The distribution of the sample correlation coefficient under variance-truncated normality

Preprint · April 2022

DOI: 10.13140/RG.2.2.19080.90883

1 author:

Haruhiko Ogasawara
Otaru University of Commerce

The distribution of the sample correlation coefficient under variance-truncated normality

April 19, 2022

Haruhiko Ogasawara*

*Otaru University of Commerce, 3-5-21 Midori, Otaru 047-8501 Japan; Email: emt-hogasa@emt.otaru-uc.ac.jp; Web: http://www.res.otaru-uc.ac.jp/~emt-hogasa/.

The distribution of the sample correlation coefficient under variance-truncated normality

Abstract: The non-null distribution of the sample correlation coefficient under bivariate normality is derived when each of the associated two sample variances is subject to stripe truncation including usual single and double truncation as special cases. The probability density function is obtained using series expressions as in the untruncated case with new definitions of weighted hypergeometric functions. Formulas of the moments of arbitrary orders are given using the weighted hypergeometric functions. It is shown that the null joint distribution of the sample correlation coefficients under multivariate untruncated normality holds also in the variance-truncated cases. Some numerical illustrations are shown.

Keywords: Wishart distribution, stripe truncation, weighted hypergeometric functions, sample variances and covariances, multivariate normality, multivariate gamma function.

1. Introduction

The non-null distribution of the sample correlation coefficient (r) under normality was first given by Fisher (1915, p. 516) using a differential expression (see Anderson, 2003, Equation (39), Section 4.2.2). Soper, Young, Cave and Pearson (1917, Equations (xiii) & (xxiv)) obtained the recursive formulas of the even and odd moments of r using the Gauss hypergeometric functions, and the results in samples of size as small as 3 or larger. Kendall (1948, Equation (14.53); Sections 14.17-14.18) gave the joint probability density function (pdf) of the sample variances and r under bivariate normality with the moments of r up to the fourth order. Hotelling (1953, Equation (25)) derived an expression of the pdf of r using the Gauss hypergeometric series with relatively fast convergence (see Muirhead, 1982, Section 5.1.2, Equation (10); Stuart & Ort, 1994, Section 16.30), which was used by Rady, Fergany and Edress (2005) to have moments of r. Anderson (1958, 2003, Theorem 4.2.2) gave a series expression of the pdf of r based on the Wishart distribution, which can be used to have the moments rather easily (see also Muirhead, 1982, Section 5.1.3, Equation (11)). Ghosh (1966) seems to have employed one of these formulas to have the moments of r. Joarder (2006, Theorem 2.1) and Romero-Padilla (2016, Theorem 1) used the joint distribution of the sample variances and r under bivariate normality for the moments of r. For other expressions of the distribution under normality and historical developments, see Stuart and Ort (1994, pp. 559-567) and Johnson, Kotz and Balakrishnan (1995, Chapter 32, Section 2).

It is known that the null distribution of the sample correlation coefficient is given from the t-distribution with $N-2$ degrees of freedom with N being the number of observations, which holds under sphericity as well as normality (see Muirhead, 1982, Section 5.1.2). The non-null distribution of the sample correlation coefficient under normality is typically given by the Wishart distribution with $n=N-1$ degrees of freedom, where the random matrix in the Wishart is (i) n times the sample covariance matrix when the scale matrix is the population covariance matrix or (ii) the sample covariance matrix when the scale matrix is n^{-1} times the population one. Both formulations can be employed to derive the distribution of the sample correlation coefficients, where the term "sample dispersion matrix" can be used in both cases. Then, the off-diagonal elements of the sample dispersion matrix are re-expressed by the sample correlation coefficients and the unchanged diagonals.

In this paper, truncation of observations based on (n times) sample variances is considered in the Wishart distribution. This corresponds to the situation when e.g., some sample variance is too large or too small, the observation is deleted due to the possible existence of outliers or distortions. Then, the distribution of the correlation coefficients in the truncated Wishart is derived, where the re-expressions of the off-diagonal elements in the sample dispersion matrix using the correlation coefficients are employed with the Jacobian.

So far, various types of truncation have been investigated, where the random vector has the multivariate normal or non-normal distribution. Observations are subject to single, double, radial (Tallis, 1961, 1963) and plane (Tallis, 1965) truncation depending on the definitions of truncation for observations. Elliptical truncation is also used when quadratic forms of observations or chi-squared variables under normality are considered (Tallis, 1963; Kotz, Balakrishnan & Johnson, 2000, Chapter 45, Section 10; Arismendi & Broda, 2017). Recently, Ogasawara (2021a) proposed stripe truncation for a single variable having a zebraic truncation pattern. Sectional truncation given by Ogasawara (2021b) is a multivariate extension of stripe truncation. In these cases, moments after truncation have been of primary interest (see Fisher, 1931; Kan & Robotti, 2017; Kirkby, Nguyen & Nguyen, 2021; Galarza, Lin, Wang & Lachos, 2021).

