alpha-CIR モデルと beta-Fleming-Viot 過程のエルゴード性*

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Based on a beautiful structure :

$$\mathcal{L}_\alpha \Psi(\eta) = \text{const.}\ \eta(E)^{-\alpha} \mathcal{A}_\alpha \Phi(\eta(E)^{-1}\eta),$$

where $0 < \alpha < 1$, $\Psi(\eta) := \Phi(\eta(E)^{-1}\eta)$,

- $\mathcal{L}_\alpha \longleftrightarrow$ a measure-valued branching process with immigration (MBI-process)
- $\mathcal{A}_\alpha \longleftrightarrow$ a jump-type version of Fleming-Viot (FV) process with 'parent-indep.' mutation

*ArXiv:1307.2407 (submitted)

Plan of talk 1. The models and their relations

2. Spectral gap for α-CIR models

3. Ergodic properties of beta-FV processes

A useful book on MBI-processes

[Measure-valued branching Markov processes, Li, Z. 2011, Springer]

Selected works on generalized FV processes

- Pioneer works are [Donnelly, Kurtz 1999], [Hiraba 2000], [Bertoin, Le Gall 2003, 2005, 2006].
- Closely related models to ours were discussed in [Alpha-stable branching and beta-coalescents, Birkner, Blath, Capaldo, Etheridge, Möhle, Schweinsberg, Wakolbinger 2005], [Foucart 2011], [Foucart, Hénard 2013]

α-CIR model (on $[0,\infty)$)

Generator [$0<\alpha<1, a>0, b\in\mathbf{R}, c\geq 0$:given]

$$L_\alpha f(z) = az\int\Big[f(z+y)-f(z)-yf'(z)\Big]n_B(dy) -\frac{b}{\alpha}zf'(z)+c\int[f(z+y)-f(z)]\,n_I(dy)$$

with $n_B(dy)=\frac{\alpha+1}{\Gamma(1-\alpha)y^{\alpha+2}}dy$, $n_I(dy)=\frac{\alpha}{\Gamma(1-\alpha)y^{\alpha+1}}dy$.
↑ branching ↑ immigration

- $L_\alpha f(z)\to Lf(z)$ $(\alpha\uparrow 1)$ generator of CIR model:

$$Lf(z):=azf''(z)+(-bz+c)f'(z).$$

- It belongs to the class of CBI-processes (continuous-state branching processes with immigration). [Kawazu, Watanabe 1971]

Measure-valued α-CIR model

E : a compact metric space

$\mathcal{M}(E) = \{$**finite** Borel measures on $E\}$

$a \in C_{++}(E), b \in C(E), m \in \mathcal{M}(E)$: given

<u>Generator</u> $\quad [\langle \eta, f \rangle := \int_E f d\eta]$

$$\mathcal{L}_\alpha \Psi(\eta) = \int n_B(dz) \int_E a(r)\eta(dr) \left[\Psi(\eta + z\delta_r) - \Psi(\eta) - z\frac{\delta \Psi}{\delta \eta}(r) \right]$$
$$-\frac{1}{\alpha}\langle b\eta, \frac{\delta \Psi}{\delta \eta} \rangle + \int n_I(dz) \int_E m(dr) [\Psi(\eta + z\delta_r) - \Psi(\eta)]$$

It is a MBI-process with

<u>branching mechanism</u> $\alpha^{-1}(a(r)f(r)^{\alpha+1} + b(r)f(r))$

<u>immigration mechanism</u> $\langle m, f^\alpha \rangle$.

‘Beta-Fleming-Viot’ process

Assume that $\sharp E \geq 2$. $B_{a,b}(du)$: **beta law**

Our beta-FV process

on $\mathcal{M}_1(E) := \{$Borel probab. measures on $E\}$ has generator

$$\mathcal{A}_\alpha \Phi(\mu) = \int_0^1 \frac{B_{1-\alpha,1+\alpha}(du)}{u^2} \int_E \mu(dr)\,[\Phi((1-u)\mu + u\delta_r) - \Phi(\mu)] + \int_0^1 \frac{B_{1-\alpha,\alpha}(du)}{(\alpha+1)u} \int_E m(dr)\,[\Phi((1-u)\mu + u\delta_r) - \Phi(\mu)].$$

(simultaneous reproduction) + (simultaneous mutation)

