「無限分解可能過程に関連する諸問題」統計数理研究所 2010/10/21-23

Continuous state branching processes with immigration and GGC (generalized gamma convolutions)

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Abbreviations:

CBI-process $\longleftarrow$ continuous state branching process with immigration

GGC $\longleftarrow$ generalized gamma convolution(s)

The main result:

GGC satisfying certain conditions are stationary distributions of CBI-processes.

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§1. The simplest case

Prototype: **Gamma distribution and Laguerre operator with** $\delta, b > 0$

$$\gamma_{\delta,b}(dx) = \frac{b^\delta}{\Gamma(\delta)} x^{\delta-1} e^{-bx} dx, \;\; x > 0 \tag{1.1}$$

$$Lf(x) = xf''(x) + (-bx + \delta)f'(x) \tag{1.2}$$

enjoy the relation $\int_0^\infty Lf(x)\gamma_{\delta,b}(dx) = 0$.

Interpretation: $\gamma_{\delta,b}$ **is a stationary distribution of the Markov process generated by** L.

Problem **Given** $\delta_1, \delta_2, b_1, b_2 > 0$**,**
generalize L **so that** $\nu := \gamma_{\delta_1,b_1} * \gamma_{\delta_2,b_2}$ **is a stationary distribution of the associated Markov process.**
(Note: $b_1 = b_2 =: b \Longrightarrow \nu = \gamma_{\delta_1+\delta_2,b}$**)**
Answer **(within the class of CBI-processes)**

$$\begin{aligned} Lf(x) &= xf''(x) + (-bx + \delta)f'(x) \\ &\quad + c\,\kappa^2 x \int_0^\infty \Big[f(x+y) - f(x) - yf'(x)\Big] e^{-\kappa y} dy \end{aligned}$$

where $\delta = \delta_1 + \delta_2$**,** $b = (\delta_1 + \delta_2)/(b_1^{-1}\delta_1 + b_2^{-1}\delta_2)$**,**

$$c = \frac{\delta_1 \delta_2 (b_1 - b_2)^2}{(\delta_1 + \delta_2)(b_2\delta_1 + b_1\delta_2)}, \;\; \kappa = \frac{b_2\delta_1 + b_1\delta_2}{\delta_1 + \delta_2}. \qquad (1.4)$$

§2 CBI-processes due to Kawazu-S.Watanabe

Limit processes from

Galton-Watson processes with immigration

Discrete model:

Markov chain $\{X_n\}$ on $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$ with

$$P(X_{n+1} \in \cdot | X_n = k) = P\left(\sum_{i=1}^{k} Y_i + Z \in \cdot\right) \tag{2.2}$$

$\{Y_i\}$: **$\mathbf{Z}_+$-valued iid,**

($Y_i \sim$ "the offspring distribution")

Z: **$\mathbf{Z}_+$-valued r.v. indep of $\{Y_i\}$**

($\sharp$ of immigrating particles)

The limit procedure (in the diffusion case)

- **Assumptions:** $E[Y_i] = 1$ (**the critical case**),

$$2a := \mathrm{Var}(Y_i) < \infty, E[Z] =: \delta, \mathrm{Var}(Z) < \infty. \tag{2.3}$$

- **Rescaled processes:** $x^{(N)}(t) := X_{[Nt]}/N \in \mathbf{R}_+$

By Taylor's theorem for each $x \in \mathbf{R}_+$ **as** $N \to \infty$

$$\begin{aligned} & E\left[f(x^{(N)}(1/N)) \middle| x^{(N)}(0) = [Nx]/N\right] \\ = \; & E\left[f((Y_1 + \cdots + Y_{[Nx]} + Z)/N)\right] \\ \sim \; & f(x) + \frac{f''(x)}{2N^2} \sum_{i=1}^{[Nx]} \mathrm{Var}(Y_i) + \frac{f'(x)}{N} E[Z] \\ = \; & f(x) + \frac{1}{N}(axf''(x) + \delta f'(x)) =: f(x) + \frac{1}{N} Lf(x) \end{aligned}$$

L: **generator of the limit process**

More generally, consider G-W processes with immigration $\{X_n^{(N)}\}$ **with offspring distr's s.t.**

$$E[Y_i^{(N)}] = 1 - \frac{b}{N} + o\left(\frac{1}{N}\right) \text{ (asympt. critical)},$$

$$2a := \lim_{N\to\infty} \mathrm{Var}\left(Y_i^{(N)}\right), E[Z] =: \delta, \mathrm{Var}(Z) < \infty.$$

Then $x^{(N)}(t) := X_{[Nt]}^{(N)}/N$ **converge as** $N \to \infty$ **to the diffusion generated by**

$$Lf(x) = axf''(x) + (-bx + \delta)f'(x). \tag{2.4}$$

Remark: It is known as the CIR(Cox-Ingersoll-Ross) model in mathematical finance.