The remainder of this paper is organized as follows. Section 2 gives the non-null distribution and moment formulas of the sample correlation coefficient under bivariate normality with variance truncation. Numerical illustrations are given in Section 3. In Section 4, it is shown that the joint null distribution of the sample correlation coefficients under multivariate normality is unchanged irrespective of variance truncation. Some discussions are given in Section 5. Proofs for lemmas, theorems and corollaries when necessary are given in the appendix.

2. The bivariate case

Let a $p \times p$ random matrix $\mathbf{V} = \{v_{ij}\}$ be Wishart distributed, which is denoted by $\mathbf{V} \sim \mathrm{W}_p(\mathbf{\Sigma}, n)$ with $\mathbf{V}$ being positive definite i.e., $\mathbf{V} > 0$, where $\mathbf{\Sigma} > 0$ is a scale matrix with n degrees of freedom. Then, the pdf of the Wishart distribution is given by

$$w_p(\mathbf{V} \mid \mathbf{\Sigma}, n) = \frac{\exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-p-1)/2}}{2^{np/2} \mid \mathbf{\Sigma} \mid^{n/2} \Gamma_p(n/2)}$$

where $\Gamma_p(t) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(t - \frac{i-1}{2}\right)$ is the multivariate gamma function (Anderson, 2003, Definition 7.2.1; Subsection 7.2, Equation (19); see also DLMF, 2021, Section 35.3, https://dlmf.nist.gov/35.3); $\mathbf{V}$ is used also as a realization of $\mathbf{V}$ for simplicity of notation. Consider the bivariate case with

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \text{ and } \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} v_{11} & \sqrt{v_{11}v_{22}}\,r \\ \sqrt{v_{22}v_{11}}\,r & v_{22} \end{pmatrix},$$

where $\sigma_{11} = \sigma_{22} = 1$ is used without loss of generality; $\mathbf{V} = \sum_{i=1}^{N} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})^{\mathrm{T}}$ with $n+1 = N$ and $\bar{\mathbf{X}} = (\bar{X}_1, \bar{X}_2)^{\mathrm{T}} = \sum_{i=1}^{N} \mathbf{X}_i / N$; and $\mathbf{X}_i = (X_{1i}, X_{2i})^{\mathrm{T}}$ $(i = 1, ..., N)$ are independent and normally distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$; and ρ and r are the population and sample correlation coefficients, respectively. Employ the change of variable from $v_{12}(= v_{21})$ to r with unchanged v_{11} and v_{22} yielding the Jacobian $\mathrm{d}v_{12} = \sqrt{v_{11}v_{22}}\,\mathrm{d}r$. Then, we have the Wishart density

$$\begin{aligned} & w_2(v_{11}, v_{22}, r \mid \mathbf{\Sigma}, n) \\ & = \exp\left[-\mathrm{tr}\left\{\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} v_{11} & \sqrt{v_{11}v_{22}}\,r \\ \sqrt{v_{22}v_{11}}\,r & v_{22} \end{pmatrix}\frac{1}{2(1-\rho^2)}\right\}\right]\frac{\{v_{11}v_{22}(1-r^2)\}^{(n-3)/2}\sqrt{v_{11}v_{22}}}{2^n(1-\rho^2)^{n/2}\Gamma_2(n/2)} \\ & = \exp\left\{-\frac{v_{11} + v_{22} - 2\rho\sqrt{v_{11}v_{22}}\,r}{2(1-\rho^2)}\right\}\frac{(v_{11}v_{22})^{(n-2)/2}(1-r^2)^{(n-3)/2}}{2^n(1-\rho^2)^{n/2}\Gamma_2(n/2)}, \end{aligned}$$

(Anderson, 2003, Section 4.2.2; Muirhead, 1982, Section 5.1.3).

Assume that $\mathbf{V}$ is truncated such that only when $\bigcap_{i=1}^{2} \bigcup_{k=1}^{K_i} \{v_{ii} \in I_{ik}\}$, the observation $\mathbf{V} = \{v_{ij}\}$ is selected otherwise truncated, where I_{ik} is an interval satisfying

$$I_{ik} = [a_{ik}, b_{ik}),\ 0 \le a_{i1} < b_{i1} < ... < a_{iK_i} < b_{iK_i} \le \infty\ (i = 1, 2; k = 1, ..., K_i),$$

where K_i is the number of intervals for v_{ii}. When $K_i = 1$ with $0 = a_{i1} < b_{i1} < \infty$ or $0 < a_{i1} < b_{i1} = \infty$, v_{ii} is singly upper- or lower-truncated, respectively while when $K_i = 1$

with $0 < a_{i1} < b_{i1} < \infty$, v_{ii} is doubly truncated. When $K_i = 2$ with $0 = a_{i1} < b_{i1} < a_{i2} < b_{i2} = \infty$, v_{ii} is inner-truncated with the two tails being selected. These cases occur e.g., when $\mathbf{V}/n$ is the sample covariance matrix; and v_{ii} (and consequently $\mathbf{V}$) is discarded if v_{ii}/n is too small, too large or both. When truncation does not occur irrespective of the value of v_{ii}, $K_i = 1$ with $0 = a_{i1} < b_{i1} = \infty$ is employed though $\mathbf{V}$ is possibly truncated due to the other value of $v_{3-i,3-i}$. The above truncation is similar to stripe truncation for univariate cases (Ogasawara, 2021a). Since a particular size of v_{ii} gives truncation, $\mathbf{V}$ or $\mathbf{u} = (v_{11}, v_{22}, r)^{\mathrm{T}}$ is said to be variance-truncated in this paper.

