon}\) to obtain
\[\hat{s} = \frac{1}{(e_2 - e_1)\beta_2} \hat{p}, \quad (14)\]
\[\hat{\varepsilon} = \frac{e_1}{(e_2 - e_1)\beta_2} \hat{p}, \quad (15)\]
where \(e_i = \varepsilon_i/\beta_i\) can be called "congestion-relevance index" since large \(e_i\), whether caused by large \(\varepsilon_i\) or small \(\beta_i\) (thus large income share of social common capital service), implies that firm \(i\) is highly relevant or contributory to congestion. It follows from (14) that

**Proposition 1**  Let import goods be the numeraire and congestion tax fixed. An improvement in the terms of trade reduces the degree of congestion if and only if the export firm has a smaller congestion-relevance index than the import firm.\(^{10}\) That is:

\[
\frac{ds}{dp} < 0 \quad \text{iff} \quad e_2 < e_1.
\]

Proposition 1 is intuitive and similar with the result of Asako (2009). Introducing non-tradable goods seems to make no difference here. Note that \(dw/dp < 0\) if and only if \(e_2 < e_1\), and note also that it is possible for \(e_2 < e_1\) when \(\alpha_2 < \alpha_1\), where \(\alpha_i\) denotes the income share of social common capital. This means that it in turn is possible to yield \(dw/dp < 0\) when \(\alpha_2 < \alpha_1\). Thus the Stolper-Samuelson theorem, which basically centers on relative capital intensity ratio and relates the terms of trade and factor price ratio, does not necessarily hold in the model.

Next, substituting (14) and (15) into (13), it can be obtained that
\[\hat{p}_3 = \frac{(e_3 - e_1)\beta_3}{(e_2 - e_1)\beta_2} \hat{p}, \quad (16)\]
indicating that we have the second proposition:
Proposition 2 Let import goods be the numeraire and congestion tax fixed. An improvement in the terms of trade increases the price of non-tradable goods if and only if the congestion-relevance indices of both the non-tradable firm and the export firm are greater than, or both less than, that of the import firm. That is:

\[ \frac{dp_3}{dp} > 0 \quad \text{iff} \quad \text{sgn} \left[ e_3 - e_1 \right] = \text{sgn} \left[ e_2 - e_1 \right] \]

where, the notation \( \text{sgn} \left[ e_2 - e_1 \right] \), for instance, means the sign of \( e_2 - e_1 \).

2.2 Effects on Social Welfare

Does or doesn't an improvement in the terms of trade increase the social welfare of a small country? To answer this question, calculate the total derivative of utility with respect to the terms of trade and obtain\(^{11}\)

\[
\frac{1}{\lambda V} \frac{dU}{dp} = \left( \theta - \frac{1}{X} \sum p_i e_i Q_i - \frac{1}{X} \frac{\varepsilon_i U}{\lambda} \right) \frac{ds}{dp} + \frac{E}{V}. \tag{17}
\]

Next, define the marginal social cost of using social common capital service as

\[
\text{MSC} = -\frac{\partial}{\partial X} \sum p_i Q_i - \frac{1}{\lambda} \frac{\partial U}{\partial X}.
\]

The first term in the right-hand side of above expression is the marginal damage to the gross output caused by the use of social common capital service. It measures the marginal social cost on the production side. The second term is exactly the value by which the household needs to increase consumption to compensate for the marginal loss of utility. This loss also comes from the use of social common capital services. The second term evaluates the marginal social cost on the consumption side.

Using the definitions of \( \varepsilon_h \) and \( \varepsilon_i \), we can rewrite the expression of \( \text{MSC} \) into

\[
\text{MSC} = \frac{1}{X} \sum p_i e_i Q_i + \frac{1}{X} \frac{\varepsilon_h U}{\lambda}. \tag{18}
\]

Then, substituting (18) into (17), we have

\[
\frac{1}{\lambda V} \frac{dU}{dp} = \left( \theta - \text{MSC} \right) \frac{ds}{dp} + \frac{E}{V}, \tag{19}
\]

which implies:

Proposition 3 Let import goods be the numeraire and congestion tax fixed. If
an improvement in the terms of trade decreases the degree of congestion, then this improvement increases social welfare if and only if the congestion tax is low enough. If an improvement in the terms of trade increases the degree of congestion, then this improvement increases social welfare if and only if the congestion tax is high enough. Specifically, if \( e_2 < e_1 \), then

\[
\frac{dU}{dp} > 0 \iff \theta < MSC + |B|.
\]

If on the contrary \( e_2 > e_1 \), then

\[
\frac{dU}{dp} > 0 \iff \theta > MSC - |B|,
\]

where

\[
B \equiv \frac{p E \beta}{X} (e_2 - e_1),
\]

and \( |B| > 0 \) denotes the absolute value of \( B \).