Kawazu-S.Watanabe (1971) determined possible limit processes (we call CBI-processes).

- **generator (in the conservative case):**

$$\begin{aligned} Lf(x) &= axf''(x) - bxf'(x) \\ &\quad + x\int_0^\infty \left[f(x+y) - f(x) - yf'(x)\right] n_1(dy) \\ &\quad + \delta f'(x) + \int_0^\infty [f(x+y) - f(x)]\, n_2(dy) \end{aligned}$$

with $a, b, \delta \geq 0$, n_i satisfying integrability (2.6).

Note: $(a, b, n_1) \longleftarrow$ **branching**

$(\delta, n_2) \longleftarrow$ **immigration**

- **transition function $P_t(x, dy)$:**

$$\int e^{-\lambda y} P_t(x, dy) = \exp\left(-x\psi(t,\lambda) - \int_0^t F(\psi(s,\lambda))ds\right),$$

where $\psi(t,\lambda)$ solves

$$\frac{\partial \psi}{\partial t}(t,\lambda) = R(\psi(t,\lambda)), \ \ \psi(0,\lambda) = \lambda, \ \ \lambda \geq 0 \tag{2.7}$$

with "branching mechanism"

$$R(\lambda) = -a\lambda^2 - b\lambda - \int_0^\infty \left(e^{-\lambda y} - 1 + \lambda y\right) n_1(dy)$$

and "immigration mechanism"

$$F(\lambda) = \delta\lambda + \int_0^\infty (1 - e^{-\lambda y}) n_2(dy). \tag{2.10}$$

Infinitely divisibility of $P_t(x, dy)$ follows from

$$\int e^{-\lambda y} P_t(x_1 + \cdots + x_k, dy) = \prod_{i=1}^{k} \exp\left(-x_i \psi(t,\lambda) - \int_0^t \frac{1}{k} F(\psi(s,\lambda))ds\right).$$

Indeed, this implies

$$P_t(x_1 + \cdots + x_k, \cdot) = P_t^{(k)}(x_1, \cdot) * \cdots * P_t^{(k)}(x_k, \cdot),$$

where $P_t^{(k)}(x, \cdot) \longleftrightarrow L$ with $(\delta/k, n_2/k)$ in place of (δ, n_2).
Interpretation: Devide the country into k districts in which the same branching mechanism applies and every immigrating person chooses one district equally likely.

§3 Stationary distributions of CBI-processes

Exploring $\lim_{t\to\infty}\int e^{\lambda y}P_t(x,dy)$ **yields**

Ogura's formula (1970): A CBI-process has a stationary distribution iff

$$\Phi(\lambda) := -\int_{0+}^{\lambda}\frac{F(u)}{R(u)}du < \infty, \qquad \lambda > 0. \tag{3.1}$$

In this case, the stationary distribution, say ν**, is unique and**

$$\int e^{-\lambda x}\nu(dx) = \exp\left(-\Phi(\lambda)\right), \qquad \lambda > 0. \tag{3.2}$$

Remarks • $b > 0 \Longrightarrow$ **(3.1) holds.**

• **(3.1)** $\Longrightarrow$ **"ergodicity" ; for every** $x \in \mathrm{R}_+$

$$P_t(x,dy) \to \nu \ \ (t\to\infty) \text{ weakly.}$$

Because of infinitely divisibility of ν, under (3.1)

$$-\int_{0+}^{\lambda} \frac{F(u)}{R(u)} du =: \Phi(\lambda) = q\lambda + \int_0^{\infty} (1 - e^{-\lambda u}) \Lambda(du)$$

for some $q \geq 0$ **and** Λ **(Lévy measure) on** $(0, \infty)$**.**

Problem: **Study the correspondence**

$$(a, b, n_1, \delta, n_2) \longleftrightarrow (q, \Lambda)$$

Example 1 (CIR model and gamma distr.)