[**Birkner et al 2005**] $m = 0$, [**Foucart, Hénard 2013**] **degenerate** m

Remark $\mathcal{A}_\alpha \Phi(\mu) \to \mathcal{A}\Phi(\mu)$ $(\alpha \uparrow 1)$, **generator of FV-process**

$$\mathcal{A}\Phi(\mu) = \frac{1}{2}\langle \mu(dr)\delta_r(ds) - \mu(dr)\mu(ds), \frac{\delta^2\Phi}{\delta\mu^2}\rangle + \frac{1}{2}\langle m - m(E)\mu, \frac{\delta\Phi}{\delta\mu}\rangle.$$

Key identity (KI)

Proposition Assume that $a \equiv 1 \equiv b$. Then

$$\mathcal{L}_\alpha \Psi(\eta) = \Gamma(\alpha + 2)\ \eta(E)^{-\alpha} \mathcal{A}_\alpha \Phi(\eta(E)^{-1}\eta),$$

where $\eta \in \mathcal{M}(E)$ with $\eta(E) > 0$, $\Psi(\eta) := \Phi(\eta(E)^{-1}\eta)$.

Remarks (i) Only $\mathcal{A}_\alpha$'s enjoy this kind of relation to MBI-proc's. (cf. [Birkner et al 2005], [Foucart, Hénard 2013])
(ii) (KI) for FV-processes ('$\alpha = 1$') is found in [Shiga 1990].

cf. $$\lim_{\alpha \uparrow 1} \mathcal{L}_\alpha \Psi(\eta) = \langle \eta, a\frac{\delta^2 \Psi}{\delta \eta^2}\rangle - \langle \eta, b\frac{\delta \Psi}{\delta \eta}\rangle + \langle m, \frac{\delta \Psi}{\delta \eta}\rangle =: \mathcal{L}\Psi(\eta)$$

Stationary distribution ; α-CIR case

$(T(t))$: semigroup generated by $\mathcal{L}_\alpha$ [$\Psi_f(\eta) := e^{-\langle \eta, f\rangle}$]

Theorem 1 (1) For $t \geq 0$

$$T(t)\Psi_f(\eta) = \exp\left[-\langle \eta, V_t f\rangle - \int_0^t \langle m, (V_s f)^\alpha\rangle ds\right], \qquad f \geq 0,$$

where $V_t f(r) = \dfrac{e^{-b(r)t/\alpha} f(r)}{\left[1+a(r)f(r)^\alpha \int_0^t e^{-b(r)s} ds\right]^{1/\alpha}}$.

(2) If $b \in C_{++}(E)$, a unique stationary distribution $\mathcal{Q}_\alpha$ satisfies

$$\int_{\mathcal{M}(E)} e^{-\langle \eta, f\rangle} \mathcal{Q}_\alpha(d\eta) = e^{-\langle m, a^{-1} \log(1+ab^{-1}f^\alpha)\rangle}, \quad f \geq 0.$$

Proof. Apply [Kawazu, Watanabe 1971]'s theory.

Stationary distribution ; β-FV case

Theorem 2 If $m(E) > 0$, then a unique stationary distribution of the process associated with $\mathcal{A}_\alpha$ is

$$\mathcal{P}_\alpha(\bullet) := \Gamma(\alpha+1)\int_{\mathcal{M}_1(E)} \mathcal{D}_m(d\mu) E^{\mathcal{S}_{\alpha,\mu}}\left[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \bullet\right],$$

where $\mathcal{D}_m$ is the law of a Dirichlet random measure with parameter m and $\int_{\mathcal{M}(E)} e^{-\langle \eta, f\rangle}\mathcal{S}_{\alpha,\mu}(d\eta) = e^{-\langle \mu, f^\alpha\rangle}$, $\quad f \geq 0$.

Proof. By **(KI)** $\mathcal{A}_\alpha$-process has a stationary distribution

$$\tilde{\mathcal{P}}_\alpha(\bullet) := E^{\mathcal{Q}_\alpha}\left[\eta(E)^{-\alpha}; \eta(E)^{-1}\eta \in \bullet\right] / E^{\mathcal{Q}_\alpha}\left[\eta(E)^{-\alpha}\right]$$

if $E^{\mathcal{Q}_\alpha}\left[\eta(E)^{-\alpha}\right] < \infty$. $\tilde{\mathcal{P}}_\alpha = \mathcal{P}_\alpha$ can be shown if $m(E) > 1$.

