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Preprint · April 2022

DOI: 10.13140/RG.2.2.17979.90400

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April 18, 2022

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Abstract: A multivariate *t*-distribution is introduced such that each element of a normal vector is scaled by using a chi-square that is independent of the chi-squares for the remaining elements of the vector with possibly distinct degrees of freedom. The integral and series expressions of the probability density function are given. The absolute values of the covariances/correlations and Mardia's multivariate measure of kurtosis are shown to be smaller than those for the corresponding usual multivariate *t* with a common chi-square. An extended multivariate *t* with common and unique chi-squares is also proposed, which takes a form like factor analysis.

Keywords: Student *t*, independent chi-squares, series expression, Mardia's multivariate kurtosis, factor analysis, multivariate gamma.

1. Introduction

Student's *t*-distribution is one of the non-normal distributions that have been most intensively investigated by researchers in spite of the non-existence of higher-order moments under finite degrees of freedom. Another intractable aspect is associated with the various definitions of the multivariate versions (see e.g., Kotz & Nadarajah, 2004, Chapters 4 and 5). Currently, the most familiar multivariate *t*-distribution (see Kotz & Nadarajah, 2004, Equation (1.1)) can be traced back to Cornish (1954) (see also Gupta, 2003, p. 360). The *p*-dimensional vector $\mathbf{Z}$ following this distribution is denoted by $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, whose probability density function (pdf) at $\mathbf{Z} = \mathbf{z}$ is

$$t_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu) = \frac{\Gamma\{(\nu + p)/2\}}{(\pi\nu)^{p/2}\Gamma(\nu/2)\,|\,\boldsymbol{\Sigma}\,|^{1/2}} \left\{1 + \frac{(\mathbf{z} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z} - \boldsymbol{\mu})}{\nu}\right\}^{-(\nu+p)/2},$$

where ν is the degrees of freedom (df), which typically takes positive integers but can be positive real values; $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}^{1/2}$ are the location and scale parameters, respectively with $\boldsymbol{\Sigma}^{1/2}$ being the symmetric matrix-square-root of positive definite $\boldsymbol{\Sigma}$; and $\Gamma(\cdot)$ is the gamma function.

The above distribution when $p = 1$ gives the pdf of the univariate *t*:

$$\begin{aligned} t(z \mid \mu, \sigma^2, \nu) &= \frac{\Gamma\{(\nu+1)/2\}}{(\pi\nu)^{1/2}\Gamma(\nu/2)\sigma} \left\{1 + \frac{(z-\mu)^2}{\nu\sigma^2}\right\}^{-(\nu+1)/2} \\ &= \frac{1}{\sqrt{\nu}\sigma\,\mathrm{B}(1/2, \nu/2)} \left\{1 + \frac{(z-\mu)^2}{\nu\sigma^2}\right\}^{-(\nu+1)/2}, \end{aligned}$$

where $\sigma^2 = \boldsymbol{\Sigma}$ when $p = 1$ and $\mathrm{B}(\cdot\,,\,\cdot)$ is the beta function. It is well known that variable Z following the univariate *t* denoted by $Z \sim \mathrm{St}(\mu, \sigma^2, \nu)$ is derived when

$$Z - \mu = (X - \mu)/\sqrt{Y/\nu}\,, \tag{1}$$

where *X* follows the normal distribution with mean μ and variance σ^2 i.e., $X \sim \mathrm{N}_1(\mu, \sigma^2) = \mathrm{N}(\mu, \sigma^2)$; and *Y* follows the chi-square with ν df independent of *X*. denoted by $Y \sim \chi^2(\nu)$.

It is known that the pdf of the above multivariate *t* for random vector $\mathbf{Z}$ is obtained when *Z* and *X* are replaced by $\mathbf{Z}$ and $\mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, respectively with unchanged $Y \sim \chi^2(\nu)$ independent of $\mathbf{X}$ as

$$\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu}\ .$$

It is also known that its pdf is given by the mixture of the precision matrix i.e., $\boldsymbol{\Sigma}^{-1}$ of the multivariate normal with the gamma weight (Stuart & Ort, 1994, Example 5.6; Bishop, 2006, Equation 2.161; Forbes & Wraith, 2014, Equation (2); Babić, Ley, & Veredas, 2019, Equation (5); Kirkby, Nguyen & Nguyen, 2019, Equation (9); 2021, Equation (3.1)). Since the mixture expression is typically used to show the pdf in many cases without referring to the equation $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu}$, the following is summarized for ease of reference, where the proof with other proofs for theorems and corollaries, when necessary, will be given in the appendix.

Lemma 1 *Let* $\phi_p(\mathbf{x} \,|\, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ *be the pdf of* $\mathbf{X} = \mathbf{x}$ *when* $\mathbf{X} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; *and* $g_{\Gamma}(y \,|\, \alpha, \beta)$ *be the pdf of Y = y when Y is gamma distributed with the shape and rate parameters* α *and* β, *respectively. Then, the pdf of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ *and its mixture expressions*

$$\int_0^{\infty} \phi_p\{\mathbf{z} \,|\, \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu / (cy)\} g_{\Gamma}(y \,|\, \nu / 2, c / 2) \mathrm{d}y \ \ (0 < c < \infty)$$

with c being an arbitrary constant, are given from $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu}$ *with associated assumptions.*

It is found that the difference of the mixture and explicit pdf expressions is how to use the Jacobian $(y / \nu)^{p/2}$ in the variable transformation from $\mathbf{X}$ to $\mathbf{Z}$. That is, in the former case the Jacobian is used to construct a modified normal pdf before the arbitrary constant *c* is introduced while in the latter case it is used to have a modified gamma to be integrated out. Note that when *c* = 1, the chi-square weight i.e., $g_{\Gamma}(y \,|\, \nu / 2, 1 / 2) = g_{\chi^2}(y \,|\, \nu)$ is unchanged while $\boldsymbol{\Sigma}^{-1} / \nu$ in the modified $\phi_p(\mathbf{z} \,|\, \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu / y)$ is the scaled precision matrix in the mixture expression. When $c = \nu$, $\phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu / (cy)\}$ becomes $\phi_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}y)$ giving the unscaled precision $\boldsymbol{\Sigma}^{-1}$ while $g_\Gamma(y \mid \nu/2, \nu/2)$ is a scaled chi-square or the gamma density with the modified rate parameter $\nu / 2$, which seems to be usually used in the mixture expression as in the references given earlier among infinitely many cases using c.

Another indeterminacy is due to the transformation of variable Y. For instance, when $Y^* = \sqrt{Y}$ is used, Y^* is chi-distributed with unchanged distribution of $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu} = (\mathbf{X} - \boldsymbol{\mu}) / (Y^* / \sqrt{\nu})$. More generally, an arbitrary γ^{-1}-th power of Y $(-\infty < \gamma < \infty,\ \gamma \neq 0)$ can be used. In other words, Y^* is defined such that $Y^{*\gamma} = (Y^*)^\gamma$ follows the chi-square with ν df. Ogasawara (2021c) defined the power-gamma distribution whose pdf is

$$g_{\text{Power-}\Gamma}(y \mid \alpha, \beta, \gamma) = \frac{\beta^\alpha y^{\gamma(\alpha-1)} \,|\gamma|\, y^{\gamma-1}}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma\right)$$

$$= \frac{\beta^\alpha \,|\gamma|\, y^{\gamma\alpha-1}}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma\right)$$

$$(0 < y < \infty, 0 < \alpha < \infty,\ 0 < \beta < \infty,\ -\infty < \gamma < \infty,\ \gamma \neq 0).$$

This is the distribution when γ-th power of Y follows the gamma with the shape and rate parameters α and β, respectively. Note that the chi-square with ν df is equal to the gamma when $\alpha = \nu / 2$ and $\beta = 1/2$. That is, the power-gamma distributed Y^* with the pdf $g_{\text{Power-}\Gamma}(y \mid \nu/2, 1/2, \gamma)$ can be used to have the multivariate *t*-distribution such that

$$\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y^{*\gamma} / \nu}$$

In the chi-distributed case, $\gamma = 2$. When $\gamma = -2$, we have the inverse-chi distributed Y^* with $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu})Y^*\sqrt{\nu}$, which gives a convenient property to obtain moments of $\mathbf{Z}$ since Y^* and $\mathbf{X}$ are independent (see Kollo, Käärik & Selart 2021; Ogasawara, 2021c). The power gamma with various powers can also be used in the mixture expression as well as various scale parameters using c.

Although the pdf of $\text{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ gives a closed-form and transparent expression, we have an intractable property. That is, when $\mathbf{\Sigma}$ is diagonal or $X_i (i = 1,..., p)$ are independently distributed under normality, it can be shown that $Z_i (i = 1,..., p)$ are uncorrelated, but mutually dependent due to the common chi-square Y used in $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu}$ (see e.g., Jones, 2002a).

2. The multivariate *t*-distribution with independent chi-squares

The usual multivariate *t*-distribution $\mathrm{St}(\boldsymbol{\mu}, \mathbf{\Sigma}, \nu)$ was defined using a single common chi-square. On the other hand a modified multivariate *t* is defined using independent chi-squares as follows:

Definition 1 A random *p*-vector $\mathbf{Z}$ is said to follow the multivariate *t*-distribution with independent chi-squares, when we use a multivariate normal vector $\mathbf{X} = (X_1,..., X_p)^{\mathrm{T}} \sim \mathrm{N}_p(\boldsymbol{\mu}, \mathbf{\Sigma})$ and *p* independent chi-squared variables $Y_i \sim \chi^2(\nu_i)$ $(i = 1,...., p)$ in $\mathbf{Y} = (Y_1,..., Y_p)^{\mathrm{T}}$ independent of $\mathbf{X}$ with possibly distinct df's $\boldsymbol{\nu} = (\nu_1,...,\nu_p)^{\mathrm{T}}$ such that

$$\mathbf{Z} - \boldsymbol{\mu} = \{(X_1 - \mu_1) / \sqrt{Y_1 / \nu_1},...,(X_p - \mu_p) / \sqrt{Y_p / \nu_p}\}^{\mathrm{T}}.$$

This multivariate *t* is denoted by $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\nu})$.

It is found that when $\mathbf{\Sigma}$ is diagonal, the marginal distributions in $\mathrm{St}(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\nu})$ become independent, which is not attained by the usual $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \mathbf{\Sigma}, \nu)$ with ν being a scalar. Shaw and Lee (2008, Equation (2.6)) considered a similar distribution in the bivariate case. Forbes and Wraith (2014, Equation (8)) gave a model using independent normal X_i's as well as independent Y_i's as

$$\mathbf{Z} - \boldsymbol{\mu} = \mathbf{B}\{(X_1 - \mu_1) / \sqrt{Y_1},...,(X_p - \mu_p) / \sqrt{Y_p}\}^{\mathrm{T}},$$

where $\mathbf{B}$ is a fixed matrix which is similar to our model, but is different in that their model reduces to the linear combinations of independent univariate *t*'s with possibly distinct df's.

Theorem 1 *The integral and expectation expressions of the pdf of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \mathbf{\Sigma}, \boldsymbol{\nu})$ *at* $\mathbf{Z} = \mathbf{z}$ *are given by*

$$C_p \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \exp(-\mathbf{u}^{\mathrm{T}} \mathbf{\Psi}_{\mathbf{z}}^{-1} \mathbf{u} / 2) \mathrm{d}\mathbf{u}$$
$$= C_p (2\pi)^{p/2} |\mathbf{\Psi}_{\mathbf{z}}|^{1/2} \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \phi_p(\mathbf{u} | \mathbf{0}, \mathbf{\Psi}_{\mathbf{z}}) \mathrm{d}\mathbf{u}$$
$$= C_p (2\pi)^{p/2} |\mathbf{\Psi}_{\mathbf{z}}|^{1/2} \ \mathrm{E}(\mathbf{u}^{\mathbf{v}}; \mathbf{0}, \mathbf{\Psi}_{\mathbf{z}}; \mathbf{A} = \mathbf{0},\ \mathbf{B} = \infty \mathbf{1}_p),$$

where

$$C_p = \frac{2^{p/2}}{\pi^{p/2} |\mathbf{\Sigma}|^{1/2}} \left\{ \prod_{i=1}^{p} \frac{1}{\sqrt{\nu_i}\, 2^{\nu_i/2} \Gamma(\nu_i / 2)} \right\},\ \ \mathbf{u}^{\mathbf{v}} = u_1^{\nu_1} \cdots u_p^{\nu_p},$$

$$\mathbf{\Psi}_{\mathbf{z}} = \left[\mathrm{diag}\{(\mathbf{z} - \boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} \mathbf{\Sigma}^{-1} \mathrm{diag}\{(\mathbf{z} - \boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} + \mathbf{I}_p \right]^{-1};$$

$\mathbf{v}^{-1/2} = (\nu_1^{-1/2}, \ldots, \nu_p^{-1/2})^{\mathrm{T}}$; $\odot$ *indicates Hadamard or elementwise product*; $\mathbf{I}_p$ *is the* $p \times p$ *identity matrix*; $\mathbf{1}_p$ *is the* $p \times 1$ vector *of* 1'*s*; *and* **A** *and* **B** *indicate the R pairs of the lower and upper endpoints, respectively for the partial product moments of* **U** *under normality with sectional truncation in the case of a single interval for selection i.e., R* = 1 *(see Ogasawara, 2021b)*:

$$\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{u}^{\mathbf{v}} \phi_p(\mathbf{u} | \mathbf{0}, \mathbf{\Psi}_{\mathbf{z}}) \mathrm{d}\mathbf{u} = \sum_{r=1}^{R} \int_{\mathbf{a}_r}^{\mathbf{b}_r} \mathbf{u}^{\mathbf{v}} \phi_p(\mathbf{u} | \mathbf{0}, \mathbf{\Psi}_{\mathbf{z}}) \mathrm{d}\mathbf{u}$$
$$= \mathrm{E}\{\mathbf{u}^{\mathbf{v}}; \mathbf{0}, \mathbf{\Psi}_{\mathbf{z}}; \mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_R),\ \mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_R)\}.$$