Suppose that $\mathbf{u}$ is Wishart distributed including change of variable and reparametrization under variance truncation with the non-null set of intervals $\bigcap_{i=1}^{2} \bigcup_{k=1}^{K_i} \{v_{ii} \in I_{ik}\}$ for selection. Then, this truncated distribution is denoted by $\mathbf{u} \sim \mathrm{W}_2(\rho, n; \mathbf{A}, \mathbf{B})$, where $\mathbf{A} = \{a_{ik}\}$ and $\mathbf{B} = \{b_{ik}\}$ are the sets of the lower and upper limits of the intervals for selection defined earlier, respectively.

Theorem 1. *Let* $\mathbf{u} = (v_{11}, v_{22}, r)^{\mathrm{T}} \sim \mathrm{W}_2(\rho, n; \mathbf{A}, \mathbf{B})$ *with* $n > 1$ *and* $|\rho| < 1$. *Then, the marginal pdf of r is*

$$f(r \mid \rho, n; \mathbf{A}, \mathbf{B}) = C_2 \sum_{l=0}^{\infty} \frac{(2\rho r)^l}{l!} (1 - r^2)^{(n-3)/2} \Gamma\left(\frac{n+l}{2}\right)^2 F_{\mathbf{A}(l)}^{\mathbf{B}} \quad \textit{with} \quad \Gamma(\cdot)^2 = \{\Gamma(\cdot)\}^2,$$

where the normalizer is given in two ways:

$$\begin{aligned} 1/C_2 &= \Gamma\left(\frac{n-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \rho^{2k} (k+1)_{(n-2)/2} F_{\mathbf{A}(2k)}^{\mathbf{B}} \\ &= 2^{2-n} \pi \, \Gamma(n-1) \, {}_1F_{0 w_k}(n/2; \, ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}}), \end{aligned}$$

$${}_1F_{0 w_k}(n/2; \, ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}}) \equiv \sum_{k=0}^{\infty} \frac{\rho^{2k} (n/2)_k}{k!} F_{\mathbf{A}(2k)}^{\mathbf{B}},$$

$$0 < F_{\mathbf{A}(l)}^{\mathbf{B}} = \prod_{i=1}^{2} \sum_{j=1}^{K_i} \left[F_{\Gamma}\left\{ \frac{b_{ij}}{2(1-\rho^2)} \,\Big|\, \frac{n+l}{2} \right\} - F_{\Gamma}\left\{ \frac{a_{ij}}{2(1-\rho^2)} \,\Big|\, \frac{n+l}{2} \right\} \right] \le 1,$$

$(a)_k = a(a+1)\cdots(a+k-1)$ *when k is a non-negative integer with* $(a)_0 = 1 \ (a \ne 0)$ *otherwise* $(a)_k = \Gamma(a+k)/\Gamma(a) \ (a > 0, k > 0)$ *is the rising or ascending factorial using* *the Pochhammer symbol; and* $F_\Gamma(x\,|\,\alpha)=\int_0^x t^{\alpha-1}\exp(-t)\,\mathrm{d}t/\Gamma(\alpha)$ *is the regularized gamma function or the distribution function of the gamma with the shape parameter* α *and the unit scale parameter.*

In Theorem 1, ${}_1F_{0w_k}(a;\,;x;w_{(k)})$ is a new weighted generalized hypergeometric series or function, where a weight $w_{(k)}$ in the *k*-th term of the series (k = 0, 1,…) is added to the corresponding usual or unweighted generalized hypergeometric function ${}_1F_0(a;\,;x)=\sum_{k=0}^{\infty}(a)_k x^k/k!$ (see Mathai, 1993, Section 3.2.1; Abadir, 1999, Equation (4)). Note that similar weighted hypergeometric functions

$${}_1F_{1w}(a;c;x;w)=\sum_{k=0}^{\infty}\frac{(a)_k x^k\gamma(w\,|\,a+k)}{(c)_k k!\,\Gamma(a+k)} \quad \text{and}$$

$${}_2F_{1w2}(a,b;c;x;w)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k x^k\gamma(w_1\,|\,a+k)\gamma(w_2\,|\,b+k)}{(c)_k k!\,\Gamma(a+k)\Gamma(b+k)}$$

corresponding to the Kummer confluent and Gauss series are defined by Ogasawara (2021b, Equations (5) and (7)), respectively.

As addressed earlier, the pdf of *r* under untruncated normality is known. The following result includes an alternative expression of the normalizing constant.