**Proof.** From (19), \( dU/dp > 0 \) is equivalent to \((\theta - MSC) ds/dp > -E/V\). Therefore, if \( e_2 < e_1 \), i.e. \( ds/dp < 0 \), then the inequality means \( \theta < MSC - (E/V)dp/ds \); if \( e_2 > e_1 \), i.e. \( ds/dp > 0 \), then the inequality means \( \theta > MSC - (E/V)dp/ds \). Substituting (14) for \( dp/ds \) gives the claimed result.

Q. E. D.

Proposition 3 has some policy implications: (i) if the export firm has a smaller congestion-relevance index than the import firm, when the terms of trade improves, a relatively low congestion tax may be appropriate; (ii) however, when the terms of trade deteriorate, a relatively high congestion tax may very well be better; (iii) if the export firm has a larger congestion-relevance index than the import firm, the policy should be taken in the opposite direction; and (iv) imposing congestion tax around \( MSC \) can ensure that, in spite of the congestion-relevance index, an improvement in the terms of trade will raise social welfare.

### 3. Operation in Congestion Tax

#### 3.1 Effects on Congestion

Now suppose that a small operation in \( \theta \), denoted by \( d\theta \), occurs while the
terms of trade \( p \) remains unchanged, i.e. \( dp = 0 \). From (4) and (6), similarly with from (11) to (13) for \( d\theta = 0 \), we have
\[
\begin{align*}
-e_1 \hat{s} + \beta_1 \hat{w} &= \hat{\theta}, \\
-e_2 \hat{s} + \beta_2 \hat{w} &= \hat{\theta}, \\
\hat{p}_3 - e_3 \hat{s} + \beta_3 \hat{w} &= \hat{\theta}.
\end{align*}
\]
where (21) and (22) can be used to solve for \( \hat{s} \) and \( \hat{w} \):
\[
\begin{align*}
\hat{s} &= \frac{\beta_2 - \beta_1}{(e_2 - e_1)\beta_1\beta_2} \hat{\theta}, \\
\hat{w} &= \frac{e_2 - e_1}{(e_2 - e_1)\beta_1\beta_2} \hat{\theta}.
\end{align*}
\]
Then, we obtain the fourth proposition that:

**Proposition 4** Let import goods be the numeraire and the terms of trade unchanged. An increase in congestion tax reduces the congestion level if and only if the export firm is social common capital intensive and has a larger congestion-relevance index, or is private capital intensive and has a smaller congestion-relevance index. That is,
\[
\frac{ds}{d\theta} < 0 \quad \text{iff} \quad \text{sgn} \left[ \beta_2 - \beta_1 \right] = -\text{sgn} \left[ e_2 - e_1 \right].
\]
Note that an irregular case \( ds/d\theta > 0 \) is possible in Proposition 4 when \( \text{sgn} \left[ \beta_2 - \beta_1 \right] = \text{sgn} \left[ e_2 - e_1 \right] \), indicating that raising congestion tax does not necessarily reduce the degree of congestion under the small open economy setting.

### 3.2 Welfare Effects and the Optimal Congestion Tax

In this subsection, we are concerned with the influence of congestion tax on social welfare. As with (17) or through the process similar with Appendix B, we obtain
\[
\frac{1}{\lambda V} \frac{dU}{d\theta} = \left( \theta - \text{MSC} + E \frac{d\theta}{V \frac{ds}{d\theta}} \right) \frac{ds}{d\theta}.
\]
Then, making use of (24) enables us to rewrite (26) into
\[
\frac{1}{\lambda V} \frac{dU}{d\theta} = \left[ \theta - \left( \text{MSC} + A \right) \right] \frac{ds}{d\theta},
\]
where
\[
A \equiv -\frac{(e_2 - e_1)\theta E \beta_1 \beta_2}{(\beta_2 - \beta_1)X}.
\]
Note that $A > 0$ when $ds/d\theta < 0$, and $A < 0$ when $ds/d\theta > 0$. Thus we obtain the fifth proposition that:

**Proposition 5** Let import goods be the numeraire and the terms of trade unchanged. If an increase in the congestion tax decreases the degree of congestion, then it improves social welfare if and only if congestion tax is low enough. If an increase in the congestion tax increases the degree of congestion, then it improves social welfare if and only if the congestion tax is high enough. Specifically, if $\text{sgn} \left( \beta_2 - \beta_1 \right) = -\text{sgn} \left( e_2 - e_1 \right)$, then

$$
\frac{dU}{d\theta} > 0 \quad \text{iff} \quad \theta < MSC + |A|.
$$

If, on the other hand, $\text{sgn} \left( \beta_2 - \beta_1 \right) = \text{sgn} \left( e_2 - e_1 \right)$, then

$$
\frac{dU}{d\theta} > 0 \quad \text{iff} \quad \theta > MSC - |A|.
$$

where $|A| > 0$ denotes the absolute value of $A$.

In the case $ds/d\theta < 0$, congestion tax

$$
\theta^* = MSC + |A| \tag{28}
$$

maximizes the utility or social welfare. Note that this optimal level of congestion tax $\theta^*$ is greater than the optimal tax under autarky, which is the marginal social cost $MSC$ to be given for $E = 0$ and thereby $A = 0$ and is written implicitly as in (18). However, in the irregular case $ds/d\theta > 0$, although $\theta^*$ is an extremum point as well, it in fact is a local minimum point that locally minimizes the utility, thus is not the optimal congestion tax.

## 4. Non-tradable Goods as the Numeraire

### 4.1 Effects of the Change in the Terms of Trade

So far, by choosing import good 1 as the numeraire, introducing non-tradable good 3 does not bring any significantly novel results compared with the model in Asako (2009). Here, we change the numeraire from good 1 to good 3, i.e. let $p_3 = 1$, to see what difference it will bring about.

Under the new numeraire setting, considering an improvement in the
terms of trade \( p \equiv p_2/p_1 \) with fixed congestion tax, equations from (11) to (13) are rewritten, respectively, to
\[
\hat{p}_1 - \varepsilon_1 \hat{s} + \beta_1 \hat{w} = 0, \\
\hat{p}_2 - \varepsilon_2 \hat{s} + \beta_2 \hat{w} = 0, \\
- \varepsilon_3 \hat{s} + \beta_3 \hat{w} = 0.
\]
(29)  
(30)  
(31)
Solving for \( \hat{s} \) and \( \hat{w} \) gives
\[
\hat{s} = \frac{\beta_3}{(\varepsilon_2 - \varepsilon_1)\beta_3 - (\beta_2 - \beta_1)\varepsilon_3} \hat{p}, \\
\hat{w} = \frac{\varepsilon_3}{(\varepsilon_2 - \varepsilon_1)\beta_3 - (\beta_2 - \beta_1)\varepsilon_3} \hat{p}.
\]
(32)  
(33)
Now, corresponding to Proposition 1, we have

**Proposition 6**  Let non-tradable goods be the numeraire and congestion tax fixed. Then, we obtain
\[
\frac{ds}{dp} < 0 \quad \text{iff} \quad (\varepsilon_2 - \varepsilon_1) \beta_3 - (\beta_2 - \beta_1) \varepsilon_3 < 0.
\]
Similarly, the effect of an improvement in the terms of trade on social welfare can be readily obtained through (19), and the corresponding version of Proposition 3 is

**Proposition 7**  Let non-tradable goods be the numeraire and congestion tax fixed. If
\[
(\varepsilon_2 - \varepsilon_1) \beta_3 - (\beta_2 - \beta_1) \varepsilon_3 < 0,
\]
then
\[
\frac{dU}{dp} > 0 \quad \text{iff} \quad \theta < MSC + |B|.
\]
If, on the contrary,
\[
(\varepsilon_2 - \varepsilon_1) \beta_3 - (\beta_2 - \beta_1) \varepsilon_3 > 0,
\]
then
\[
\frac{dU}{dp} > 0 \quad \text{iff} \quad \theta > MSC - |B|.
\]
Proposition 7 implies that, by changing the numeraire, non-tradable good 3 becomes crucial in determining the effects of the change in the terms of trade on the degree of congestion and social welfare. For instance, if good 3 has a value of \( \varepsilon_3 \) much greater than \( \beta_3 \), the degree of congestion is likely to decrease with a rise in the terms of trade.
### 4.2 Effects of the Operation in Congestion Tax