Theorem 1 can be used to have the pdf at **Z** = **z** using e.g., a numerical integral. Let $\alpha = \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \exp(-\mathbf{u}^{\mathrm{T}} \mathbf{\Psi}_{\mathbf{z}}^{-1} \mathbf{u} / 2) \mathrm{d}\mathbf{u}$. Then, it is found that α is the normalizer of a special case of the multivariate basic parabolic cylinder (bpc) distribution for $p \times 1$ random vector **U** introduced by Ogasawara (2021c), whose pdf when **U** = **u** is $\alpha^{-1} \mathbf{u}^{\mathbf{v}} \exp(-\mathbf{u}^{\mathrm{T}} \mathbf{\Psi}_{\mathbf{z}}^{-1} \mathbf{u} / 2)$. The multivariate bpc is a multivariate extension of the univariate bpc distributions of the first and second kinds (Ogasawara, 2021a, Definitions 1 and 2). The pdf of the general case of the multivariate bpc is

$$\frac{\mathbf{u}^{\mathbf{v}} \exp\{-(\mathbf{u} - \mathbf{d})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{u} - \mathbf{d}) / 2\}}{\int_{\mathbf{0}}^{\infty} \mathbf{t}^{\mathbf{v}} \exp\{-(\mathbf{t} - \mathbf{d})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{t} - \mathbf{d}) / 2\} \mathrm{d}\mathbf{t}},$$

where **d** and **C** are fixed quantities of appropriate sizes. It is seen that in the case of

Theorem 1, $\mathbf{d} = \mathbf{0}$ and $\mathbf{C} = \mathbf{\Psi}_{\mathbf{z}}$. An alternative expression of the pdf in Theorem 1 is derived as follows:

Theorem 2 *A series expression of the pdf in Theorem 1 is given by*

$$C_p \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \exp(-\mathbf{u}^{\mathrm{T}} \mathbf{\Psi}_{\mathbf{z}}^{-1} \mathbf{u} / 2) \mathrm{d}\mathbf{u}$$

$$= \frac{1}{\pi^{p/2} |\mathbf{\Sigma}|^{1/2}} \left\{ \prod_{i=1}^{p} \frac{1}{\sqrt{\nu_i} \Gamma(\nu_i / 2) (\psi_{\mathbf{z}}^{ii})^{(\nu_i+1)/2}} \right\}$$

$$\times \sum_{\mathbf{u}=\mathbf{0}}^{\infty} \left\{ \prod_{i=1}^{p} \Gamma\left(\frac{\nu_i + 1 + u_i^*}{2} \right) \right\} \prod_{g<h} \left(\frac{-2\psi_{\mathbf{z}}^{gh}}{\sqrt{\psi_{\mathbf{z}}^{gg} \psi_{\mathbf{z}}^{hh}}} \right)^{u_{gh}} \frac{1}{u_{gh}!},$$

where $\psi_{\mathbf{z}}^{gh}$ *is the (g, h)th element of*

$$\mathbf{\Psi}_{\mathbf{z}}^{-1} = \mathrm{diag}\{(\mathbf{z} - \boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} \mathbf{\Sigma}^{-1} \mathrm{diag}\{(\mathbf{z} - \boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} + \mathbf{I}_p$$

$(g, h = 1, ..., p)$; $\sum_{\mathbf{u}=\mathbf{0}}^{\infty}(\cdot) = \sum_{u_{12}=0}^{\infty} \cdots \sum_{u_{p-1,p}=0}^{\infty}(\cdot)$; $u_i^* = \sum_{g=1}^{p-1} \sum_{h=g+1}^{p} u_{gh}(\delta_{gi} + \delta_{hi})$ $= \sum_{g<h} u_{gh}(\delta_{gi} + \delta_{hi})$; δ_{gi} *is the Kronecker delta; and* $\prod_{g<h}$ *is defined similarly to* $\sum_{g<h}$.

The series expression of Theorem 2 is simpler than that of the general case due to $\mathbf{d} = \mathbf{0}$. We have no parabolic cylinder function employed in the series expression since the function becomes 1 when its argument is 0 due to $\mathbf{d} = \mathbf{0}$. Theorem 2 can be used to have the pdf of $\mathbf{Z} = \mathbf{z}$ by the numerical method, where ∞ in the infinite series $\sum_{\mathbf{u}=\mathbf{0}}^{\infty}(\cdot) = \sum_{u_{12}=0}^{\infty} \cdots \sum_{u_{p-1,p}=0}^{\infty}(\cdot)$ is replaced by a finite integer, which is increased until the value of the pdf is unchanged or the difference becomes within a reasonably small positive value.

Corollary 1 *The pdf at* $\mathbf{Z} = \mathbf{z}$ *of the multivariate t with independent chi-squares is expressed by the multiple mixtures of the precision matrix of the multivariate normal density for* $\mathbf{Z}$ *using p independent chi-squared or variously gamma-distributed variables*:

$$\int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}\left\{\prod_{i=1}^{p} g_{\chi^2}(y_i \mid \nu_i)\right\}\mathrm{d}\mathbf{y}$$

$$= \int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y}/\mathbf{v})\}$$

$$\times\left\{\prod_{i=1}^{p} g_{\Gamma}(y_i \mid \nu_i/2, c_i/2)\right\}\mathrm{d}\mathbf{y},$$

$$\mathbf{c} = (c_1,\ldots,c_p)^{\mathrm{T}},\ 0 < c_i < \infty \ \ (i = 1,\ldots,p),$$

where c_i*'s are arbitrary fixed constants.*

Though the result of Corollary 1 is insightful, the actual computation is obtained by those in Theorems 1 and 2 using numerical integral or approximations of the infinite series by finite ones, respectively. The next results are immediate consequences of Lemma 1 and Corollary 1.

Corollary 2 *The mixture expression of the pdf at* $\mathbf{Z} = \mathbf{z}$ *of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and that of $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ *under sectional truncation using R pairs of selection points* $\mathbf{A}$ *and* $\mathbf{B}$ *defined in Theorem 1, which are denoted by* $t_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu; \mathbf{A}, \mathbf{B})$ *and* $t_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v}; \mathbf{A}, \mathbf{B})$*, respectively, are*

$$t_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu; \mathbf{A}, \mathbf{B}) = \int_0^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy); \mathbf{A}, \mathbf{B}\} g_{\Gamma}(y \mid \nu/2, c/2)\mathrm{d}y \ \ (0 < c < \infty),$$

where

$$\phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy); \mathbf{A}, \mathbf{B}\} = \frac{\phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy)\}}{\int_{\mathbf{A}}^{\mathbf{B}} \phi_p\{\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy)\}\mathrm{d}\mathbf{x}}$$

and;

$$t_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v}; \mathbf{A}, \mathbf{B}) = \int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y}/\mathbf{v}); \mathbf{A}, \mathbf{B}\}$$

$$\times\left\{\prod_{i=1}^{p} g_{\Gamma}(y_i \mid \nu_i/2, c_i/2)\right\}\mathrm{d}\mathbf{y}$$

$$\mathbf{c} = (c_1,\ldots,c_p)^{\mathrm{T}},\ 0 < c_i < \infty \ \ (i = 1,\ldots,p),$$

where

$$\phi_p\{\mathbf{z}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v}); \mathbf{A},\ \mathbf{B}\}$$
$$= \frac{\phi_p\{\mathbf{z}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\}}{\int_{\mathbf{A}}^{\mathbf{B}}\phi_p\{\mathbf{x}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\}\mathrm{d}\mathbf{x}},$$

respectively.

Note that Kirkby et al. (2021, Theorem 3.1) used a special case of the result of $t_p(\mathbf{z}\,|\,\boldsymbol{\mu},\boldsymbol{\Sigma},\nu;\mathbf{A},\mathbf{B})$ under double truncation i.e., $\mathbf{A}=\mathbf{a}_1$ and $\mathbf{B}=\mathbf{b}_1$ when $R = 1$ and $c=\nu$.

Theorem 3 *The product moments of the multivariate t with independent chi-squares are given by*

$$\mathrm{E}(\mathbf{Z}^{\mathbf{k}}|\boldsymbol{\mu},\boldsymbol{\Sigma},\mathbf{v}) = \mathrm{E}(Z_1^{k_1}\cdots Z_p^{k_p}|\boldsymbol{\mu},\boldsymbol{\Sigma},\mathbf{v}) = \boldsymbol{\mu}^{\mathbf{k}}\prod\nolimits_{i=1}^{p}\frac{\nu_i^{k_i/2}\Gamma\{(\nu_i-k_i)/2\}}{2^{k_i/2}\Gamma(\nu_i/2)}$$
$$(k_i=0,1,...;\ \nu_i>k_i,\ i=1,...,p),$$

where $\boldsymbol{\mu}^{\mathbf{k}} = \mathrm{E}(\mathbf{X}^{\mathbf{k}}|\boldsymbol{\mu},\boldsymbol{\Sigma})$ *is the k^*-th order p-variate product moment of* $\mathbf{X}\sim\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ *with* $k^*=\sum\nolimits_{i=1}^{p}k_i$.

The mixture expression of the above result under sectional truncation denoted by $\mathrm{E}(\mathbf{Z}^{\mathbf{k}}|\boldsymbol{\mu},\boldsymbol{\Sigma},\mathbf{v};\ \mathbf{A},\mathbf{B})$ *is given by*

$$\mathrm{E}(\mathbf{Z}^{\mathbf{k}}|\boldsymbol{\mu},\boldsymbol{\Sigma},\mathbf{v};\ \mathbf{A},\mathbf{B}) = \int_{\mathbf{0}}^{\infty}\boldsymbol{\mu}^{\mathbf{k}}_{\mathbf{y},\mathbf{c},\mathbf{v},\mathbf{A},\mathbf{B}}\left\{\prod\nolimits_{i=1}^{p}g_{\Gamma}(y_i\,|\,\nu_i/2, c_i/2)\right\}\mathrm{d}\mathbf{y},$$

where

$$\boldsymbol{\mu}^{\mathbf{k}}_{\mathbf{y},\mathbf{c},\mathbf{v},\mathbf{A},\mathbf{B}} = \int_{\mathbf{A}}^{\mathbf{B}}\mathbf{z}^{\mathbf{k}}\phi_p\{\mathbf{z}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{c}\odot\mathbf{y}/\mathbf{v}); \mathbf{A},\mathbf{B}\}\mathrm{d}\mathbf{z}$$
$$\mathbf{c}=(c_1,...,c_p)^{\mathrm{T}},\ 0<c_i<\infty\ \ (i=1,...,p).$$

In Theorem 3, $\boldsymbol{\mu}^{\mathbf{k}}$ may be obtained by e.g., using the moment generating function $\mathrm{M}_{\mathbf{X}}(\mathbf{t})=\exp\{\mathbf{t}^{\mathrm{T}}\boldsymbol{\mu}+(\mathbf{t}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{t}/2)\}$ of $\mathbf{X}\sim\mathrm{N}_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$. For $\boldsymbol{\mu}^{\mathbf{k}}_{\mathbf{y},\mathbf{c},\mathbf{v},\mathbf{A},\mathbf{B}}$, see Ogasawara (2021b).

Corollary 3 *Let* $\mathbf{Z}=(\mathbf{Z}_1^{\mathrm{T}},\mathbf{Z}_2^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{Z}_1=(Z_1,...,Z_{p_1})^{\mathrm{T}}$, $\mathbf{Z}_2=(Z_{p_1+1},...,Z_p)^{\mathrm{T}}$ $(p_1=1,...,p-1;\ p_1+p_2=p)$. *Partition* $\mathbf{X}=(\mathbf{X}_1^{\mathrm{T}},\mathbf{X}_2^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\mu}=(\boldsymbol{\mu}_1^{\mathrm{T}},\boldsymbol{\mu}_2^{\mathrm{T}})^{\mathrm{T}}$,

$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$ *and* $\mathbf{v} = (\mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}})^{\mathrm{T}}$ *as for* $\mathbf{Z} = (\mathbf{Z}_1^{\mathrm{T}}, \mathbf{Z}_2^{\mathrm{T}})^{\mathrm{T}}$ *. Then, the* p_j *-variate moments of* $\mathbf{Z}_j$ *when* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ *are equal to those when* $\mathbf{Z}_j \sim \mathrm{St}(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_{jj}, \mathbf{v}_j)\,(j = 1,\ 2)$.

The covariance and correlation of Z_i *and* Z_j $(i, j = 1,..., p;\ i \neq j)$ *are*

$$\mathrm{cov}(Z_i, Z_j \mid \mathbf{v}) = \frac{\sigma_{ij}\sqrt{\nu_i \nu_j}}{2} \frac{\Gamma\{(\nu_i - 1)/2\}}{\Gamma(\nu_i/2)} \frac{\Gamma\{(\nu_j - 1)/2\}}{\Gamma(\nu_j/2)}$$
$$(\nu_i > 1,\ \nu_j > 1;\ i, j = 1,..., p;\ i \neq j)$$

and

$$\mathrm{cor}(Z_i, Z_j \mid \mathbf{v}) = \frac{\rho_{ij}}{2} \frac{\Gamma\{(\nu_i - 1)/2\}\sqrt{\nu_i - 2}}{\Gamma(\nu_i/2)} \frac{\Gamma\{(\nu_j - 1)/2\}\sqrt{\nu_j - 2}}{\Gamma(\nu_j/2)}$$
$$(\nu_i > 2,\ \nu_j > 2;\ i, j = 1,..., p;\ i \neq j),$$

where $\rho_{ij} = \sigma_{ij} / \sqrt{\sigma_{ii}\sigma_{jj}}$.