Corollary 1. *Let* $(v_{11},v_{22},r)^{\mathrm{T}}\sim \mathrm{W}_2(\rho,n)$ *with* $n>1$, *which indicates the untruncated Wishart. Then, the marginal pdf of r is*

$$f(r\,|\,\rho,n)=\frac{\sum_{l=0}^{\infty}\Gamma\left(\frac{n+l}{2}\right)^2\frac{(2\rho r)^l}{l!}(1-r^2)^{(n-3)/2}}{\Gamma\left(\frac{n-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\rho^{2k}(k+1)_{(n-2)/2}}.$$

Note that Anderson (1958, 2003, Theorem 4.2.2) and Muirhead (1982, Equation (11), Section 5.1.3) employ the following expression corresponding to that of Corollary 1:

$$f(r\,|\,\rho,n)=\frac{2^{n-2}(1-\rho^2)^{n/2}}{\Gamma(n-1)\pi}\sum_{l=0}^{\infty}\Gamma\left(\frac{n+l}{2}\right)^2\frac{(2\rho r)^l}{l!}(1-r^2)^{(n-3)/2}.$$

which is obtained from the original normalizer of the Wishart density $w_p(\mathbf{S}\,|\,\boldsymbol{\Sigma},n)$ shown earlier using the Legendre duplication formula for the bivariate gamma function. While the above expression of the normalizing constant is simpler than that of Corollary 1, the two expressions give the following formula.

Result 1. When $n > 1$,

$$\sum_{k=0}^{\infty}\rho^{2k}(k+1)_{(n-2)/2}=\frac{\Gamma(n-1)\pi^{1/2}}{2^{n-2}(1-\rho^2)^{n/2}}/\Gamma\left(\frac{n-1}{2}\right)=\frac{\Gamma(n/2)}{(1-\rho^2)^{n/2}}.$$

In Result 1, the Legendre duplication formula is used. When n is a positive even number, this equality can be proved without using Corollary 1. That is, when $n = 2$, the series becomes a usual geometric one $\sum_{k=0}^{\infty}\rho^{2k}=1/(1-\rho^2)\ (-1<\rho<1)$ with the definitions $(a)_0=1\ (a\neq 0)$ and $0^0=1$. The cases $n = 4, 6,\ldots$ are given by differentiating the series by ρ^2 successively. When $n>1$ is real-valued including even n, the equality can also be obtained without using Corollary 1. Divide the left-hand side of the above equality by $\Gamma(n/2)$, giving

$$\sum_{k=0}^{\infty}\frac{\rho^{2k}(k+1)_{(n-2)/2}}{\Gamma(n/2)}=\sum_{k=0}^{\infty}\frac{\rho^{2k}\Gamma\{(n+2k)/2\}}{k!\Gamma(n/2)}=\sum_{k=0}^{\infty}\frac{\rho^{2k}(n/2)_k}{k!}$$
$$=1+\left(-\frac{n}{2}\right)(-\rho^2)+\frac{1}{2!}\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)(-\rho^2)^2+\frac{1}{3!}\left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)\left(-\frac{n}{2}-2\right)(-\rho^2)^3+\ldots$$
$$=\sum_{k=0}^{\infty}\binom{-n/2}{k}(-\rho^2)^k=\frac{1}{(1-\rho^2)^{n/2}},$$

which is equal to $\Gamma(n/2)^{-1}$ times the right-hand side of the equality to be derived, where the binomial expansion is used (see Abadir, 1999, Equations (2) and (5)).

Define

$$1/D_2=(1/C_2)\Gamma\left(\frac{n-1}{2}\right)^{-1}\pi^{-1/2}=\Gamma\left(\frac{n}{2}\right){}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})$$
$$=\sum_{k=0}^{\infty}\rho^{2k}(k+1)_{(n-2)/2}F^{\mathbf{B}}_{\mathbf{A}(2k)},$$

${}_2F_{1w_k}(\cdot)$ and ${}_3F_{2w_k}(\cdot)$ similarly to ${}_1F_{0w_k}(\cdot)$. Then, we have the following results.

Lemma 1. *Some moments of r under variance truncation as in Theorem 1 using different series expressions are*

(i) $\mathrm{E}\{r^{2i+1}(1-r^2)^j\mid i,j=0,1,\ldots;\rho,n;\mathbf{A},\mathbf{B}\}$

$$=D_2\{(n-1)/2\}_j\sum_{k=0}^{\infty}\rho^{2k+1}\frac{\{(2k+3)/2\}_i(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{(2i+2j+1)/2}}F^{\mathbf{B}}_{\mathbf{A}(2k+1)}$$

$$= \frac{\{(n-1)/2\}_j (n/2)_{1/2} (3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n+2j-1)/2}} \rho$$

$$\times \frac{{}_3F_{2w_k}\{(n+1)/2, (n+1)/2, (2i+3)/2; (n+2i+2j+2)/2,\ 3/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})},$$

$$\mathrm{E}\{r(1-r^2)^j \mid j = 0,1,...; \rho, n; \mathbf{A}, \mathbf{B}\}$$

$$= D_2\{(n-1)/2\}_j \sum_{k=0}^{\infty} \rho^{2k+1} \frac{(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{(2j+1)/2}} F^{\mathbf{B}}_{\mathbf{A}(2k+1)}$$

$$= \frac{\{(n-1)/2\}_j (n/2)_{1/2} (3/2)_{(n-2)/2}}{(3/2)_{(n+2j-1)/2}} \rho$$

$$\times \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+2j+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