Ergodic property ; α-CIR case

Theorem 3 If $b \in C_{++}(E)$ and $m(E) > 0$, then

(SG) $\mathrm{var}_{\mathcal{Q}_\alpha}(\Psi) \leq 2 \operatorname*{ess\,sup}_{(E,m)}(b^{-1}) \ E^{\mathcal{Q}_\alpha}[(-\mathcal{L}_\alpha)\Psi \cdot \Psi]$.

Moreover, the constant is optimal.

Remarks (i) In case of '$\alpha = 1$', the optimal constant is $\operatorname{ess\,sup}_{(E,m)}(b^{-1})$. [Stannat 2003]

(ii) Non-symmetric Dirichlet form:

$$\frac{E^{Q_\alpha}\left[(-\mathcal{L}_\alpha)\Psi_f \cdot \Psi_g\right]}{E^{Q_\alpha}\left[\Psi_{f+g}\right]} = \left\langle m, f^\alpha - \frac{(f+g)^{\alpha-1}(af^{\alpha+1}+bf)}{b+a(f+g)^\alpha}\right\rangle$$

Proof of Theorem 3 (1/3)

Mimicking [Stannat 2005]'s argument:

• Since $\mathcal{L}_\alpha$ is regarded as a 'direct sum' of the one-dimensional version (generating an α-CIR model)

$$\begin{aligned} L_\alpha f(z) &= az\int \left[f(z+y) - f(z) - yf'(z)\right] n_B(dy) \\ &\quad -\frac{b}{\alpha} z f'(z) + c\int [f(z+y) - f(z)]\, n_I(dy) \end{aligned}$$

with $a, b, c > 0$, (SG) for $\mathcal{L}_\alpha$ is reduced to (SG) for L_α.

• Thanks to infinite divisibility of the stationary distribution Q_α for L_α, (SG) is reduced to its 'Lévy measure version' (given below).

Proof of Theorem 3 (2/3)

Λ_α: **Lévy measure of Q_α with $a = b = c = 1$, i.e.,**

$$\int (1 - e^{-\lambda z}) \Lambda_\alpha(dz) = \log(1 + \lambda^\alpha), \quad \lambda > 0$$

Showing (SG) for L_α can be reduced to

$$\begin{aligned} &\int (F(z) - F(0))^2 \Lambda_\alpha(dz) \\ &\quad \le \int \Lambda_\alpha(dz) z \int n_B(dy)(F(z+y) - F(z))^2 \\ &\qquad + \int n_I(dy)(F(y) - F(0))^2, \qquad F \in D, \end{aligned}$$

where $D := \text{L.S.}\{c\ e^{-\lambda z} : c \in \mathbb{R}, \lambda > 0\}$.

Showing this is based on explicit representations for $F(z) = \sum c_i e^{-\lambda_i z}$. (rather difficult !)

Proof of Theorem 3 (3/3)

To prove optimality of (SG) we only have to show that there exists a Ψ such that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathrm{var}_{\mathcal{Q}_\alpha}(T(t)\Psi) = -\frac{1}{2} \operatorname*{ess\,inf}_{(E,m)} b.$$

This can be shown for $\Psi(\eta) = \Psi_1(\eta) = e^{-\eta(E)}$, for which an explicit form of $T(t)\Psi(\eta)$ is available.

Remark For '$\alpha = 1$', the corresponding limit is evaluated as $-\operatorname*{ess\,inf}_{(E,m)} b$.

Ergodic property ; β-FV case

$(S(t))$: semigroup on $L^2(\mathcal{P}_\alpha)$ generated by $\mathcal{A}_\alpha$

Theorem 4 **Assume that** $\theta := m(E) > 1$.

Then $\exists\ C = C(\alpha, \theta) < \infty$ such that for $\forall\ \Phi \in L^2(\mathcal{P}_\alpha)$

$$\limsup_{t\to\infty} \mathrm{var}_{\mathcal{P}_\alpha}(S(t)\Phi)\frac{t^{\theta-1}}{\log t} \le C\ \mathrm{osc}^2(\Phi),$$

where $\mathrm{osc}^2(\Phi)$ is the ess. sup. of $|\Phi(\mu_1) - \Phi(\mu_2)|^2$ with respect to $\mathcal{P}_\alpha \otimes \mathcal{P}_\alpha$.