3. Some comparisons of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ with $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and numerical illustrations

3.1 Covariances and kurtoses

It is of interest to compare the results of Corollary 3 to those of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$. For ease of comparison, let $\mathbf{v} = \mathbf{v}_0 = (\nu_0, ..., \nu_0)^{\mathrm{T}}$ in $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$. Then, Corollary 3 gives

$$\mathrm{cov}(Z_i, Z_j \mid \mathbf{v}_0) = \sigma_{ij}\left[\frac{\sqrt{\nu_0}\Gamma\{(\nu_0 - 1)/2\}}{\sqrt{2}\Gamma(\nu_0/2)}\right]^2 = \sigma_{ij}\frac{\nu_0\Gamma^2\{(\nu_0 - 1)/2\}}{2\Gamma^2(\nu_0/2)}$$
$$(\nu_0 > 1;\ i, j = 1,..., p;\ i \neq j)$$

and

$$\mathrm{cor}(Z_i, Z_j \mid \mathbf{v}_0) = \rho_{ij}\frac{\Gamma^2\{(\nu_0 - 1)/2\}(\nu_0 - 2)}{2\Gamma^2(\nu_0/2)}$$
$$(\nu_0 > 2;\ i, j = 1,..., p;\ i \neq j).$$

On the other hand, the corresponding results for $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ are

$$\operatorname{cov}(Z_i, Z_j \mid \nu) = \sigma_{ij} \mathrm{E}\{(\sqrt{\nu / Y})^2 \mid Y \sim \chi^2(\nu)\}$$
$$= \sigma_{ij} \nu \frac{\Gamma\{(\nu-2)/2\}}{2\Gamma(\nu/2)} = \sigma_{ij} \frac{\nu}{\nu-2} \quad (\nu > 2;\, i, j = 1, \ldots, p;\, i \neq j)$$

and

$$\operatorname{cor}(Z_i, Z_j \mid \nu) = \sigma_{ij} \frac{\nu}{\nu-2} \Big/ \sqrt{\sigma_{ii} \frac{\nu}{\nu-2} \sigma_{jj} \frac{\nu}{\nu-2}} = \rho_{ij}$$
$$(\nu > 2;\, i, j = 1, \ldots, p;\, i \neq j).$$

The last pleasantly simple result $\operatorname{cor}(Z_i, Z_j \mid \nu) = \operatorname{cor}(X_i, X_j \mid \boldsymbol{\Sigma}) = \rho_{ij}$ is known as a general property which holds for elliptical distributions (see e.g., Muirhead, 1982, p. 34). The last result is also found to be a special case of the following general case.

Lemma 2 *Let Z be a random variable independent of X and Y with* $\mathrm{E}(X) = \mathrm{E}(Y) = 0$, *which are possibly correlated. Then, the correlation coefficient of ZX and ZY is equal to that of X and Y when they exist.*

In Lemma 2, the condition $\mathrm{E}(X) = \mathrm{E}(Y) = 0$ is necessary. That is, in the non-central case i.e., $\mathbf{Z} = \mathbf{X} / \sqrt{Y / \nu}$ rather than $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu}) / \sqrt{Y / \nu}$ when $\mathbf{X} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, Lemma 2 does not hold unless $\boldsymbol{\mu} = \mathbf{0}$. For the central case, we have the following inequalities.

Result 1 *When* $v_0 = v$,

$$|\operatorname{cov}(Z_i, Z_j \mid \nu)| - |\operatorname{cov}(Z_i, Z_j \mid \mathbf{v} = \mathbf{v}_0 = \mathbf{1}_p \nu)|$$
$$= |\sigma_{ij}| \left[\frac{\nu}{\nu-2} - \frac{\nu \Gamma^2\{(\nu-1)/2\}}{2\Gamma^2(\nu/2)} \right] = |\sigma_{ij}| \operatorname{var}\{\sqrt{\nu / Y} \mid Y \sim \chi^2(\nu)\} \geq 0$$

and

$$|\operatorname{cor}(Z_i, Z_j \mid \nu)| - |\operatorname{cor}(Z_i, Z_j \mid \mathbf{v} = \mathbf{v}_0 = \mathbf{1}_p \nu)|$$
$$= |\rho_{ij}| \left[1 - \frac{\Gamma^2\{(\nu-1)/2\}}{\Gamma(\nu/2)\Gamma\{(\nu-2)/2\}} \right] \geq 0$$
$$(\nu > 2;\, i, j = 1, \ldots, p;\, i \neq j).$$

The equalities hold if and only if $\sigma_{ij} = 0$.

Next, Mardia's (1970, Equation (3.5)) multivariate non-excess kurtosis $\beta_{2,p}$ is considered. Note that Mardia's (1970, Equation (2.19)) multivariate analogue of squared skewness is zero.

Result 2 *Using* $\mathbf{v} = \mathbf{v}_0 = (\nu_0, \dots, \nu_0)^{\mathrm{T}}$ *in* $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ *as before, we obtain*

$$\beta_{2,p}(\mathbf{Z} \mid \mathbf{v}_0) = \mathrm{E}[\{(\mathbf{Z}-\boldsymbol{\mu})^{\mathrm{T}} \mathrm{cov}^{-1}(\mathbf{Z})(\mathbf{Z}-\boldsymbol{\mu})\}^2 \mid \mathbf{v}_0]$$
$$= p(p+2) + \sum_{i=1}^{p} \frac{6(\sigma^{ii})^2 \sigma_{ii}^2}{\nu_0 - 4}$$
$$(\nu_0 > 4).$$

The corresponding result of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ *is*

$$\beta_{2,p}(\mathbf{Z} \mid \nu) = \mathrm{E}[\{(\mathbf{Z}-\boldsymbol{\mu})^{\mathrm{T}} \mathrm{cov}^{-1}(\mathbf{Z})(\mathbf{Z}-\boldsymbol{\mu})\}^2 \mid \nu] = \frac{p(p+2)(\nu-2)}{\nu-4}$$
$$> \beta_{2,p}(\mathbf{Z} \mid \nu \mathbf{1}_p) = p(p+2) + \sum_{i=1}^{p} \frac{6(\sigma^{ii})^2 \sigma_{ii}^2}{\nu - 4}$$
$$= \beta_{2,p}(\mathbf{X}) + \sum_{i=1}^{p} \frac{6(\sigma^{ii})^2 \sigma_{ii}^2}{\nu - 4} > \beta_{2,p}(\mathbf{X}) \ (\nu > 4).$$

The factor $6/(\nu-4)$ in the second term of $\beta_{2,p}(\mathbf{Z} \mid \mathbf{v})$ is the excess kurtosis of the usual univariate *t*, which is obtained by $\sigma^{11} = 1/\sigma_{11}$ when $p = 1$ in the above result with $p(p+2) = 3$ being the non-excess kurtosis under univariate normality. The inequality $\beta_{2,p}(\mathbf{Z} \mid \nu) > \beta_{2,p}(\mathbf{Z} \mid \nu \mathbf{1}_p)$ is expected due to the independent chi-squares used in $\beta_{2,p}(\mathbf{Z} \mid \nu \mathbf{1}_p)$ rather than a common chi-square. The second equality for $\beta_{2,p}(\mathbf{Z} \mid \nu)$ shows its positive excess kurtosis with $\beta_{2,p}(\mathbf{Z} \mid \nu) / \beta_{2,p}(\mathbf{X}) = (\nu-2)/(\nu-4) > 1$ as well as that of $\beta_{2,p}(\mathbf{Z} \mid \mathbf{v}_0)$, which is again a special case of the general result:

Lemma 3 Let *Y be a random variable independent of arbitrarily distributed* $\mathbf{X}$ *with* $\mathrm{E}(\mathbf{X}) = \boldsymbol{\mu}$. *Define* $\mathbf{Z} = (\mathbf{X} - \boldsymbol{\mu})Y + \boldsymbol{\mu}^*$. *Then,*

$$\beta_{2,p}(\mathbf{Z}) = \beta_{2,p}(\mathbf{X}) \mathrm{E}(Y^4) / \mathrm{E}^2(Y^2) \geq \beta_{2,p}(\mathbf{X}),$$

where $\mathrm{E}^2(\cdot) = \{\mathrm{E}(\cdot)\}^2$, *when* $\beta_{2,p}(\cdot)$'s *exist. The equality in the inequality holds if and* *only if* $\mathrm{var}(Y^2) = 0$.

Note that the equality in the inequality can happen even when $\mathrm{var}(Y) > 0$ as in the 2-point symmetric distribution about zero with $\mathrm{var}(Y^2) = 0$.

3.2 Numerical illustrations

Differences of the pdf's of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\nu})$ are numerically illustrated in this subsection. While $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ has the pdf of closed form, the corresponding expressions of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\nu})$ are given by the integral and series in Theorems 1 and 2, respectively. The latter series is generally $p(p+1)/2$-fold, which is the number of non-duplicated off-diagonal elements of $\boldsymbol{\Sigma}$, where the number is reduced by the number of zero σ_{ij}'s if any. Though this number soon becomes very large as *p* increases, the computation is quite reasonable when *p* is relatively small e.g., *p* = 2 and 3. Note that when *p* = 2 giving the bivariate case, the series is uni-fold.

In actual computation, the infinite series is replaced by a finite one. Our rule is as follows. When the sum of current absolute four differences of the series divided by the newest value of the series is less than or equal to a constant, computation is stopped. The four differences are used considering that consecutive equal two values in similar situations (see e.g., Pearson, Olver & Porter, 2015) can happen before convergence. The value 'eps' of 1e-6 = 10^{-6} or 0 is used in bivariate cases. Note that eps = 0 indicates the largest accuracy under machine precision employed. For the integral expression, the R-function 'cubintegrate' using the method 'hcubature' in R-package 'cubature version 2.04' (Narasimhan, Koller, Johnson, Hahn, Bouvier, Kiêu & Gaure, 2019) is used with the default arguments.

For comparison of the integral and series expressions, the bivariate cases of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\nu})$ with $\boldsymbol{\mu} = \mathbf{0}$, $\boldsymbol{\nu} = (6,\ 6)^{\mathrm{T}}, (6,\ 12)^{\mathrm{T}}, (12,\ 12)^{\mathrm{T}}$, $\sigma_{12} = 0,\ 0.5,\ 0.8$ ($\sigma_{11} = \sigma_{22} = 1$) and $\mathbf{z} = (-2, -2)^{\mathrm{T}},\ (-2, 1)^{\mathrm{T}},\ (1,\ 1)^{\mathrm{T}},\ (1, -2)^{\mathrm{T}}$ are used. Over the $3 \times 3 \times 4 = 36$ pairs of the pdf's by the integral and series expressions, the summary statistics of the pdf's by the series when eps = 0 are unchanged by those when eps = 10^{-6} up to the third decimal places both as min = 6.262e-5, median = 1.270e-2, mean = 2.717e-2 and max = 1.304e-1. The user cpu times for the two methods with eps = 0 and 10^{-6} including the common numerical integral are the same i.e., 0.55 seconds up to the second place using intel Core: 7-6700 CPU@3.40GHz with the default double precision of the R-language. The absolute differences of interest between the pdf'd by the series and the corresponding numerical integral when eps = 10^{-6} give min = 1.230e-10, median = 2.952e-9, mean = 4.847e-8 and max = 4.193e-7 while when eps = 0, the corresponding absolute differences yield min = 1.222e-10, median = 2.754e-9, mean = 4.848e-8 and max = 4.198e-7, which are almost the same as those when eps = 10^{-6} although when eps = 10^{-6}, the required numbers of terms in the series i.e., the maximum values of u_{12} until convergence over the 36 cases give min = 0, median = 16, mean = 15.33 and max = 41, which are much smaller than the corresponding values when eps = 0 i.e., min = 0, median = 28.5, mean = 28.0 and max = 81. When we look at the last set of results, eps = 10^{-6} may be the better choice. However, recall that eps = 10^{-6} does not save the cpu time up to the second place over eps = 0. Over all, the series expressions of the pdf of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ with eps = 10^{-6} or 0, seem to give reasonable results as long as in similar situations as above.

Based on these results, comparisons of the bivariate pdf's of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ with $\nu = 6, 9, 30$ and $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ with $\mathbf{v} = (6, 6)^{\mathrm{T}}, (2, 16)^{\mathrm{T}}, (30, 30)^{\mathrm{T}}$ are made using the series expression with eps = 0 for $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$, where $\nu = 9$ in $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ is the mean of $\nu_1 = 2$ and $\nu_2 = 16$. Note that Student *t* with 2 df has a finite mean and an infinite variance (for this distribution, see Jones, 2002b). The 30 df in ν and $\mathbf{v}$ is employed to give cases similar to normal variables in a practical sense. The same values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as before are used.

Each of Figures 1 to 3 shows 3 pairs of the contour plots when $\sigma_{12} = 0, 0.5, 0.8$ from left to right under different conditions of ν and $\mathbf{v}$ over the figures. In Figure 1, $\mathbf{v} = (6, 6)^{\mathrm{T}}$ when $\sigma_{12} = 0$ seems to yield a relatively squared shape over that of $\nu = 6$. It is of interest to find that when $\sigma_{12} = 0.8$, the contour of $\mathbf{v} = (6, 6)^{\mathrm{T}}$ has valleys in the north-west and south-east slopes. When we look at the apexes of the contours, they give unsmooth shapes e.g., an octangle in the case of $\sigma_{12} = 0$ and $\mathbf{v} = (6, 6)^{\mathrm{T}}$, which is due to an approximation using equally spaced 31^2 points in the grid of $Z_1 = -3\ (0.2)\ 3$ and $Z_2 = -3\ (0.2)\ 3$. In Figure 2, the three pdfs of different df's i.e., $\mathbf{v} = (2,\ 16)^{\mathrm{T}}$ are not exchangeable with respect to Z_1 and Z_2 while all the 15 remaining pdf's are exchangeable, which is due to $\sigma_{11} = \sigma_{22}$ in $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and $\nu_1 = \nu_2$ as well as $\sigma_{11} = \sigma_{22}$ in $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ each when $\mu_1 = \mu_2$. Figure 3 gives the cases with $\nu = 30$ or $\mathbf{v} = (30,\ 30)^{\mathrm{T}}$. It is seen that the valleys found in the pdf's when $\mathbf{v} = (6,\ 6)^{\mathrm{T}}$, and $\mathbf{v} = (2,\ 16)^{\mathrm{T}}$ in Figures 1 and 2, respectively each with $\sigma_{12} = 0.8$ disappear. Three pairs of the pdf's in Figure 3 are almost indistinguishable. This corresponds to the summary statistics of the absolute differences of the pdf's at the 31^2 points. For instance, when $\sigma_{12} = 0.8$ in Figure 2, they are min = 1.4e-7, median = 2.1e-3, mean = 6.1e-3 and max = 4.4e-2 while the corresponding values in Figure 3 are min = 0.0, median = 3.0e-4, mean = 9.6e-4 and max = 5.7e-3. These values indicate the convergence to the common normal density of $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ when the df's in $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ and $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ go to infinity.