(ii) $\mathrm{E}\{r^{2i}(1-r^2)^j \mid i,\ j = 0,1,...; \rho, n; \mathbf{A}, \mathbf{B}\}$

$$= D_2\{(n-1)/2\}_j \sum_{k=0}^{\infty} \frac{\{(2k+1)/2\}_i (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_{i+j}} F^{\mathbf{B}}_{\mathbf{A}(2k)} \rho^{2k}.$$

$$= \frac{\{(n-1)/2\}_j (1/2)_{(n-1)/2}}{\{(2i+1)/2\}_{(n+2j-1)/2}}$$

$$\times \frac{{}_3F_{2w_k}\{n/2, n/2, (2i+1)/2; (n+2i+2j)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})},$$

$$\mathrm{E}\{(1-r^2)^j \mid j = 0,1,...; \rho, n; \mathbf{A}, \mathbf{B}\}$$

$$= D_2\{(n-1)/2\}_j \sum_{k=0}^{\infty} \rho^{2k} \frac{(k+1)_{(n-2)/2}}{\{(n+2k)/2\}_j} F^{\mathbf{B}}_{\mathbf{A}(2k)}$$

$$= \frac{\{(n-1)/2\}_{1/2}}{\{(n+2j-1)/2\}_{1/2}} \frac{{}_2F_{1w_k}\{n/2, n/2; (n+2j)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

Remark 1. It is found that when hypergeometric functions are used, Lemma 1 gives two expressions of moments i.e., non-recursive and recursive, where the former includes ${}_3F_{2w_k}(\cdot)$ while the latter ${}_2F_{1w_k}(\cdot)$. The expressions of the moments of odd powers using the non-recursive method (non-recursive moments for short) are given by Lemma 1 (i):

$$\mathrm{E}\{r^{2k+1} = r^{2k+1}(1-r^2)^j \mid k = 1,2,...; j = 0; \rho, n; \mathbf{A}, \mathbf{B}\}$$

while the corresponding recursive moments are

$$\mathrm{E}(r^{2k+1} \mid \rho, n; \mathbf{A}, \mathbf{B}) \ \ (k = 1, 2, ...)$$
$$= (-1)^k \left[\mathrm{E}\{r(1-r^2)^k \mid \rho, n; \mathbf{A}, \mathbf{B}\} - \sum_{i=0}^{k-1} \binom{k}{i} (-1)^i \mathrm{E}(r^{2i+1} \mid \rho, n; \mathbf{A}, \mathbf{B}) \right].$$

On the other hand, the corresponding non-recursive moments of even powers are given by Lemma 1 (ii):

$$\mathrm{E}\{r^{2k} = r^{2k}(1-r^2)^j \mid k = 1, 2, ...; j = 0; \rho, n; \mathbf{A}, \mathbf{B}\}$$

while the recursive moments are

$$\mathrm{E}(r^{2k} \mid \rho, n; \mathbf{A}, \mathbf{B}) \ \ (k = 1, 2, ...)$$
$$= (-1)^k \left[\mathrm{E}\{(1-r^2)^k \mid \rho, n; \mathbf{A}, \mathbf{B}\} - \sum_{i=0}^{k-1} \binom{k}{i} (-1)^i \mathrm{E}(r^{2i} \mid \rho, n; \mathbf{A}, \mathbf{B}) \right].$$

Theorem 2. *The raw moments of r up to the fourth order under variance truncation as in Theorem 1 using different series expressions are*

$$\mathrm{E}(r \mid \rho, n; \mathbf{A}, \mathbf{B}) = D_2 \sum_{k=0}^{\infty} \rho^{2k+1} \frac{(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}}$$
$$= \frac{2}{n} (n/2)_{1/2}^2 \rho \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+2)/2;\ \rho^2;\ F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})},$$

$$\mathrm{E}(r^2 \mid \rho, n; \mathbf{A}, \mathbf{B}) = D_2 \sum_{k=0}^{\infty} \frac{\rho^{2k}(2k+1)(k+1)_{(n-2)/2}}{n+2k} F_{\mathbf{A}(2k)}^{\mathbf{B}}$$
$$= n^{-1} \frac{{}_3F_{2w_k}\{n/2,\ n/2, 3/2;\ (n+2)/2,\ 1/2;\ \rho^2;\ F_{\mathbf{A}(2k)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})}$$
$$= 1 - (1-n^{-1}) \frac{{}_2F_{1w_k}\{n/2,\ n/2;\ (n+2)/2;\ \rho^2;\ F_{\mathbf{A}(2k)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})},$$

$$\mathrm{E}(r^3 \mid \rho, n; \mathbf{A}, \mathbf{B}) = D_2 \sum_{k=0}^{\infty} \rho^{2k+1} \frac{\{(2k+3)/2\}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{3/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}}$$
$$= \frac{(n/2)_{1/2}(3/2)_{(n-2)/2}}{(5/2)_{(n-1)/2}} \rho \frac{{}_3F_{2w_k}\{(n+1)/2, (n+1)/2, 5/2; (n+4)/2,\ 3/2;\ \rho^2;\ F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})},$$
$$= \mathrm{E}(r \mid \rho, n; \mathbf{A}, \mathbf{B})$$
$$- \frac{\{(n-1)/2\}(n/2)_{1/2}(3/2)_{(n-2)/2}}{(3/2)_{(n+1)/2}} \rho \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+4)/2;\ \rho^2;\ F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})}$$