Now consider a small operation in congestion tax $d\theta$ without altering the terms of trade, i.e. $dp = 0$. In this setting, equations (21)-(23) are replaced by

\[
\begin{align*}
\dot{p}_1 - \epsilon_1 \dot{s} + \beta_1 \dot{w} &= \hat{\theta}, \\
\dot{p}_2 - \epsilon_2 \dot{s} + \beta_2 \dot{w} &= \hat{\theta}, \\
-\epsilon_3 \dot{s} + \beta_3 \dot{w} &= \hat{\theta}.
\end{align*}
\]

(34) (35) (36)

The solutions can be expressed as

\[
\dot{s} = \frac{\beta_2 - \beta_1}{\beta_3 (\epsilon_2 - \epsilon_1) - \epsilon_3 (\beta_2 - \beta_1)} \hat{\theta},
\]

\[
\dot{w} = \frac{\epsilon_2 - \epsilon_1}{\beta_3 (\epsilon_2 - \epsilon_1) - \epsilon_3 (\beta_2 - \beta_1)} \hat{\theta}.
\]

Therefore, the corresponding versions of Propositions 4 and 5 should be replaced by:

**Proposition 8** Let non-tradable goods be the numeraire and the terms of trade unchanged. Then, we obtain

\[
\frac{ds}{d\theta} < 0 \quad \text{iff} \quad \text{sgn} \left[ \beta_2 - \beta_1 \right] = - \text{sgn} \left[ (\epsilon_2 - \epsilon_1) \beta_3 - (\beta_2 - \beta_1) \epsilon_3 \right].
\]

**Proposition 9** Let non-tradable good be the numeraire and the terms of trade unchanged. If

\[
\text{sgn} \left[ \beta_2 - \beta_1 \right] = - \text{sgn} \left[ (\epsilon_2 - \epsilon_1) \beta_3 - (\beta_2 - \beta_1) \epsilon_3 \right],
\]

then

\[
\frac{dU}{d\theta} > 0 \quad \text{iff} \quad \theta < MSC + |A|.
\]

And if

\[
\text{sgn} \left[ \beta_2 - \beta_1 \right] = \text{sgn} \left[ (\epsilon_2 - \epsilon_1) \beta_3 - (\beta_2 - \beta_1) \epsilon_3 \right],
\]

then

\[
\frac{dU}{d\theta} > 0 \quad \text{iff} \quad \theta > MSC - |A|.
\]

By changing the numeraire, the non-tradable good now becomes crucial to the effects of operating congestion tax. However, in the case of $ds/d\theta < 0$, the optimal congestion tax that maximizes social welfare equals

\[
\theta^* = MSC + |A|
\]

which is the same as that obtained in (28) with (27) for $|A|$ with good 1 set as the numeraire. The irrelevance property of the numeraire good to the
optimal level of congestion tax stems from the fact that $\theta^*$ is determined first as a result of policy operation, and the real economy adjusts to that endogenously with the remaining degree of freedom.

**Concluding Remarks**

By introducing non-tradable goods, this paper discusses how the degree of congestion, the price of non-tradable goods and social welfare are affected by an improvement in the terms of trade, as well as by an operation in congestion tax. The main results are concluded as Propositions 1 to 5. We find that the presence of non-tradable goods has little influence on these results when the numeraire is the import good.

However, when the numeraire is changed to the non-tradable goods, the results of comparative statics change considerably in that, for example, the income share of social common capital and the sensitivity over the degree of congestion become different from those obtained under the case of the import good numeraire. The detailed alterations are summarized as Propositions 6 to 9.

Why does the setting of the numeraire matter? This is because in the model the price variables are not all determined endogenously, since congestion tax $\theta$ to begin with is an exogenously determined policy parameter. Thus, different numeraire settings mean different real levels of congestion tax in terms of the numeraire good, which in turn implies that relative prices do matter beyond the proportional scale effects. When a certain good is used as the numeraire, congestion tax has to be measured in the unit of that good, and setting different numeraires will yield different policy effects, depending on the properties of the numeraire goods. Therefore, choice of an appropriate price numeraire, if possible, can be seen as another policy-making tool.