4. The multivariate *t*-distribution with patterned chi-squares

The bivariate/multivariate *t*-distributions with common and uncommon chi-squares have been given by Bulgren, Dykstra and Hewett (1974, Equation (1.1)), Jones (2002a, Equation (2)), and Shaw and Lee (2006, Equation (3.6); 2008, Equations (2.1) and (2.6)) (for reviews, see Nadarajah & Kotz, 2004; Nadarajah & Dey, 2005). The following multivariate *t* is a generalized one including $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ i.e., the usual multivariate *t*, $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ introduced in Section 2 and the distributions given by the above authors as special cases. Note that in Jones (2002a), an uncorrelated normal vector with common/uncommon chi-squares is considered.

Definition 2 A random p-vector $\mathbf{Z}$ is defined to follow the multivariate *t*-distribution with patterned chi-squares using a multivariate normal vector $\mathbf{X} = (X_1, \ldots, X_p)^{\mathrm{T}}$ $\sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; a $p \times q$ matrix $\boldsymbol{\Lambda}$ for a 0/1 pattern; q independent chi-squared variables $\mathbf{F} = (F_1, \ldots, F_q)^{\mathrm{T}}$ with possibly distinct df's $\mathbf{n} = (n_1, \ldots, n_q)^{\mathrm{T}}$, whose partial sums are used for $\mathbf{Y} = (Y_1, \ldots, Y_p)^{\mathrm{T}}$ as $\mathbf{Y} = \mathbf{\Lambda F}$; and

$$\mathbf{Z} - \boldsymbol{\mu} = \{(X_1 - \mu_1) / \sqrt{Y_1 / \nu_1}, \ldots, (X_p - \mu_p) / \sqrt{Y_p / \nu_p}\}^{\mathrm{T}},$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_p)^{\mathrm{T}} = \mathbf{\Lambda n}$. This multivariate *t* is denoted by $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Lambda}, \mathbf{n})$.

Using our notation, Jones' (2002a, Equation (2)) bivariate *t* is given by

$$\mathbf{Z} = \{X_1 / \sqrt{F_1 / n_1}, X_2 / \sqrt{(F_1 + F_2)/(n_1 + n_2)}\}^{\mathrm{T}} \text{ with } \mathbf{X} \sim \mathrm{N}_2(\mathbf{0}, \mathbf{I}_2),$$

when $\mathbf{\Lambda n} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$. Jones (2002a, Equations (12) and (13)) suggested multivariate *t* distributions:

$$\mathbf{Z} = \{X_1 / \sqrt{F_1 / n_1}, X_2 / \sqrt{(F_1 + F_2)/(n_1 + n_2)}, \ldots, X_p / \sqrt{(F_1 + \ldots + F_p)/(n_1 + \ldots + n_p)}\}^{\mathrm{T}}$$

and $\mathbf{Z} = \{X_1 / \sqrt{F_1 / n_1}, X_2 / \sqrt{(F_1 + F_2)/(n_1 + n_2)}, \ldots, X_p / \sqrt{(F_1 + F_p)/(n_1 + n_p)}\}^{\mathrm{T}}$

each with $\mathbf{X} \sim \mathrm{N}_p(\mathbf{0}, \mathbf{I}_p)$. It is found that these two models are derived when

$\mathbf{\Lambda n} = \begin{pmatrix} 1\,0\,0 \cdots 0 \\ 1\,1\,0 \cdots 0 \\ \vdots \\ 1\,1\,1 \cdots 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_p \end{pmatrix}$ and $\mathbf{\Lambda n} = \begin{pmatrix} 1\,0\,0 \cdots 0 \\ 1\,1\,0 \cdots 0 \\ \vdots \\ 1\,0\,0 \cdots 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_p \end{pmatrix}$, respectively. Jones (2002a, p.170) also suggested adding independent Y_i's for $\mathbf{Z}$ in the square roots of the denominators of the above equations. These modifications can be easily employed by adapting $\boldsymbol{\Lambda}$ and $\mathbf{n}$.

Note that $\mathbf{Y} = \mathbf{\Lambda F}$ is similar to the factor analysis model, where $\boldsymbol{\Lambda}$ is a factor pattern/loading matrix and **F** is the vector of common and unique factors. It is seen that Jones' first multivariate model has $p-1$ common factors (chi-squares) $F_1, \ldots, F_{p-1}$ and a single unique factor F_p while the second model has a single common factor F_1 and $p-1$ unique factors $F_2, \ldots, F_p$ yielding a one-factor model in factor analysis. Note that in our case the elements of $\boldsymbol{\Lambda}$ are restricted to 0 or 1 to form univariate marginal *t*-distributions.

Theorem 4 *The integral expression of the pdf of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Lambda}, \mathbf{n})$ *at* **z** *is given by*

$$C_q \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{n}-\mathbf{1}_q} \left(\prod_{i=1}^{p} \sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}} (\mathbf{u} \odot \mathbf{u})} \right)$$

$$\times \exp\left(-\frac{1}{2} \left[\{\boldsymbol{\Lambda}(\mathbf{u} \odot \mathbf{u})\}^{1/2\mathrm{T}} \boldsymbol{\Omega}_{\mathbf{z}}^{-1} \{\boldsymbol{\Lambda}(\mathbf{u} \odot \mathbf{u})\}^{1/2} + \mathbf{u}^{\mathrm{T}}\mathbf{u} \right] \right) \mathrm{d}\mathbf{u},$$

where

$$C_q = \frac{2^q}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \left(\prod_{i=1}^{p} 1/v_i \right) \prod_{i=1}^{q} \frac{1}{2^{n_i/2} \Gamma(n_i/2)} \quad \textit{and}$$

$$\boldsymbol{\Omega}_{\mathbf{z}} = \left[\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} \boldsymbol{\Sigma}^{-1} \mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu}) \odot \mathbf{v}^{-1/2}\} \right]^{-1} = (\boldsymbol{\Psi}_{\mathbf{z}}^{-1} - \mathbf{I}_p)^{-1}.$$

It is found that the pdf is not given by the normalizer of the multivariate bpc. Some numerical method is required for actual computation.

Corollary 4 *The pdf of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Lambda}, \mathbf{n})$ *at* $\mathbf{z}$ *is expressed by the multiple mixtures of the precision matrix of the multivariate normal density using q independent chi-squared or variously gamma-distributed variables*:

$$\int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})\} \left\{ \prod_{i=1}^{q} g_{\chi^2}(f_i \mid n_i) \right\} \mathrm{d}\mathbf{f}$$

$$= \int_{\mathbf{0}}^{\infty} \phi_p[\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}\{\boldsymbol{\Lambda}(\mathbf{c} \odot \mathbf{f})/\mathbf{v}\} \boldsymbol{\Sigma} \mathrm{diag}^{-1/2}\{\boldsymbol{\Lambda}(\mathbf{c} \odot \mathbf{f})/\mathbf{v}\}]$$

$$\times \left\{ \prod_{i=1}^{q} g_{\Gamma}(f_i \mid n_i/2, c_i/2) \right\} \mathrm{d}\mathbf{f} \quad .$$

$$\mathbf{c} = (c_1, \ldots, c_q)^{\mathrm{T}},\ 0 < c_i < \infty \quad (i = 1, \ldots, q),$$

where c_i *'s are arbitrary constants*.

While the integral expression in Theorem 4 was complicated, the corresponding mixture in Corollary 3 is simple and transparent though for the actual computation of the latter, some numerical method is required.

As addressed earlier in Jones' model $\boldsymbol{\Lambda}\mathbf{n} = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 1 & 1 & 0 \cdots 0 \\ & \vdots & \\ 1 & 0 & 0 \cdots 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_p \end{pmatrix}$, which is a special case of the model of Definition 2, the number of variables $Y_i (i = 1, \ldots, p)$ is the same as that for $f_i (i = 1, \ldots, p)$. A model more similar to the one-factor model is given by

$$\mathbf{\Lambda n} = \begin{pmatrix} 1\,1\,0\,0\cdots 0 \\ 1\,0\,1\,0\cdots 0 \\ \vdots \\ 1\,0\,0\,0\cdots 1 \end{pmatrix} \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_p \end{pmatrix},$$

where $F_i (i = 0, 1, ..., p)$ are used with F_0 for a single "common factor" while $F_i (1, ..., p)$ for *p* "unique factors". It is to be noted that the dimension size of $F_i (i = 0, 1, ..., p)$ is inflated by 1 over the observable variables $Y_i (i = 1, ..., p)$, which is a property in the latent variable model like factor analysis while the dimension size in Jones' model is unchanged as in the principal component model when considering minor components as well as principal ones. The above model like the one-factor model (one-factor model for short) has a property: when $F_i (i = 1, ..., p)$ are missing, we have $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$ while when F_0 is omitted, $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$ with independent chi-squares follows. That is, the one-factor model is situated between these two models. Note also that Jones' above model is obtained when F_1 is removed in the one-factor model.

Alternative expressions of Corollary 4 for the one-factor model are given as follows:

Corollary 5 *Let* $\mathbf{f}^* = (f_0, f_1, ..., f_p)^{\mathrm{T}} = (f_0, \mathbf{f}^{\mathrm{T}})^{\mathrm{T}}$. *The pdf of* $\mathbf{Z} \sim \mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Lambda}, \mathbf{n})$ *at* **z** *for the one-factor model is expressed by the multiple mixtures of the precision matrix of the multivariate normal density using p* + 1 *independent chi-squared or variously gamma-distributed variables*:

$$\int_{\mathbf{0}}^{\infty} \phi_p \{\mathbf{z} \,|\, \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\boldsymbol{\Lambda}\mathbf{f}^* / \mathbf{v}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2}(\boldsymbol{\Lambda}\mathbf{f}^* / \mathbf{v})\} \left\{ \prod\nolimits_{i=0}^{p} g_{\chi^2}(f_i \,|\, n_i) \right\} \mathrm{d}\mathbf{f}^*$$

$$= \int_{\mathbf{0}}^{\infty} \phi_p \left[\mathbf{z} \,|\, \boldsymbol{\mu}, \mathrm{diag}^{-1/2}\{(\mathbf{1}_p c_0 f_0 + \mathbf{c} \odot \mathbf{f}) / \mathbf{v}\} \boldsymbol{\Sigma} \mathrm{diag}^{-1/2}\{(\mathbf{1}_p c_0 f_0 + \mathbf{c} \odot \mathbf{f}) / \mathbf{v}\} \right]$$

$$\times \left\{ \prod\nolimits_{i=0}^{p} g_{\Gamma}(f_i \,|\, n_i / 2, c_i / 2) \right\} \mathrm{d}\mathbf{f}^*$$

$$= \int_{f_0 \mathbf{1}_p}^{\infty} \int_0^{\infty} \phi_p \{\mathbf{z} \,|\, \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y} / \mathbf{v}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2}(\mathbf{c} \odot \mathbf{y} / \mathbf{v})\}$$

$$\times \left\{ \prod\nolimits_{i=1}^{p} g_{\Gamma}(y_i - f_0 \,|\, n_i / 2, c_i / 2) \right\} g_{\Gamma}(f_0 \,|\, n_0 / 2, c_0 / 2) \mathrm{d}f_0 \mathrm{d}\mathbf{y}$$

$$\mathbf{c} = (c_1, ..., c_p)^{\mathrm{T}},\ 0 < c_i < \infty \ \ (i = 0, 1, ..., p),$$

where $\int_{f_0 \mathbf{1}_p}^{\infty} (\cdot) \mathrm{d}\mathbf{y} = \int_{f_0}^{\infty} \cdots \int_{f_0}^{\infty} (\cdot) \mathrm{d}\mathbf{y}$; *and* c_i *'s are arbitrary constants.*

The last expression of Corollary 5 is associated with the following result.

Result 3 *In the one-factor model, the joint density of* $Y_i = F_0 + F_i$ *at* $y_i \ (i = 1, ..., p)$ *is*

$$\int_0^{\min(\mathbf{y})} \left\{ \prod_{i=1}^{p} g_{\chi^2}(y_i - f_0 \mid n_i) \right\} g_{\chi^2}(f_0 \mid n_0) \mathrm{d}f_0$$

$$= \frac{\exp(-\mathbf{1}_p^{\mathrm{T}} \mathbf{y} / 2)}{2^{(n_0 + \mathbf{1}_p^{\mathrm{T}} \mathbf{n})/2} \prod_{i=0}^{p} \Gamma(n_i / 2)}$$

$$\times \int_0^{\min(\mathbf{y})} f_0^{(n_0/2)-1} \left\{ \prod_{i=1}^{p} (y_i - f_0)^{(n_i/2)-1} \right\} \exp\left\{ \frac{(p-1) f_0}{2} \right\} \mathrm{d}f_0 .$$

where $\min(\mathbf{y}) = \min\{y_1, ..., y_p\}$ and $\mathbf{n} = (n_1, ..., n_p)^{\mathrm{T}}$.

The expression in Result 3 taking a form of convolution for the dependent chi-squares $Y_i \ (i = 1, ..., p)$ is obtained by the variable transformation from $\mathbf{F} = (F_1, ..., F_p)^{\mathbf{T}}$ to $\mathbf{Y} = (Y_1, ..., Y_p)^{\mathbf{T}}$ with unchanged common F_0 and unit Jacobian (for the remaining part of the proof see the appendix). Alternatively, it is immediately derived by noting the local (conditional) independence of $Y_i (i = 1, ..., p)$ when F_0 is given, as was used in Corollary 5. Local independence is a typical assumption employed in latent variable models e.g., the item response model with a random variable for the ability parameter (see e.g., Bock & Aitkin, 1981) and the exploratory factor analysis model under normality (see e.g., Ogasawara, 2016).