and

$$\mathrm{E}(r^4 \mid \rho, n; \mathbf{A}, \mathbf{B}) = D_2 \sum_{k=0}^{\infty} \frac{\rho^{2k} \{(2k+1)/2\}_2 (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_2} F_{\mathbf{A}(2k)}^{\mathbf{B}}$$

$$= \frac{3}{(n+2)n} \frac{{}_3F_{2w_k}\{n/2, n/2, 5/2; (n+4)/2,\ 1/2;\ \rho^2;\ F_{\mathbf{A}(2k)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})}$$

$$= -1 + 2\mathrm{E}(r^2 \mid \rho, n; \mathbf{A}, \mathbf{B}) + \frac{(n+1)(n-1)}{(n+2)n} \frac{{}_2F_{1w_k}\{n/2, n/2; (n+4)/2;\ \rho^2;\ F_{\mathbf{A}(2k)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})}.$$

Result 2. The skewness (sk) and excess kurtosis (kt) of r under variance truncation as in Theorem 1 are given by Theorem 2 in various ways. A set of simple expressions is shown as

$$\mathrm{sk}(r \mid \rho, n; \mathbf{A}, \mathbf{B}) = \left[D_2 \sum_{k=0}^{\infty} \rho^{2k+1} \frac{\{(2k+3)/2\}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{3/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right.$$

$$- 3D_2^2 \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k}(2k+1)(k+1)_{(n-2)/2}}{n+2k} F_{\mathbf{A}(2k)}^{\mathbf{B}} \right\} \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}$$

$$\left. + 2D_2^3 \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}^3 \right] \{\mathrm{var}(r \mid \rho, n; \mathbf{A}, \mathbf{B})\}^{-3/2}$$

and

$$\mathrm{kt}(r \mid \rho, n; \mathbf{A}, \mathbf{B}) = \left[D_2 \sum_{k=0}^{\infty} \frac{\rho^{2k}\{(2k+1)/2\}_2 (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_2} F_{\mathbf{A}(2k)}^{\mathbf{B}} \right.$$

$$- 4D_2^2 \left\{ \sum_{k=0}^{\infty} \rho^{2k+1} \frac{\{(2k+3)/2\}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{3/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\} \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}$$

$$+ 6D_2^3 \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k}(2k+1)(k+1)_{(n-2)/2}}{n+2k} F_{\mathbf{A}(2k)}^{\mathbf{B}} \right\} \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}^2$$

$$\left. - 3D_2^4 \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}^4 \right] \{\mathrm{var}(r \mid \rho, n; \mathbf{A}, \mathbf{B})\}^{-2} - 3,$$

where

$$\mathrm{var}(r \mid \rho, n; \mathbf{A}, \mathbf{B})$$

$$= D_2 \sum_{k=0}^{\infty} \frac{\rho^{2k}(2k+1)(k+1)_{(n-2)/2}}{n+2k} F_{\mathbf{A}(2k)}^{\mathbf{B}} - D_2^2 \left\{ \sum_{k=0}^{\infty} \frac{\rho^{2k+1}(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{1/2}} F_{\mathbf{A}(2k+1)}^{\mathbf{B}} \right\}^2.$$

In the untruncated case, noting that $F_{\mathbf{A}(\cdot)}^{\mathbf{B}} = 1$ and

$${}_1F_{0w_k}(n/2;\ ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}}) = {}_1F_0(n/2;\ ; \rho^2) = \sum_{k=0}^{\infty} \frac{\rho^{2k}(n/2)_k}{k!} = \frac{1}{(1-\rho^2)^{n/2}},$$

as shown after Result 1, the mean using the hypergeometric functions becomes

$$\mathrm{E}(r\,|\,\rho,n) = \frac{2}{n}(n/2)_{1/2}^{2}\,\rho\,\frac{{}_2F_1\{(n+1)/2,(n+1)/2;(n+2)/2;\ \rho^2\}}{{}_1F_0(n/2;\,;\rho^2)}$$
$$= \frac{2}{n}(n/2)_{1/2}^{2}\,\rho(1-\rho^2)^{n/2}\,{}_2F_1\{(n+1)/2,(n+1)/2;(n+2)/2;\ \rho^2\}$$
$$= \frac{2}{n}(n/2)_{1/2}^{2}\,\rho\,{}_2F_1\{1/2,1/2;(n+2)/2;\ \rho^2\},$$

where the last result is given by the Euler transform

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;x)$$

(Erdélyi, 1953, Section 2.1.4, Equation (22); Abramowitz & Stegun, 1972, Equation 15.3.3).