Remarks (i) We have NOT seen that exponential convergence does NOT hold.

(ii) Nothing has been shown for the case $\theta \le 1$.

Proof of Theorem 4 (1/2)

(a) (KI) implies

$$E^{\mathcal{Q}_\alpha}[(-\mathcal{L}_\alpha)\Psi \cdot \Psi] = C_{\alpha,\theta}\ E^{\mathcal{P}_\alpha}[(-\mathcal{A}_\alpha)\Phi \cdot \Phi],$$

where $\Psi(\eta) := \Phi(\eta(E)^{-1}\eta)$.

(b) (SG) for $\mathcal{L}_\alpha$ **with** $b \equiv 1$ **:**

$$\mathrm{var}_{\mathcal{Q}_\alpha}(\Psi) \le 2E^{\mathcal{Q}_\alpha}[(-\mathcal{L}_\alpha)\Psi \cdot \Psi] \qquad \text{(Theorem 3)}$$

(c) By Hölder's inequality

$$\mathrm{var}_{\mathcal{P}_\alpha}(\Phi) \le \tilde{C}_{\alpha,\theta}\left(E^{\mathcal{Q}_\alpha}\left[\eta(E)^{-\alpha q}\right]\mathrm{osc}^2(\Phi)\right)^{\frac{1}{q}}\left(\mathrm{var}_{\mathcal{Q}_\alpha}(\Psi)\right)^{\frac{1}{p}},$$

where $q \in (1,\theta)$ **is arbitrary and** $1/p + 1/q = 1$.

Proof of Theorem 4 (2/2)

(1) By (a)-(c)

$$\mathrm{var}_{\mathcal{P}_\alpha}(\Phi) \le C_{\alpha,\theta,q}\mathrm{osc}^2(\Phi)^{\frac{1}{q}}\left(E^{\mathcal{P}_\alpha}[(-\mathcal{A}_\alpha)\Phi\cdot\Phi]\right)^{\frac{1}{p}}.$$

In addition, $\mathrm{osc}^2(S(t)\Phi) \le \mathrm{osc}^2(\Phi)$ $(t > 0)$.

(2) [Liggett 1991] 's theorem yields

$$\mathrm{var}_{\mathcal{P}_\alpha}(S(t)\Phi) \le \tilde{C}_{\alpha,\theta,q}\mathrm{osc}^2(\Phi)t^{1-q}, \qquad t > 0.$$

(3) Optimizing the value of $q \in (1,\theta)$ **for each** $t \gg 1$ **gives the result;** $\log t$ **appears as a 'trade-off'.**

[Liggett 1991]'s theorem

P : a stationary distribution of a Markov process

$(S(t))$: strongly continuous semigroup on $L^2(P)$

A : generator of $(S(t))$

$\mathcal{V} : L^2(P) \to [0, \infty]$ satisfies $\mathcal{V}(cf + d) = c^2\mathcal{V}(f)$ $(\forall c, d \in \mathbf{R})$

& $\mathcal{V}(S(t)f) \le \mathcal{V}(f)$ $(\forall t > 0)$. $p, q > 1$ satisfy $1/p + 1/q = 1$.

Then

$$\mathrm{var}_P(f) \le C \left(E^P\left[(-Af)f\right]\right)^{1/p} \mathcal{V}(f)^{1/q}, \ \forall f \in D(A)$$

implies

$$\mathrm{var}_P(S(t)f) \le \mathcal{V}(f) C^q \left(\frac{q-1}{2t}\right)^{q-1}, \ \forall f \in L^2(P), t > 0.$$

Characterization of the $\mathcal{A}_\alpha$-process

Proposition The closure of $\mathcal{A}_\alpha$ defined on

$$\left\{\Phi_f(\mu) = \langle f, \mu^{\otimes n}\rangle : f \in C(E^n), n \in \mathbf{N}\right\}$$

generates a **Feller semigroup** on $C(\mathcal{M}_1(E))$.

Proof uses a concrete form of $\mathcal{A}_\alpha \Phi_f(\mu)$ and Hille-Yosida's theorem.

Remark The same result holds for '$\alpha = 1$' as proved in [Ethier, Kurtz 1993].