The joint distribution of $Y_i \ (i = 1, ..., p)$ in Result 3 is seen as a multivariate chi-square with multiple df's. It is well known that the Wishart is a multivariate version of the chi-square, where both distributions have single df's. Furman (2008, Definition 2.1) gave the multivariate distribution corresponding to Result 3 using $p + 1$ independently distributed variables. A variation of Result 3 is the case of Jones' one-factor like model with $Y_1 = F_0$ and $Y_i = F_0 + F_i \ (i = 2, ..., p)$, which gives the joint density without integral

$$\left\{\prod_{i=2}^{p} g_{\chi^2}(y_i - y_1 \mid n_i)\right\} g_{\chi^2}(y_1 \mid n_1)$$
$$= \frac{\exp[\{(p-1)y_1 - y_2 \cdots - y_p\}/2]}{2^{\mathbf{1}_p^{\mathrm{T}}\mathbf{n}/2} \prod_{i=1}^{p} \Gamma(n_i/2)} y_1^{(n_1/2)-1} \left\{\prod_{i=2}^{p} (y_i - y_1)^{(n_i/2)-1}\right\}$$
$$(0 \le y_1 < \infty;\ y_1 \le y_i < \infty,\ i = 2, \ldots, p).$$

A generalized version of Result 3 may be obtained for the model of Definition 2 with patterned chi-squares. In this case, when the number q of independent chi-squares is greater than p as in the one-factor model, generally the $(q-p)$-fold multiple integral with the range of integral giving $y_i \ge 0\,(i = 1, \ldots, p)$ is required while when $q \le p$, no integral is used. An example of the latter case of $q = p$ is found in Jones' (2002a) another multivariate t with $Y_i = \sum_{j=1}^{i} F_j\,(i = 1, \ldots, p)$ as mentioned earlier, where only F_p is the "unique factor" with the remaining $F_i\,(i = 1, \ldots, p-1)$ being the "common factors". It is seen that the joint distribution of $Y_i\,(i = 1, \ldots, p)$ in this model is a special case of the multivariate gamma given by Mathai and Moschoupoulos (1992, Theorem 1.1), where independent p gammas are replaced by independent p chi-squares

Appendix

Proof of Lemma 1

The pdf of the joint distribution of $\mathbf{X}$ and Y is given by $\phi_p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) g_{\Gamma}(y \mid \nu/2, 1/2)$. Use the variable transformation $\mathbf{Z} - \boldsymbol{\mu} = (\mathbf{X} - \boldsymbol{\mu})/\sqrt{Y/\nu}$ with unchanged Y. Noting that $\mathbf{x} - \boldsymbol{\mu} = (\mathbf{z} - \boldsymbol{\mu})\sqrt{y/\nu}$ and the Jacobian is $\prod_{i=1}^{p} \mathrm{d}x_i / \mathrm{d}z_i = (y/\nu)^{p/2}$, the pdf of $\mathbf{Z}$ is given from the joint pdf of $\mathbf{Z}$ and Y, when Y is integrated out:

$$\int_0^{\infty} \phi_p\{(\mathbf{z} - \boldsymbol{\mu})\sqrt{y/\nu} + \boldsymbol{\mu} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\} g_{\Gamma}(y \mid \nu/2, 1/2)(y/\nu)^{p/2} \mathrm{d}y$$

$$= \int_0^\infty \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma} \nu / y|^{1/2}} \exp\left\{ -\frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} (\boldsymbol{\Sigma}^{-1} y/\nu)(\mathbf{z}-\boldsymbol{\mu})}{2} \right\}$$
$$\times g_{\Gamma}(y \mid \nu/2, 1/2)\mathrm{d}y$$

$$= \int_0^\infty \phi_p(\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/y) g_{\Gamma}(y \mid \nu/2, 1/2)\mathrm{d}y$$
$$= \int_0^\infty \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy^*)\} g_{\Gamma}(cy^* \mid \nu/2, 1/2)\, c\, \mathrm{d}y^* \quad (0 < c < \infty)$$
$$= \int_0^\infty \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}\nu/(cy^*)\} g_{\Gamma}(y^* \mid \nu/2, c/2)\mathrm{d}y^*,$$

where the variable transformation $y^* = y/c$ is used. Redefining y^* as y gives the mixture expression of the pdf of **Z**.

On the other hand, the actual expression of the pdf is given by the above intermediate result as

$$\int_0^\infty \phi_p(\mathbf{z}\sqrt{y/\nu} + \boldsymbol{\mu} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) g_{\Gamma}(y \mid \nu/2, 1/2)(y/\nu)^{p/2}\mathrm{d}y$$
$$= \int_0^\infty \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} (\boldsymbol{\Sigma}^{-1} y/\nu)(\mathbf{z}-\boldsymbol{\mu})}{2} \right\}$$
$$\times g_{\Gamma}(y \mid \nu/2, 1/2)(y/\nu)^{p/2}\mathrm{d}y$$

$$= \int_0^\infty \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} (\boldsymbol{\Sigma}^{-1} y/\nu)(\mathbf{z}-\boldsymbol{\mu})}{2} \right\}$$
$$\times \frac{y^{(\nu/2)-1}(y/\nu)^{p/2}}{2^{\nu/2}\Gamma(\nu/2)} \exp\left(-\frac{y}{2} \right)\mathrm{d}y$$

$$= \frac{\Gamma\{(\nu+p)/2\}}{(2\pi\nu)^{p/2} 2^{\nu/2} \Gamma(\nu/2) |\boldsymbol{\Sigma}|^{1/2}} \left\{ \frac{1}{2}\left(1 + \frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\nu} \right) \right\}^{-(\nu+p)/2}$$
$$\times \int_0^\infty \frac{y^{(\nu+p)/2-1}}{\Gamma\{(\nu+p)/2\}} \left\{ \frac{1}{2}\left(1 + \frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\nu} \right) \right\}^{(\nu+p)/2}$$
$$\times \exp\left\{ -\frac{1}{2}\left(1 + \frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\nu} \right) y \right\}\mathrm{d}y$$

$$= \frac{\Gamma\{(\nu+p)/2\}}{(2\pi\nu)^{p/2} 2^{\nu/2} \Gamma(\nu/2) |\boldsymbol{\Sigma}|^{1/2}} \left\{ \frac{1}{2} \left(1 + \frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\nu} \right) \right\}^{-(\nu+p)/2}$$

$$= \frac{\Gamma\{(\nu+p)/2\}}{(\pi\nu)^{p/2} \Gamma(\nu/2) |\boldsymbol{\Sigma}|^{1/2}} \left(1 + \frac{(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{z}-\boldsymbol{\mu})}{\nu} \right)^{-(\nu+p)/2},$$

which is the required result.

Proof of Theorem 1

The pdf of the joint distribution of $\mathbf{X}$ and $\mathbf{Y}$ is

$$\phi_p(\mathbf{X}=\mathbf{x} \,|\, \boldsymbol{\mu}, \boldsymbol{\Sigma}) \prod_{i=1}^{p} g_{\chi^2}(Y_i = y_i \,|\, \nu_i)$$
$$= \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\boldsymbol{\mu})}{2} \right\}$$
$$\times \prod_{i=1}^{p} \frac{y_i^{(\nu_i/2)-1}}{2^{\nu_i/2} \Gamma(\nu_i/2)} \exp\left(-\frac{y_i}{2} \right).$$

where $g_{\chi^2}(Y_i = y_i \,|\, \nu_i)$ is the pdf of $Y_i \sim \chi^2(\nu_i)$ $(i = 1, \ldots, p)$. Define $\mathrm{diag}^{-1/2}(\mathbf{Y}/\mathbf{v}) \equiv \mathrm{diag}^{-1/2}(\mathbf{Y})\mathrm{diag}^{1/2}(\mathbf{v})$, where $\mathrm{diag}^{-1/2}(\mathbf{Y}) = \mathrm{diag}(Y_1^{-1/2}, \ldots, Y_p^{-1/2})$ and $\mathrm{diag}^{1/2}(\mathbf{v}) = \mathrm{diag}(\nu_1^{1/2}, \ldots, \nu_p^{1/2})$. Use the variable transformation from $\mathbf{X}$ to $\mathbf{Z} = \mathrm{diag}^{-1/2}(\mathbf{Y}/\mathbf{v})(\mathbf{X}-\boldsymbol{\mu}) + \boldsymbol{\mu}$ with unchanged $\mathbf{Y}$ and the Jacobian $\prod_{i=1}^{p} \frac{\mathrm{d}x_i}{\mathrm{d}z_i} = \prod_{i=1}^{p} \sqrt{\frac{y_i}{\nu_i}}$. Then, noting that $\mathbf{x}-\boldsymbol{\mu} = \mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu})$ with $\mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v}) = \{\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}^{-1}$, the pdf of the joint distribution of $\mathbf{Z}$ and $\mathbf{Y}$ becomes

$$\frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v}) \boldsymbol{\Sigma}^{-1} \mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu}) \right\}$$
$$\times \prod_{i=1}^{p} \frac{y_i^{(\nu_i/2)-1} \sqrt{y_i/\nu_i}}{2^{\nu_i/2} \Gamma(\nu_i/2)} \exp\left(-\frac{y_i}{2} \right)$$

$$= \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}}\left\{\prod\nolimits_{i=1}^{p}\frac{1}{\sqrt{\nu_i}2^{\nu_i/2}\Gamma(\nu_i/2)}\right\}\left\{\prod\nolimits_{i=1}^{p}y_i^{(\nu_i-1)/2}\right\}$$

$$\times\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}}\mathrm{diag}^{1/2}(\mathbf{y}/\boldsymbol{\nu})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\mathbf{y}/\boldsymbol{\nu})(\mathbf{z}-\boldsymbol{\mu})-\frac{\mathbf{1}_p^{\mathrm{T}}\mathbf{y}}{2}\right\}.$$

The pdf of $\mathbf{Z}$ is given by the above result, when $\mathbf{Y}$ is integrated out over its distribution. Let $C=\frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}}\prod\nolimits_{i=1}^{p}\frac{1}{\sqrt{\nu_i}2^{\nu_i/2}\Gamma(\nu_i/2)}$. Use the variable transformations $U_i=\sqrt{Y_i}\ (i=1,...,p)$ with $\mathbf{U}=(U_1,...,U_p)^{\mathrm{T}}$ and the other notations defined similarly. Then, the pdf of $\mathbf{Z}$ is given as

$$C\int_0^{\infty}\cdots\int_0^{\infty}\left\{\prod\nolimits_{i=1}^{p}y_i^{(\nu_i-1)/2}\right\}$$

$$\times\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}}\mathrm{diag}^{1/2}(\mathbf{y}/\boldsymbol{\nu})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\mathbf{y}/\boldsymbol{\nu})(\mathbf{z}-\boldsymbol{\mu})-\frac{\mathbf{1}_p^{\mathrm{T}}\mathbf{y}}{2}\right\}\mathrm{d}y_1\cdots\mathrm{d}y_p$$

$$\equiv C\int_{\mathbf{0}}^{\infty}\left\{\prod\nolimits_{i=1}^{p}y_i^{(\nu_i-1)/2}\right\}$$

$$\times\exp\left[-\frac{1}{2}\mathbf{y}^{1/2\mathrm{T}}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\boldsymbol{\nu}^{-1/2}\}\boldsymbol{\Sigma}^{-1}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\boldsymbol{\nu}^{-1/2}\}\mathbf{y}^{1/2}-\frac{\mathbf{1}_p^{\mathrm{T}}\mathbf{y}}{2}\right]\mathrm{d}\mathbf{y}$$

$$= C2^p\int_{\mathbf{0}}^{\infty}\left(\prod\nolimits_{i=1}^{p}u_i^{\nu_i}\right)$$

$$\times\exp\left(-\frac{1}{2}\mathbf{u}^{\mathrm{T}}\left[\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\boldsymbol{\nu}^{-1/2}\}\boldsymbol{\Sigma}^{-1}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\boldsymbol{\nu}^{-1/2}\}+\mathbf{I}_p\right]\mathbf{u}\right)\mathrm{d}\mathbf{u}$$

$$\equiv C2^p\int_{\mathbf{0}}^{\infty}\mathbf{u}^{\boldsymbol{\nu}}\exp(-\mathbf{u}^{\mathrm{T}}\boldsymbol{\Psi}_{\mathbf{z}}^{-1}\mathbf{u}/2)\mathrm{d}\mathbf{u}$$

$$= C_p\int_{\mathbf{0}}^{\infty}\mathbf{u}^{\boldsymbol{\nu}}\exp(-\mathbf{u}^{\mathrm{T}}\boldsymbol{\Psi}_{\mathbf{z}}^{-1}\mathbf{u}/2)\mathrm{d}\mathbf{u},$$

where $\mathbf{y}^{1/2}=(y_1^{1/2},...,y_p^{1/2})^{\mathrm{T}}$. The expression using the expectation is given by the definition of $\phi_p(\cdot|\cdot)$.