Similarly, the untruncated variance using $\mathrm{E}(1-r^2\,|\,\rho,n)$ with the hypergeometric functions is given by

$$\mathrm{var}(r\,|\,\rho,n) = D_2\sum_{k=0}^{\infty}\frac{\rho^{2k}(k+1)_{(n-2)/2}}{(n+2k)/2} - \{\mathrm{E}(r\,|\,\rho,n)\}^2$$
$$= 1-(1-n^{-1})(1-\rho^2)^{n/2}\,{}_2F_1\{n/2,\ n/2;\ (n+2)/2;\ \rho^2\} - \{\mathrm{E}(r\,|\,\rho,n)\}^2$$
$$= 1-(1-n^{-1})(1-\rho^2)\,{}_2F_1\{1,\ 1;\ (n+2)/2;\ \rho^2\} - \{\mathrm{E}(r\,|\,\rho,n)\}^2,$$

which is simpler than the corresponding expression using $\mathrm{E}(r^2\,|\,\rho,n)$ in that the usual Gauss hypergeometric function ${}_2F_1\{\cdot\}$ can be used rather than the generalized higher-order function ${}_3F_2\{\cdot\}$.

The expressions of truncated sk and kt are given without using hypergeometric functions for simplicity. The corresponding untruncated results are given by substituting $F_{\mathbf{A}(\cdot)}^{\mathbf{B}} = 1$ for truncated ones. The untruncated raw moments up to the second order using the Gauss hypergeometric functions were derived by Kendall (1948) and up to the fourth order by Ghosh (1966) as well as Soper et al. (1917, Section (3)) with somewhat different expressions. It is found that the untruncated expectation and variance shown above are equal to the corresponding results given by Kendall (1948, Equations (14.55) & (14.56)), Ghosh (1966, Equation (1)) and Muirhead (1982, p. 155, the last equation; p. 156, Equation (20)).

Further, in Lemma 1, the odd and even moments without truncation using hypergeometric functions become

$$\mathrm{E}(r^{2i+1} \mid i=0,1,...;\rho,n) = \frac{(n/2)_{1/2}(3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n-1)/2}}\rho(1-\rho^2)^{n/2}$$
$$\times {}_3F_2\{(n+1)/2,(n+1)/2,(2i+3)/2;(n+2i+2)/2,\ 3/2;\ \rho^2\}$$

and

$$\mathrm{E}(r^{2i} \mid i=1,2,...;\rho,n)$$
$$= \frac{(1/2)_{(n-1)/2}}{\{(2i+1)/2\}_{(n-1)/2}}(1-\rho^2)^{n/2}\,{}_3F_2\{n/2,\ n/2,(2i+1)/2;\ (n+2i)/2,\ 1/2;\ \rho^2\},$$

which are found to be algebraically equal to the known ones (Romero-Padilla, 2016, Corollary 1, Equations (16) and (15)), respectively. The last expression of $\mathrm{E}(r^3 \mid \rho,n;\mathbf{A},\mathbf{B})$ in Theorem 2 using ${}_2F_1\{\cdot\}$ when without truncation becomes

$$\mathrm{E}(r^3 \mid \rho,n)$$
$$=\mathrm{E}(r \mid \rho,n) - \frac{\{(n-1)/2\}(n/2)_{1/2}(3/2)_{(n-2)/2}}{(3/2)_{(n+1)/2}}\rho\frac{{}_2F_1\{(n+1)/2,(n+1)/2;(n+4)/2;\ \rho^2\}}{{}_1F_0(n/2;\ ;\rho^2)}$$
$$= \mathrm{E}(r \mid \rho,n) - \frac{2(n-1)(n/2)_{1/2}^2}{n(n+2)}\rho(1-\rho^2)\,{}_2F_1\{3/2,3/2;(n+4)/2;\ \rho^2\},$$

which is found to be algebraically equal to Romero-Padilla (2016, Equation (22)), who obtained this result by re-expressing Soper et al. (1917, Equation (xxvi)) without using the recursive formula in Lemma 1. The above expression $\mathrm{E}(r^3 \mid \rho,n)$ is different from the corresponding one using ${}_2F_1\{\cdot\}$ by Ghosh (1966, Equation (1)). Romero-Padilla (2016, p.20) stated that Ghosh's formula for $\mathrm{E}(r^3 \mid \rho,n)$ is incorrect. The author also found that the numerical values of $\mathrm{E}(r^3 \mid \rho,n)$ using Ghosh's formula in the numerical examples given later are much smaller than those obtained by the formulas of Theorem 2, which are algebraically equal to Romero-Padilla (2016, Equation (22)).

The last expression of $\mathrm{E}(r^4 \mid \rho,n;\mathbf{A},\mathbf{B})$ in Theorem 2 using ${}_2F_1\{\cdot\}$ when without truncation becomes

$$\mathrm{E}(r^4 \mid \rho,n) = -1 + 2\mathrm{E}(r^2 \mid \rho,n) + \frac{(n+1)(n-1)}{(n+2)n}\frac{{}_2F_1\{n/2,n/2;(n+4)/2;\ \rho^2\}}{{}_1F_0(n/2;\ ;\rho^2)}$$
$$= 1 - 2(1-n^{-1})(1-\rho^2)\,{}_2F_1\{1,\ 1;\ (n+2)/2;\ \rho^2\}$$
$$+ \frac{(n+1)(n-1)}{(n+2)n}(1-\rho^2)^2\,{}_2F_1\{2,\ 2;(n+4)/2;\ \rho^2\}.$$

Though this formula is different from the corresponding expression of Ghosh (1966, Equation (1)), the two expressions give the same numerical values, which are also similar to the corresponding simulated values, suggesting the algebraic equality.