$a, b, \delta > 0, n_1 \equiv 0 \equiv n_2$

$\Longrightarrow R(\lambda) = -a\lambda^2 - b\lambda$, $F(\lambda) = \delta\lambda$ **and**

$$\begin{aligned} \Phi(\lambda) &= \int_0^{\lambda} \frac{\delta}{au + b} du = \frac{\delta}{a} \log\left(1 + \frac{a}{b}\lambda\right) \\ &= \int_0^{\infty} (1 - e^{-\lambda u}) \delta \frac{e^{-bu/a}}{au} du. \end{aligned}$$

Example 2 (stable distribution [Ogura, 1971])

Given $0 < \alpha < \beta < 1$, **set** $a = b = \delta = 0$

$$n_1(dy) = y^{-(2+\alpha)}dy, \quad n_2(dy) = y^{-(1+\beta)}dy.$$

$\Longrightarrow R(\lambda) = -c_1\lambda^{1+\alpha}$, $F(\lambda) = c_2\lambda^{\beta}$, $\Phi(\lambda) = c'\lambda^{\beta-\alpha}$

Example 3 (Interpolating stable and gamma)

Given $0 < \alpha < 1$, $\kappa \geq 0$, **set** $a = 0$, $b = \kappa^{\alpha}$, $\delta > 0$,

$$n_1(dy) = \frac{\alpha}{\Gamma(1-\alpha)}\left(-\frac{e^{-\kappa y}}{y^{1+\alpha}}\right)' dy, \quad n_2 \equiv 0. \tag{3.7}$$

$\Longrightarrow R(\lambda) = -\lambda(\lambda+\kappa)^{\alpha}$, $F(\lambda) = \delta\lambda$ **and**

$$\begin{aligned} \Phi(\lambda) &= \delta(1-\alpha)^{-1}\left[(\lambda+\kappa)^{1-\alpha} - \kappa^{1-\alpha}\right] \\ &= \frac{\delta}{\Gamma(\alpha)}\int_0^{\infty}(1-e^{-\lambda y})\frac{e^{-\kappa y}}{y^{1+(1-\alpha)}}dy. \end{aligned}$$

From now on **let** $n_2 \equiv 0$ **and** $\delta > 0$. **Thus**

$$\begin{aligned} Lf(x) &= axf''(x) - bxf'(x) + \delta f'(x) \\ &\quad + x\int_0^\infty \left[f(x+y) - f(x) - yf'(x) \right] n(dy) \end{aligned}$$

with n **s.t.** $\int \min\{y, y^2\} n(dy) < \infty$.

Remark: Stannat (2005) gave, under the assumptions $a, b > 0, c := \int yn(dy) < \infty$, **a formula**

$$\frac{\Lambda(du)}{du} = \frac{e^{-(b+c)u}}{au} + \frac{1}{u}\sum_{k=1}^{\infty} (k\text{-fold integral} \geq 0).$$

$\longleftrightarrow$ **'perturbation' of a gamma distribution**

§4 GGC as stationary distributions

Consider the "inverse problem": Given Λ, find $(a, b, n, \delta, n_2 \equiv 0) \longleftrightarrow (q = 0, \Lambda)$.

Lévy measures for GGC:

$$\Lambda_m(dy) := \left(\int e^{-uy} m(du) \right) dy/y \tag{4.1}$$

with m (called the Thorin measure) satisfying certain integrability conditions.

Remarks: • The Laplace exponent ($q = 0$) is

$$\Phi_m(\lambda) := \int \log \left(1 + \frac{\lambda}{u} \right) m(du), \qquad \lambda \geq 0 \tag{4.2}$$

• m: degenerate $\Longleftrightarrow$ The GGC : a gamma

Notation & convention: $\overline{m}_\alpha = \int x^\alpha dm$, $\frac{1}{\infty} = 0$.

Theorem (the main result):

m: **a Thorin measure s.t.** $0 < \overline{m}_0 < \infty$, $\overline{m}_1 < \infty$

$\Longrightarrow$ $\exists$ **a unique** $M(du)$ **on** $(0, \infty)$ **s.t.**

$(q = 0, \Lambda_m(dy)) \longleftrightarrow$

$$\left(a = \frac{1}{\overline{m}_0}, b = \frac{1}{\overline{m}_{-1}}, n(dy) = dy \int u^2 e^{-yu} M(du), \delta = 1\right)$$

Moreover $M(\mathrm{R}_+) = \dfrac{\overline{m}_1}{\overline{m}_0^2} - \dfrac{1}{\overline{m}_{-1}} \geq 0$ **(4.4)**

Example 6 (trivial case) m: **degenerate**

$\Longleftrightarrow$ $M \equiv 0$ $\Longleftrightarrow$ $n \equiv 0 \longleftrightarrow$ **a CIR model**

Essence of the proof: We want to find R s.t.

$$-\int_0^\lambda \frac{u}{R(u)} du = \int \log\left(1 + \frac{\lambda}{u}\right) m(du). \qquad (*)$$

Recall that R is the branching mechanism;

$$R(u) = -au^2 - bu - \int_0^\infty \left(e^{-uy} - 1 + uy\right) n(dy).$$

Make the 'ansatz' $n(dy) = dy \int u^2 e^{-yu} M(du)$ to rewrite $d(*)/d\lambda$ as

$$\frac{1}{a\lambda + b + \lambda \int \frac{M(du)}{\lambda + u}} = \int \frac{m(du)}{\lambda + u}, \ \ \lambda > 0. \qquad (**)$$

Use a characterization of Stieltjes transforms to prove the existence of M solving $(**)$.