Proof of Theorem 2

Use the variable transformations: $y_i = u_i^2 \psi_{\mathbf{z}}^{ii} / 2$ with $\mathrm{d}u_i / \mathrm{d}y_i = 1 / \sqrt{2\psi_{\mathbf{z}}^{ii} y_i}$ $(i = 1, ..., p)$. Then, we obtain

$$C_p \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \exp(-\mathbf{u}^{\mathrm{T}} \mathbf{\Psi}_{\mathbf{z}}^{-1} \mathbf{u} / 2) \mathrm{d}\mathbf{u}$$

$$= C_p \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{v}} \exp\left(-\sum_{i=1}^{p} \frac{u_i^2 \psi_{\mathbf{z}}^{ii}}{2} - \sum_{i<j} u_i u_j \psi_{\mathbf{z}}^{ij} \right) \mathrm{d}\mathbf{u}$$

$$= C_p \int_{\mathbf{0}}^{\infty} \left\{ \prod_{i=1}^{p} \frac{(2 y_i / \psi_{\mathbf{z}}^{ii})^{\nu_i/2}}{\sqrt{2\psi_{\mathbf{z}}^{ii} y_i}} \right\} \exp\left(-\sum_{i=1}^{p} y_i - \sum_{i<j} \frac{2\psi_{\mathbf{z}}^{ij}}{\sqrt{\psi_{\mathbf{z}}^{ii} \psi_{\mathbf{z}}^{jj}}} \sqrt{y_i y_j} \right) \mathrm{d}\mathbf{y}$$

$$= C_p \int_{\mathbf{0}}^{\infty} \left\{ \prod_{i=1}^{p} \frac{2^{(\nu_i - 1)/2} y_i^{(\nu_i - 1)/2}}{(\psi_{\mathbf{z}}^{ii})^{(\nu_i + 1)/2}} \right\} \exp\left(-\sum_{i=1}^{n} y_i \right)$$

$$\times \left\{ \prod_{i<j} \sum_{u=0}^{\infty} \left(\frac{-2\psi_{\mathbf{z}}^{ij}}{\sqrt{\psi_{\mathbf{z}}^{ii} \psi_{\mathbf{z}}^{jj}}} \sqrt{y_i y_j} \right)^{u} \frac{1}{u!} \right\} \mathrm{d}\mathbf{y}$$

$$= C_p 2^{-p/2} \left\{ \prod_{i=1}^{p} \frac{2^{\nu_i/2}}{(\psi_{\mathbf{z}}^{ii})^{(\nu_i+1)/2}} \right\} \sum_{u_{12}=0}^{\infty} \cdots \sum_{u_{n-1,n}=0}^{\infty} \int_{\mathbf{0}}^{\infty} \left\{ \prod_{i=1}^{p} y_i^{(\nu_i - 1 + u_i^*)/2} e^{-y_i} \right\} \mathrm{d}\mathbf{y}$$

$$\times \left\{ \prod_{g<h} \left(\frac{-2\psi_{\mathbf{z}}^{gh}}{\sqrt{\psi_{\mathbf{z}}^{gg} \psi_{\mathbf{z}}^{hh}}} \right)^{u_{gh}} \frac{1}{u_{gh}!} \right\}$$

$$= C_p 2^{-p/2} \left\{ \prod_{i=1}^{p} \frac{2^{\nu_i/2}}{(\psi_{\mathbf{z}}^{ii})^{(\nu_i+1)/2}} \right\}$$

$$\times \sum_{\mathbf{u}=\mathbf{0}}^{\infty} \left\{ \prod_{i=1}^{p} \Gamma\left(\frac{\nu_i + 1 + u_i^*}{2} \right) \right\} \prod_{g<h} \left(\frac{-2\psi_{\mathbf{z}}^{gh}}{\sqrt{\psi_{\mathbf{z}}^{gg} \psi_{\mathbf{z}}^{hh}}} \right)^{u_{gh}} \frac{1}{u_{gh}!},$$

where noting that $C_p 2^{-p/2} = \frac{1}{\pi^{p/2} |\mathbf{\Sigma}|^{1/2}} \left\{ \prod_{i=1}^{p} \frac{1}{\sqrt{\nu_i} 2^{\nu_i/2} \Gamma(\nu_i / 2)} \right\}$, the required result follows.

Proofs of Corollary 1

Proof 1 In the proof of Theorem 1, moving the Jacobian forward, the pdf becomes

$$\int_{\mathbf{0}}^{\infty} \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu})\right\}$$
$$\times\left\{\prod_{i=1}^{p} \frac{y_i^{(\nu_i/2)-1}\sqrt{y_i/\nu_i}}{2^{\nu_i/2}\Gamma(\nu_i/2)} \exp\left(-\frac{y_i}{2}\right)\right\} \mathrm{d}\mathbf{y}$$

$$= \int_{\mathbf{0}}^{\infty} \frac{1}{(2\pi)^{p/2} |\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})|^{1/2}}$$
$$\times \exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}} \{\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}^{-1}(\mathbf{z}-\boldsymbol{\mu})\right\}$$
$$\times\left\{\prod_{i=1}^{p} \frac{y_i^{(\nu_i/2)-1}}{2^{\nu_i/2}\Gamma(\nu_i/2)} \exp\left(-\frac{y_i}{2}\right)\right\} \mathrm{d}\mathbf{y}$$
$$= \int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}\left\{\prod_{i=1}^{p} g_{\chi^2}(y_i\,|\,\nu_i)\right\}\mathrm{d}\mathbf{y},$$

where

$$\{\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}^{-1} = \mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\mathbf{y}/\mathbf{v}),$$

is the precision matrix in $\mathrm{N}_p\{\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}$.

Noting that $g_{\chi^2}(y_i\,|\,\nu_i) = g_{\Gamma}(y_i\,|\,\nu_i/2, 1/2)$ $(i = 1,\ldots,p)$ and using the variable transformations $y_i^* = y_i/c_i$ $(0 < c_i < \infty;\ i = 1,\ldots,p)$ as in Lemma 1, the redefinition of y_i^* as y_i $(i = 1,\ldots,p)$ gives the remaining result.

Proof 2 When $Y_i = y_i$ $(i = 1,\ldots,p)$ are given, $\mathbf{Z}-\boldsymbol{\mu} = \{(X_1-\mu_1)/\sqrt{y_1/\nu_1},\ldots,(X_p-\mu_p)/\sqrt{y_p/\nu_p}\}^{\mathrm{T}}$ is found to be a scaled $\mathbf{X}-\boldsymbol{\mu}$, where each $X_i - \mu_i$ is multiplied by $1/\sqrt{y_i/\nu_i}$ $(i = 1,\ldots,p)$, Then, the pdf of the conditional distribution of **Z** given **y** becomes

$$\phi_p\{\mathbf{z}\,|\,\boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\mathbf{v})\}$$

Since independent $Y_i \sim \chi^2(\nu_i)$ $(i = 1,\ldots, p)$ are also independent of **X**, we obtain the pdf

$$\int_{\mathbf{0}}^{\infty} \phi_p\{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2}(\mathbf{y}/\boldsymbol{\nu})\boldsymbol{\Sigma}\mathrm{diag}^{-1/2}(\mathbf{y}/\boldsymbol{\nu})\}\left\{\prod_{i=1}^{p} g_{\chi^2}(y_i \mid \nu_i)\right\}\mathrm{d}\mathbf{y}\,,$$

giving the first result. The remaining result is given as in Proof 1.

Proof of Theorem 3

Recall that

$$\mathbf{Z} - \boldsymbol{\mu} = \{(X_1 - \mu_1)/\sqrt{Y_1/\nu_1}, \ldots, (X_p - \mu_p)/\sqrt{Y_p/\nu_p}\}^{\mathrm{T}},$$

where $Y_i \sim \chi^2(\nu_i)$ $(i = 1,\ldots., p)$ independent of **X**. Employ the variable transformations $Y_i^* = 1/\sqrt{Y_i}$ $(i = 1,\ldots., p)$. As addressed earlier, Y_i^* follows the inverse-chi distribution with ν_i df denoted by $Y_i^* \sim \chi^{-1}(\nu_i)$ $(i = 1,\ldots., p)$. Then, we have

$$\mathbf{Z} - \boldsymbol{\mu} = \{(X_1 - \mu_1)Y_1^*\sqrt{\nu_1}, \ldots, (X_p - \mu_p)Y_p^*\sqrt{\nu_p}\}^{\mathrm{T}}.$$

Since Y_i^*'s are independent of **X**,

$$\mathrm{E}(\mathbf{Z}^{\mathbf{k}}) = \mathrm{E}(\mathbf{X}^{\mathbf{k}} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})\prod_{i=1}^{p} \mathrm{E}\{Y_i^{*k_i} \mid \chi^{-1}(\nu_i)\}\nu_i^{k_i/2},$$

where $\mathrm{E}\{Y_i^{*k_i} \mid \chi^{-1}(\nu_i)\}$ is the k_i-th order raw moment of $Y_i^* \sim \chi^{-1}(\nu_i)$, which is

$$\mathrm{E}\{Y_i^{*k_i} \mid \chi^{-1}(\nu_i)\} = \frac{\Gamma\{(\nu_i - k_i)/2\}}{2^{k_i/2}\Gamma(\nu_i/2)}$$
$$(k_i = 0, 1, \ldots;\ \nu_i > k_i,\ i = 1, \ldots, p)$$

(see Kollo et al., 2021, Lemma 1). These results give the first required one.

The remaining result of the mixture expression under sectional truncation is given by Corollary 2.

Proof of Corollary 3

The first result is due to the equality of the distribution of $\mathbf{X}_j \sim \mathrm{N}_{p_j}(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_{jj})$ $(j = 1,\ 2)$ and the corresponding marginal one of $\mathbf{X} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The remaining results are given as follows. Noting that $\mathrm{E}(\mathbf{Z} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\nu}) = \boldsymbol{\mu}$ and using Theorem 3, we have

$$\operatorname{cov}(Z_i, Z_j \mid \mathbf{v}) = \mathrm{E}\left\{ Z_i Z_j \mid (\mu_i, \mu_j)^{\mathrm{T}} = \mathbf{0}, \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}, (\nu_i, \nu_j)^{\mathrm{T}} \right\}$$

$$= \sigma_{ij} \frac{\nu_i^{1/2} \Gamma\{(\nu_i - 1)/2\}}{2^{1/2} \Gamma(\nu_i / 2)} \frac{\nu_j^{1/2} \Gamma\{(\nu_j - 1)/2\}}{2^{1/2} \Gamma(\nu_j / 2)}$$

$$= \frac{\sigma_{ij} \sqrt{\nu_i \nu_j}}{2} \frac{\Gamma\{(\nu_i - 1)/2\}}{\Gamma(\nu_i / 2)} \frac{\Gamma\{(\nu_j - 1)/2\}}{\Gamma(\nu_j / 2)}$$

$$(\nu_i > 1,\ \nu_j > 1;\ i, j = 1, \ldots, p;\ i \neq j).$$

Theorem 3 shows that the variances of Z_i and Z_j are equal to those for the usual univariate *t* (see e.g., Johnson, Kotz & Balakrishnan, 1995, Equation 28.7a) or can also be obtained from Theorem 3 as

$$\operatorname{var}(Z_i) = \frac{\sigma_{ii} \nu_i}{\nu_i - 2} \ (\nu_i > 2;\ i = 1, \ldots, p)$$

which gives

$$\operatorname{cor}(Z_i, Z_j \mid \mathbf{v}) = \frac{\sigma_{ij} \sqrt{\nu_i \nu_j}}{2} \frac{\Gamma\{(\nu_i - 1)/2\}}{\Gamma(\nu_i / 2)} \frac{\Gamma\{(\nu_j - 1)/2\}}{\Gamma(\nu_j / 2)} \sqrt{\frac{(\nu_i - 2)(\nu_j - 2)}{\sigma_{ii} \sigma_{jj} \nu_i \nu_j}}$$

$$= \frac{\rho_{ij}}{2} \frac{\Gamma\{(\nu_i - 1)/2\} \sqrt{\nu_i - 2}}{\Gamma(\nu_i / 2)} \frac{\Gamma\{(\nu_j - 1)/2\} \sqrt{\nu_j - 2}}{\Gamma(\nu_j / 2)}$$

$$(\nu_i > 2,\ \nu_j > 2;\ i, j = 1, \ldots, p;\ i \neq j).$$

Proof of Lemma 2

$$\operatorname{cor}(XZ, YZ) = \frac{\operatorname{cov}(XZ, YZ)}{\sqrt{\operatorname{var}(XZ)} \sqrt{\operatorname{var}(YZ)}} = \frac{\operatorname{cov}(X, Y) \mathrm{E}(Z^2)}{\sqrt{\operatorname{var}(X) \mathrm{E}(Z^2)} \sqrt{\operatorname{var}(Y) \mathrm{E}(Z^2)}}$$

$$= \frac{\operatorname{cov}(X, Y)}{\sqrt{\operatorname{var}(X)} \sqrt{\operatorname{var}(Y)}} = \operatorname{cor}(X, Y).$$

Proof of Result 1

The inequality for covariances is easily obtained, giving the proof for correlations. The second proof for correlations without using the inequality for covariances is given as follows:

$$
\begin{aligned}
&|\operatorname{cor}(Z_i, Z_j \mid \nu)| - |\operatorname{cor}(Z_i, Z_j \mid \mathbf{v} = \mathbf{v}_0 = \mathbf{1}_p \nu)| \\
&= |\rho_{ij}| \left[1 - \frac{\Gamma^2\{(\nu-1)/2\}(\nu-2)}{2\Gamma^2(\nu/2)} \right] = |\rho_{ij}| \left[1 - \frac{\Gamma^2\{(\nu-1)/2\}}{\Gamma(\nu/2)\Gamma\{(\nu-2)/2\}} \right] \geq 0 \\
&(\nu > 2;\ i, j = 1, \ldots, p;\ i \neq j).
\end{aligned}
$$

where the inequality is due to the convex property of the gamma function with

$$
\frac{\Gamma\{(\nu-1)/2\}}{\Gamma(\nu/2)} < \frac{\Gamma\{(\nu-2)/2\}}{\Gamma\{(\nu-1)/2\}}
$$

when its argument is positive.

Proof of Result 2

For the result of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{v})$, define $\mathbf{Y}_0^* = \sqrt{\nu_0}(1/\sqrt{Y_1}, \ldots, 1/\sqrt{Y_p})^{\mathrm{T}} = \sqrt{\nu_0}\mathbf{Y}^*$.

Then, we have

$$
\begin{aligned}
&\beta_{2,p}(\mathbf{Z} \mid \mathbf{v}_0) = \mathrm{E}[\{(\mathbf{Z}-\boldsymbol{\mu})^{\mathrm{T}} \operatorname{cov}^{-1}(\mathbf{Z})(\mathbf{Z}-\boldsymbol{\mu})\}^2 \mid \mathbf{v}_0] \\
&= \mathrm{E}\left(\{(\mathbf{X}-\boldsymbol{\mu}) \odot \mathbf{Y}_0^*\}^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \{(\mathbf{X}-\boldsymbol{\mu}) \odot \mathbf{Y}_0^*\} \left[\frac{\nu_0 \Gamma\{(\nu_0-2)/2\}}{2\Gamma(\nu_0/2)} \right]^{-1} \right)^2 \\
&= \mathrm{E}\left[\{(\mathbf{X}-\boldsymbol{\mu}) \odot \mathbf{Y}^*\}^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \{(\mathbf{X}-\boldsymbol{\mu}) \odot \mathbf{Y}^*\} \right]^2 (\nu_0 - 2)^2,
\end{aligned}
$$

where $\mathrm{E}(\cdot)^2 = \mathrm{E}\{(\cdot)^2\}$.