The stock of private capital and social common capital are fixed in this paper. It is quite likely that new results and implications would result if this model were to be extended to long-term and dynamic cases. This would be an interesting topic for future research.
Appendix A
By logarithmically differentiating, it directly follows from (6) that
\[
\frac{d\theta}{\theta} = \frac{dp_i}{p_i} - \varepsilon_i \frac{ds}{s} + \frac{df_i'}{f_i}.
\] (A1)

On the other hand, from (4) and (6), we have \( w = f_i' / (f_i - x_i f_i') \), thus
\[
\frac{d^2}{dw} = \frac{df_i'}{f_i'} - \frac{d(f_i - x_i f_i')}{f_i - x_i f_i'} = \frac{df_i'}{f_i'} \left[ 1 + \frac{x_i f_i'}{f_i - x_i f_i'} \right].
\] (A2)

Note that \( x_i f_i' / (f_i - x_i f_i') \) is merely the ratio of the income share of social common capital to private capital in firm \( i \), thus \( x_i f_i'/ (f_i - x_i f_i') = \alpha_i / \beta_i \), where \( \alpha_i \) and \( \beta_i \) denote the income share of social common capital and that of private capital respectively. Since \( \alpha_i + \beta_i = 1 \), the above relation can be written as \( df_i'/ f_i' = \beta_i dw/w \). Substituting it into (A1) for \( df_i'/ f_i' \) gives
\[
\frac{d\theta}{\theta} = \frac{dp_i}{p_i} - \varepsilon_i \frac{ds}{s} + \beta_i \frac{dw}{w}.
\]

Appendix B
Taking the total derivative of \( U(C_1, C_2, C_3, X_h; s) \) with respect to \( p \) gives
\[
\frac{dU}{dp} = \sum_i \frac{dU_i}{dp} + U_h \frac{dX_h}{dp} + \frac{\partial U}{\partial s} \frac{ds}{dp}.
\] (B1)

From equations (3), (5), and (7)-(9), it can be rewritten into
\[
\frac{dU}{dp} = \lambda p_i \left( \frac{dQ_i}{dp} + E + p \frac{dE}{dp} \right) + \lambda p_2 \left( \frac{dQ_2}{dp} - \frac{dE}{dp} \right) + \lambda p_3 \frac{dQ_3}{dp} + \lambda \theta \frac{dX_h}{dp} + \frac{\partial U}{\partial s} \frac{ds}{dp}.
\] (B2)

Solving for \( dQ_i / dp \) and \( dX_h / dp \) to analyze the sign of \( dU / dp \) is tedious and not necessary. We can use (4), (6), (10) and the definition of \( \varepsilon_h \) and \( \varepsilon_i \) to obtain
\[
\frac{1}{\lambda} \frac{dU}{dp} = \sum_i p_i \left( \frac{\partial Q_i}{\partial s} \frac{ds}{dp} + \frac{\partial Q_i}{\partial X_i} \frac{dX_i}{dp} + \frac{\partial Q_i}{\partial K_i} \frac{dK_i}{dp} \right) + E + \theta \frac{dX_h}{dp} + \frac{\partial U}{\partial s} \frac{ds}{dp}
\]
\[
= -\frac{1}{s} \frac{ds}{dp} \sum_i p_i \varepsilon_i Q_i + \theta \frac{dX_h}{dp} + \frac{\partial U}{\partial s} \frac{ds}{dp}
\]
\[
= -\frac{1}{s} \frac{ds}{dp} \sum_i p_i \varepsilon_i Q_i + \theta \frac{dX_h}{dp} + \frac{\partial U}{\partial s} \frac{ds}{dp}
\]
\[
= V \frac{ds}{dp} \left( \theta - \frac{1}{X} \sum_i p_i \varepsilon_i Q_i - \frac{1}{X} \frac{\varepsilon_i U}{\lambda} \right) + E,
\] (B3)
and substituting \( dX = V ds \) and \( dK = 0 \) yields (17).
Notes

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2. Faculty of Economics, Rissho University (kasako@ris.ac.jp.)
3. Faculty of Economics, Toyo University
4. Later, Uzawa (2005) shifts to prefer the term “social common capital” over “social overhead capital.”
5. This setting can be readily extended to the case of $n$ non-tradable goods, and the main results still remain.
6. Smaller $g_i(s)$ implies greater damage on firm $i$ and vice versa.
7. See Edwards (1990) for some other types of congestion.
8. We do not explicitly solve for $\lambda$ as the mere knowledge that $\lambda > 0$ is sufficient to obtain the propositions of this paper.
9. Refer to Appendix A for the detailed analytical derivations.
10. In Proposition 1 and in what follows as well, the notation “iff” is the abbreviation of “if and only if” and it implies the equivalence of conditions or that one is both necessary and sufficient condition for the other.
11. See Appendix B for the details.

References