<u>Characterization of Stieltjes transforms</u>:

Let $0 < c < \infty$. **An analytic function** $g(z)$ **on** $\{\text{Im}\, z > 0\}$ **is of the form**

$$g(z) = \int \frac{1}{z-u} M(du) =: G_M(z), \;\; \text{Im}\, z > 0$$

for some $M(du)$ **on** R **s.t.** $M(\mathrm{R}) = c$ **iff**

$$\begin{cases} \bullet \;\; \text{Im}\, z > 0 \Longrightarrow \;\text{Im}\, g(z) \le 0 \\ \bullet \;\; (iy)g(iy) \to c \;\; (0 < y \to \infty). \end{cases}$$

<u>Our equation $(**)$</u> reads

$$\frac{1}{az - b - zG_M(z)} = G_m(z), \;\; \text{Im}\, z \neq 0. \qquad \text{(ST)}$$

Example 7 (Solving the 'simplest problem'):
Consider $m(du) = \delta_1 \delta_{b_1}(du) + \delta_2 \delta_{b_2}(du)$ **and suppose** $M(du) = c\delta_\kappa(du)$**.**
Then $(**)$ **becomes**

$$\frac{\delta_1}{\lambda + b_1} + \frac{\delta_2}{\lambda + b_2} = \frac{1}{a\lambda + b + \dfrac{c\lambda}{\lambda + \kappa}}, \quad \lambda > 0, \tag{4.6}$$

which suffices to determine a, b, c, κ **uniquely.**
A generalization:
$m(du) = \sum_{i=1}^{l} \delta_i \delta_{b_i}(du)$ **with** $\delta_i > 0, 0 < b_1 < \cdots < b_l \Longrightarrow M(du) = \sum_{i=1}^{l-1} c_i \delta_{\kappa_i}(du)$ **with** $c_i > 0$ **and**

$$b_1 < \kappa_1 < b_2 < \kappa_2 < \cdots < b_{l-1} < \kappa_{l-1} < b_l$$

Absolutely continuous cases:

• **For some examples of $m(du) = f(u)du$ for which dM/dx are calculated by the Stieltjes-Perron inversion formula**

$$\begin{aligned} & M((s,t)) + \frac{M(\{s\}) + M(\{t\})}{2} \\ = & -\frac{1}{\pi} \lim_{y \downarrow 0} \int_s^t \mathrm{Im}\, G_M(x+iy)dx, \ -\infty < s < t < \infty, \end{aligned}$$

see 例8 (4.7), (4.9) in the extended abstract.

• **Conversely, given $a > 0, b \geq 0$ and a finite $M(du)$, we can find $m(du)$ via (ST). See 例9 (4.11), (4.13) for an example.**

(Unexpected) connection with other contexts

(1) Free probability theory. Let $a > 0, b \geq 0$. What is the fixed point of $m \mapsto M$? i.e.

$$G_m(z) = \frac{1}{az - b - zG_m(z)} \qquad \textbf{by (ST)}$$

or

$$G_m(z) = \frac{(az-b) - \sqrt{(az-b)^2 - 4z}}{2z}.$$

By inversion

$$m(du) = \frac{a}{2\pi(1+ab)u}\sqrt{(\beta - u)(u - \alpha)}\mathbf{1}_{(\alpha,\beta)}(u)du$$

$\propto$ **Marchenko-Pastur (free Poisson) law,**

where $\alpha, \beta = a^{-2}(ab + 2 \pm 2\sqrt{ab+1}) \geq 0$.

(2) **Boolean convolution**. For prob. distr's m_1 and m_2, the Boolean convolution $m_1 \oplus m_2$, **by definition**, linearizes '**the self-energy**':

$$K_{m_1 \oplus m_2}(z) = K_{m_1}(z) + K_{m_2}(z),$$

where $K_m(z) := z - G_m(z)^{-1}$.

Proposition Let $m_i (i = 1, 2)$ be Thorin measures s.t. $(q = 0, \Lambda_{m_i}) \longleftrightarrow (a = 1, b_i, n_i, \delta = 1)$. Then $m_1 \oplus m_2$ is a Thorin measure and

$(q = 0, \Lambda_{m_1 \oplus m_2}) \longleftrightarrow (a = 1, b_1 + b_2, n_1 + n_2, \delta = 1)$. **Indeed**, a version of **(ST)** with $a = 1$ reads

$$G_M(z) = z^{-1}(z - b - G_m(z)^{-1}) = z^{-1}(K_m(z) - b).$$