Using $\boldsymbol{\Sigma}^{-1} = \{\sigma^{ij}\}$, the above result becomes

$$
\begin{aligned}
&(\nu_0-2)^{-2} \beta_{2,p}(\mathbf{Z} \mid \mathbf{v}_0) = \sum_{i=1}^{p} (\sigma^{ii})^2 \mathrm{E}(X_i - \mu_i)^4 \mathrm{E}(Y_i^{*4}) \\
&\quad + \sum_{i \neq j}^{(p^2-p)} (\sigma^{ii}\sigma^{jj} + \sigma^{ij}\sigma^{ij} + \sigma^{ij}\sigma^{ji}) \mathrm{E}(X_i - \mu_i)^2 \mathrm{E}(X_j - \mu_j)^2 \mathrm{E}(Y_i^{*2}) \mathrm{E}(Y_j^{*2}) \\
&= \sum_{i=1}^{p} 3(\sigma^{ii})^2 \sigma_{ii}^2 \frac{\Gamma\{(\nu_0-4)/2\}}{2^2 \Gamma(\nu_0/2)} + \sum_{i \neq j}^{(p^2-p)} \{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj} \frac{\Gamma^2\{(\nu_0-2)/2\}}{2^2 \Gamma^2(\nu_0/2)} \\
&= \sum_{i=1}^{p} \frac{3(\sigma^{ii})^2 \sigma_{ii}^2}{(\nu_0-2)(\nu_0-4)} + \sum_{i \neq j}^{(p^2-p)} \frac{\{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj}}{(\nu_0-2)^2} \quad (\nu_0 > 4),
\end{aligned}
$$

where $\sum_{i \neq j}^{(p^2-p)} (\cdot) = \sum_{i=1}^{p} \sum_{j=1,\ j \neq i}^{p} (\cdot)$ is the sum of $p^2 - p$ terms with $i \neq j$. Consequently, we have

$$\beta_{2,p}(\mathbf{Z} \mid \mathbf{v}_0) = \sum_{i=1}^{p} \frac{3(\sigma^{ii})^2 \sigma_{ii}^2 (\nu_0 - 2)}{\nu_0 - 4} + \sum_{i \neq j}^{(p^2-p)} \{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj}$$

$$= \sum_{i=1}^{p} 3(\sigma^{ii})^2 \sigma_{ii}^2 \left(1 + \frac{2}{\nu_0 - 4}\right) + \sum_{i \neq j}^{(p^2-p)} \{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj}$$

$$= p(p+2) + \sum_{i=1}^{p} \frac{6(\sigma^{ii})^2 \sigma_{ii}^2}{\nu_0 - 4}$$

$$(\nu_0 > 4).$$

For the corresponding result of $\mathrm{St}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)$, define $Y_\nu^* = \sqrt{\nu} / \sqrt{Y} = \sqrt{\nu} Y^*$. Then, we obtain

$$\beta_{2,p}(\mathbf{Z} \mid \nu) = \mathrm{E}[\{(\mathbf{Z} - \boldsymbol{\mu})^{\mathrm{T}} \operatorname{cov}^{-1}(\mathbf{Z})(\mathbf{Z} - \boldsymbol{\mu})\}^2 \mid \nu]$$

$$= \mathrm{E}\left\{(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu}) Y_\nu^{*2} \left(\frac{\nu}{\nu - 2}\right)^{-1}\right\}^2$$

$$= \mathrm{E}\left\{(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu}) Y^{*2} \left(\frac{1}{\nu - 2}\right)^{-1}\right\}^2$$

$$= p(p+2) \frac{\Gamma\{(\nu - 4)/2\}(\nu - 2)^2}{2^2 \Gamma(\nu/2)}$$

$$= \frac{p(p+2)(\nu - 2)}{\nu - 4} = p(p+2)\left(1 + \frac{2}{\nu - 4}\right) (\nu > 4).$$

Using the above results, we have

$$\beta_{2,p}(\mathbf{Z} \mid \nu) - \beta_{2,p}(\mathbf{Z} \mid \nu \mathbf{1}_p)$$

$$= p(p+2)\left(1 + \frac{2}{\nu - 4}\right) - p(p+2) - \sum_{i=1}^{p} 2 \frac{3(\sigma^{ii})^2 \sigma_{ii}^2}{\nu - 4}$$

$$= p(p+2) \frac{2}{\nu - 4} - \frac{2}{\nu - 4}\left[p(p+2) - \sum_{i \neq j}^{(p^2-p)} \{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj}\right]$$

$$= \frac{2}{\nu - 4} \sum_{i \neq j}^{(p^2 - p)} \{\sigma^{ii}\sigma^{jj} + 2(\sigma^{ij})^2\} \sigma_{ii}\sigma_{jj} > 0,$$

where $\sigma^{ii} > 0 \ (i = 1, \ldots, p)$ are used for the last inequality.

Proof of Lemma 3

By definition

$$\begin{aligned}
&\beta_{2,p}(\mathbf{Z}) = \mathrm{E}\{(\mathbf{Z} - \boldsymbol{\mu}^*)^{\mathrm{T}} \mathrm{cov}^{-1}(\mathbf{Z})(\mathbf{Z} - \boldsymbol{\mu}^*)\}^2 \\
&= \mathrm{E}\{(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}} Y [\mathrm{E}\{(\mathbf{X} - \boldsymbol{\mu}) Y^2 (\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}\}]^{-1} (\mathbf{X} - \boldsymbol{\mu}) Y\}^2 \\
&= \mathrm{E}\{(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}} [\mathrm{E}\{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}\}]^{-1} (\mathbf{X} - \boldsymbol{\mu})\}^2 \mathrm{E}(Y^4) / \mathrm{E}^2(Y^2) \\
&= \beta_{2,p}(\mathbf{X}) \mathrm{E}(Y^4) / \mathrm{E}^2(Y^2) \\
&= \beta_{2,p}(\mathbf{X}) [\{\mathrm{var}(Y^2) / \mathrm{E}^2(Y^2)\} + 1] \geq \beta_{2,p}(\mathbf{X}).
\end{aligned}$$

Proof of Theorem 4

The pdf of the joint distribution of $\mathbf{X}$ and $\mathbf{F}$ is

$$\begin{aligned}
&\phi_p(\mathbf{X} = \mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \prod\nolimits_{i=1}^{q} g_{\chi^2}(F_i = f_i \mid n_i) \\
&= \frac{1}{(2\pi)^{p/2} \mid \boldsymbol{\Sigma} \mid^{1/2}} \exp\left\{ -\frac{(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})}{2} \right\} \\
&\times \prod\nolimits_{i=1}^{q} \frac{f_i^{(n_i/2)-1}}{2^{n_i/2} \Gamma(n_i / 2)} \exp\left(-\frac{f_i}{2} \right).
\end{aligned}$$

Recalling the notation $\mathrm{diag}^{-1/2}(\mathbf{Y} / \mathbf{v}) \equiv \mathrm{diag}^{-1/2}(\mathbf{Y}) \mathrm{diag}^{1/2}(\mathbf{v})$ employed earlier, use the variable transformation from $\mathbf{X}$ to $\mathbf{Z} = \mathrm{diag}^{-1/2}(\mathbf{Y} / \mathbf{v})(\mathbf{X} - \boldsymbol{\mu}) + \boldsymbol{\mu}$ $= \mathrm{diag}^{-1/2}(\boldsymbol{\Lambda}\mathbf{F} / \mathbf{v})(\mathbf{X} - \boldsymbol{\mu}) + \boldsymbol{\mu}$ with unchanged $\mathbf{F}$ and the Jacobian

$$\prod_{i=1}^{p} \frac{\mathrm{d}x_i}{\mathrm{d}z_i} = \prod_{i=1}^{p} \sqrt{\frac{y_i}{\nu_i}} = \prod_{i=1}^{p} \sqrt{\frac{\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{f}}{\nu_i}},$$

where $\boldsymbol{\lambda}_i^{\mathrm{T}}$ is the i-th row of $\boldsymbol{\Lambda} \ (i = 1, \ldots, p)$. Then, noting that $\mathbf{x} - \boldsymbol{\mu} = \mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f} / \mathbf{v})(\mathbf{z} - \boldsymbol{\mu})$, the pdf of the joint distribution of $\mathbf{Z}$ and $\mathbf{F}$ becomes

$$\frac{1}{(2\pi)^{p/2}\,|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu})\right\}$$
$$\times\left(\prod_{i=1}^{p}\sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{f}/v_i}\right)\prod_{i=1}^{q}\frac{f_i^{(n_i/2)-1}}{2^{n_i/2}\Gamma(n_i/2)}\exp\left(-\frac{f_i}{2}\right)$$
$$=\frac{1}{(2\pi)^{p/2}\,|\boldsymbol{\Sigma}|^{1/2}}\left(\prod_{i=1}^{p}\sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{f}/v_i}\right)\left\{\prod_{i=1}^{q}\frac{f_i^{(n_i/2)-1}}{2^{n_i/2}\Gamma(n_i/2)}\right\}$$
$$\times\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu})-\frac{\mathbf{1}_q^{\mathrm{T}}\mathbf{f}}{2}\right\}.$$

The pdf of $\mathbf{Z}$ is given by the above result, when $\mathbf{F}$ is integrated out over its support. Define

$$C^{*}=\frac{1}{(2\pi)^{p/2}\,|\boldsymbol{\Sigma}|^{1/2}}\left(\prod_{i=1}^{p}1/v_i\right)\prod_{i=1}^{q}\frac{1}{2^{n_i/2}\Gamma(n_i/2)}$$

and use the variable transformation $U_i=\sqrt{F_i}\ (i=1,\dots,q)$ with $\mathbf{U}=(U_1,\dots,U_q)^{\mathrm{T}}$. Then, the pdf of $\mathbf{Z}$ is given as

$$C^{*}\int_{\mathbf{0}}^{\infty}\left\{\prod_{i=1}^{q}f_i^{(n_i/2)-1}\right\}\left(\prod_{i=1}^{p}\sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{f}}\right)$$
$$\times\exp\left\{-\frac{1}{2}(\mathbf{z}-\boldsymbol{\mu})^{\mathrm{T}}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})\boldsymbol{\Sigma}^{-1}\mathrm{diag}^{1/2}(\boldsymbol{\Lambda}\mathbf{f}/\mathbf{v})(\mathbf{z}-\boldsymbol{\mu})-\frac{\mathbf{1}_q^{\mathrm{T}}\mathbf{f}}{2}\right\}d\mathbf{f}$$
$$=C^{*}\int_{\mathbf{0}}^{\infty}\left\{\prod_{i=1}^{q}f_i^{(n_i/2)-1}\right\}\left(\prod_{i=1}^{p}\sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{f}}\right)\exp\left[-\frac{1}{2}(\boldsymbol{\Lambda}\mathbf{f})^{1/2\mathrm{T}}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\mathbf{v}^{-1/2}\}\right.$$
$$\left.\times\boldsymbol{\Sigma}^{-1}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\mathbf{v}^{-1/2}\}(\boldsymbol{\Lambda}\mathbf{f})^{1/2}-\frac{\mathbf{1}_q^{\mathrm{T}}\mathbf{f}}{2}\right]d\mathbf{f}$$

$$=C^{*}2^{q}\int_{\mathbf{0}}^{\infty}\left\{\prod_{i=1}^{q}u_i^{n_i-1}\right\}\left(\prod_{i=1}^{p}\sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}}(\mathbf{u}\odot\mathbf{u})}\right)$$
$$\times\exp\left(-\frac{1}{2}\left[\{\boldsymbol{\Lambda}(\mathbf{u}\odot\mathbf{u})\}^{1/2\mathrm{T}}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\mathbf{v}^{-1/2}\}\right.\right.$$
$$\left.\left.\times\boldsymbol{\Sigma}^{-1}\mathrm{diag}\{(\mathbf{z}-\boldsymbol{\mu})\odot\mathbf{v}^{-1/2}\}\{\boldsymbol{\Lambda}(\mathbf{u}\odot\mathbf{u})\}^{1/2}+\mathbf{u}^{\mathrm{T}}\mathbf{u}\right]\right)d\mathbf{u}$$

$$\equiv C_q \int_{\mathbf{0}}^{\infty} \mathbf{u}^{\mathbf{n}-\mathbf{1}_q} \left(\prod_{i=1}^{p} \sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}} (\mathbf{u} \odot \mathbf{u})} \right)$$

$$\times \exp\left(-\frac{1}{2} \left[\{\boldsymbol{\Lambda}(\mathbf{u} \odot \mathbf{u})\}^{1/2\mathrm{T}} \boldsymbol{\Omega}_{\mathbf{z}}^{-1} \{\boldsymbol{\Lambda}(\mathbf{u} \odot \mathbf{u})\}^{1/2} + \mathbf{u}^{\mathrm{T}} \mathbf{u} \right] \right) \mathrm{d}\mathbf{u}.$$

Proof of Corollary 4

From an equation of the proof of Theorem 4, moving the Jacobian forward, the pdf becomes

$$\frac{1}{(2\pi)^{p/2} \mid \boldsymbol{\Sigma} \mid^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{z} - \boldsymbol{\mu})^{\mathrm{T}} \mathrm{diag}^{1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) \boldsymbol{\Sigma}^{-1} \mathrm{diag}^{1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) (\mathbf{z} - \boldsymbol{\mu}) \right\}$$

$$\times \left(\prod_{i=1}^{p} \sqrt{\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{f} / \nu_i} \right) \prod_{i=1}^{q} \frac{f_i^{(n_i/2)-1}}{2^{n_i/2} \Gamma(n_i / 2)} \exp\left(-\frac{f_i}{2} \right)$$

$$= \frac{1}{(2\pi)^{p/2} \mid \mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) \mid^{1/2}}$$

$$\times \exp\left[-\frac{1}{2} (\mathbf{z} - \boldsymbol{\mu})^{\mathrm{T}} \{\mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu})\}^{-1} (\mathbf{z} - \boldsymbol{\mu}) \right]$$

$$\times \prod_{i=1}^{q} \frac{f_i^{(n_i/2)-1}}{2^{n_i/2} \Gamma(n_i / 2)} \exp\left(-\frac{f_i}{2} \right)$$

$$= \int_{\mathbf{0}}^{\infty} \phi_p \{\mathbf{z} \mid \boldsymbol{\mu}, \mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu}) \boldsymbol{\Sigma} \mathrm{diag}^{-1/2} (\boldsymbol{\Lambda}\mathbf{f} / \boldsymbol{\nu})\} \left\{ \prod_{i=1}^{q} g_{\chi^2} (f_i \mid n_i) \right\} \mathrm{d}\mathbf{f}.$$

The expression using c_i 's is given in a way similar to that in Corollary 1.

Proof of Result 3

$$\int_0^{\min(\mathbf{y})} \left\{ \prod_{i=1}^{p} g_{\chi^2} (y_i - f_0 \mid n_i) \right\} g_{\chi^2} (f_0 \mid n_0) \mathrm{d}f_0$$

$$= \int_0^{\min(\mathbf{y})} \frac{f_0^{(n_0/2)-1}}{2^{n_0/2} \Gamma(n_0 / 2)} \exp(-f_0 / 2) \left\{ \prod_{i=1}^{p} \frac{(y_i - f_0)^{(n_i/2)-1}}{2^{n_i/2} \Gamma(n_i / 2)} \exp\left(-\frac{y_i - f_0}{2} \right) \right\} \mathrm{d}f_0$$

$$= \frac{\exp(-\mathbf{1}_p^{\mathrm{T}}\mathbf{y}/2)}{2^{(n_0+\mathbf{1}_p^{\mathrm{T}}\mathbf{n})/2}\prod_{i=0}^{p}\Gamma(n_i/2)}$$

$$\times \int_0^{\min(\mathbf{y})} f_0^{(n_0/2)-1}\left\{\prod_{i=1}^{p}(y_i - f_0)^{(n_i/2)-1}\right\}\exp\left\{\frac{(p-1)f_0}{2}\right\}\mathrm{d}f_0.$$

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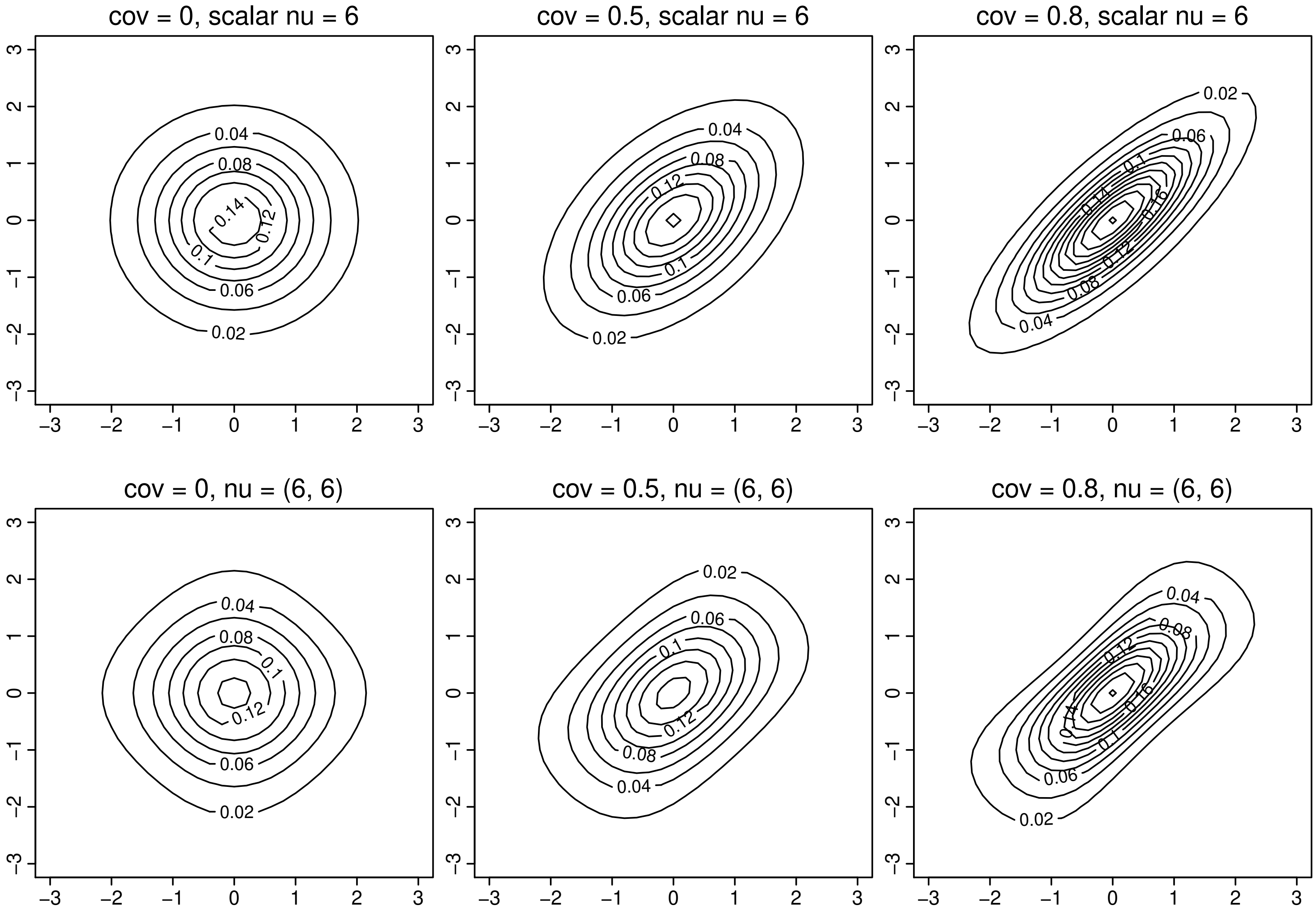

Figure 1. Density contours of the bivariate t−distributions
with common (upper) / independent (lower) chi−squares
(Z_1 = the horizontal axis, Z_2 = the vertical axis, cov = σ_{12})

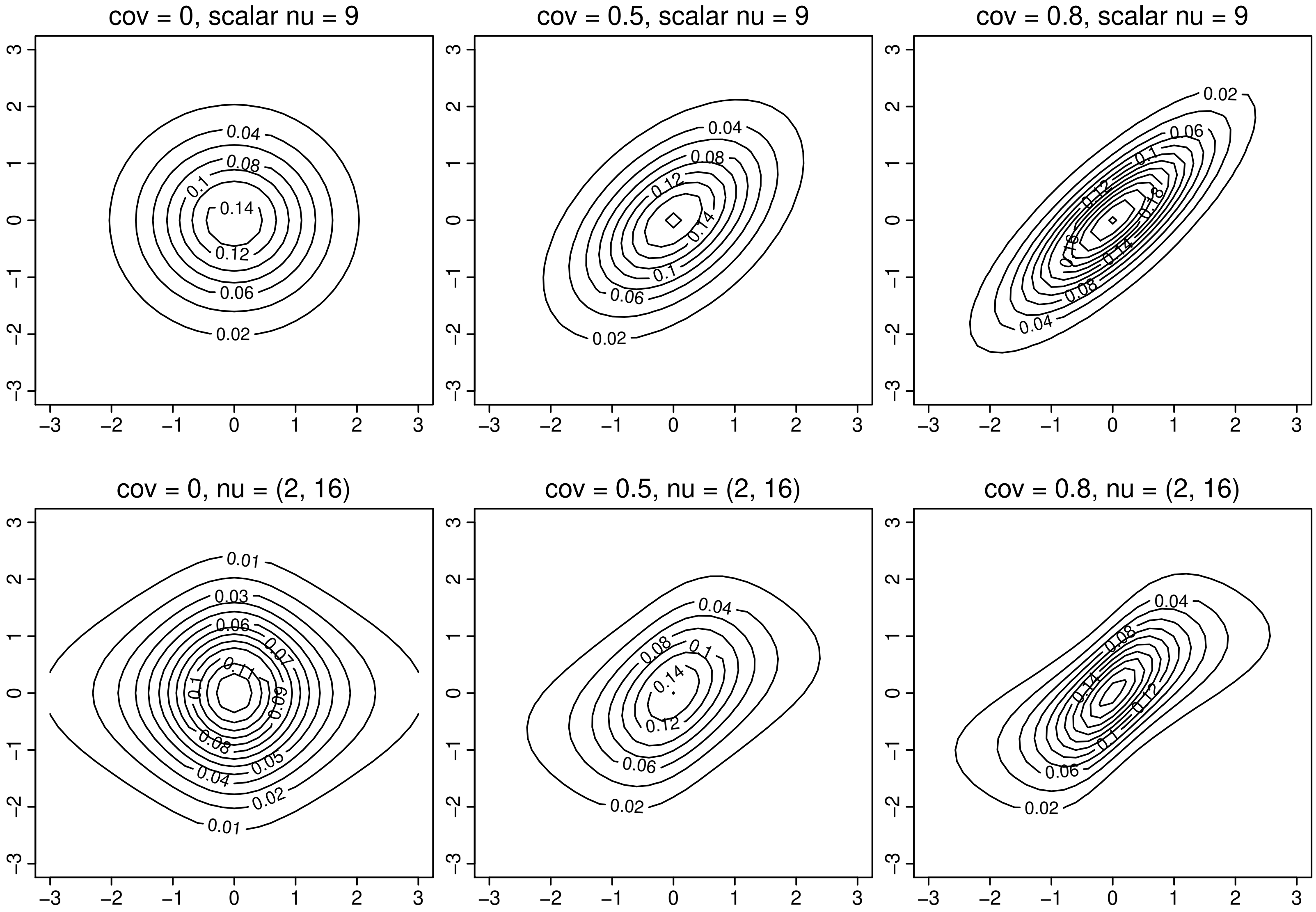

Figure 2. Density contours of the bivariate t−distributions with common (upper) / independent (lower) chi−squares (Z_1 = the horizontal axis, Z_2 = the vertical axis, cov = σ_{12})

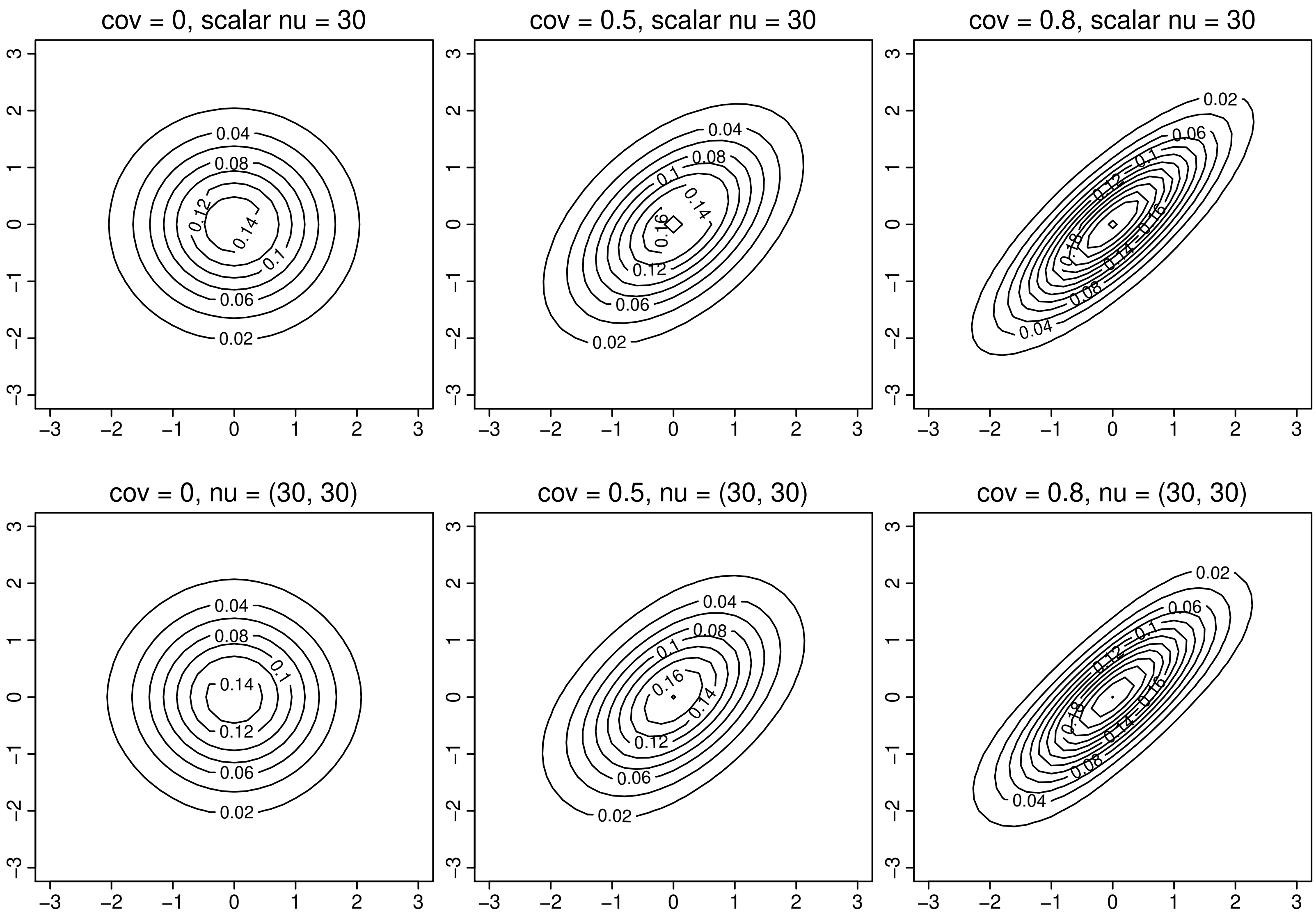

Figure 3. Density contours of the bivariate t−distributions with common (upper) / independent (lower) chi−squares (Z_1 = the horizontal axis, Z_2 = the vertical axis, cov = σ_{12})