These findings partially support our results under variance truncation.

Remark 2. Note that the different expressions of the moments using hypergeometric functions in Lemma 1 give some relationships between weighted ${}_2F_1\{\cdot\}$ and ${}_3F_2\{\cdot\}$. When without truncation, the last expression in Lemma 1 (ii) using hypergeometric functions becomes

$$\begin{aligned}
&\mathrm{E}\{(1-r^2)^j \mid j=1,2,...;\rho,n\} \\
&= \frac{\{(n-1)/2\}_{1/2}}{\{(n+2j-1)/2\}_{1/2}}(1-\rho^2)^{n/2}\,{}_2F_1\{n/2,\ n/2;\ (n+2j)/2;\ \rho^2\} \\
&= \frac{\{(n-1)/2\}_{1/2}}{\{(n+2j-1)/2\}_{1/2}}(1-\rho^2)^{j}\,{}_2F_1\{j,\ j;\ (n+2j)/2;\ \rho^2\},
\end{aligned}$$

which was used by Muirhead (1982, Section 5.1.3, Equation (19)), when j = 1, to have $\mathrm{E}(r^2 \mid \rho,n)$ and possibly by Ghosh (1966, Equation (1)). Though the last results are easily derived, they give some relationships between unweighted ${}_3F_2\{\cdot\}$ and ${}_2F_1\{\cdot\}$ when they are re-expressed by $\mathrm{E}(r^{2j} \mid j=1,2,...;\rho,n)$ as in Remark 1. For instance, when j = 1, we have

$$\begin{aligned}
&n^{-1}(1-\rho^2)^{n/2}\,{}_3F_2\{n/2,\ n/2, 3/2;\ (n+2)/2,\ 1/2;\ \rho^2\} \\
&= 1-(1-n^{-1})(1-\rho^2)\,{}_2F_1\{1,\ 1;\ (n+2)/2;\ \rho^2\}\ \ (n>1).
\end{aligned}$$

Similar relations are also given by comparing $\mathrm{E}(r^{2j+1} \mid j=1,2,...;\rho,n)$ and its recursive expression $\mathrm{E}\{r(1-r^2)^j \mid j=1,...;\rho,n\}$ in Lemma 1 (ii) when without truncation.

3. Numerical illustrations

In this section, numerical illustrations of some moments of r under variance-truncated bivariate normality are shown with simulated moments. Since the moments derived in the previous section are exact ones, the simulations may be unnecessary as long as the formulas are correct. However, the algebraic expressions include infinite series, which are approximated by finite ones in actual computation. So, the corresponding simulated values are given. Table 1 shows a summary of 11 examples under various types of variance truncation. Example (Ex.) 1 is the untruncated case included for comparison. Ex. 2 to 4 are under lower-truncation for v_{11} and 3-way truncation for v_{22}. Ex. 5 to 7 are under double truncation for v_{11} and 3-way for v_{22}. Ex. 8 to 11 are under inner-truncation for v_{11} and 4-way for v_{22}. For the intervals of selection under lower- and upper-truncation, $[\mathrm{CL}, \mathrm{Inf})$ and $[0, \mathrm{CR})$ are used for illustration, respectively, where

$$\mathrm{CL} = \mathrm{E}(v_{ii}) - \sqrt{\mathrm{var}(v_{ii})} = n - \sqrt{2n}\ , \ \ \mathrm{CR} = \mathrm{E}(v_{ii}) + \sqrt{\mathrm{var}(v_{ii})} = n + \sqrt{2n} \ \ (i = 1, 2)$$

and $\mathrm{Inf} = \infty$. These selection (truncation) points are also used under double and inner truncation.

Tables 2 and 3 give the simulated and theoretical (exact) moments/cumulants up to the fourth order. The theoretical values are given by the finite series when the relative size of an added term is smaller than or equal to a predetermined value denoted by 'eps'. The zero value of eps corresponding to the highest machine precision is used in the numerical illustrations, which is computationally attained in the series under the default double precision of the R-language (R Core Team, 2020). For the series expressions, both of modified (not including ${}_pF_{qw_k}(\cdot)$) and standard (including ${}_2F_{1w_k}(\cdot)$ or ${}_3F_{2w_k}(\cdot)$) hypergeometric functions (see Theorem 2) are used for comparison. The results showed reasonably close values. The truncated probability $\alpha = {}_1F_{0w_k}(n/2;\, ;\rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})(1-\rho^2)^{n/2}$ in the tables is the relative size of the reciprocal of the normalizer C_2 over the untruncated counterpart. Note that when both of v_{11} and v_{22} are upper-truncated, α gives the cumulative distribution function (cdf) for the marginal Wishart distribution of v_{11} and v_{22}. The asymptotic values of α are given by the asymptotic normal distributions of v_{11} and v_{22} with their exact means, variances shown earlier and $\mathrm{cov}(v_{11}, v_{22}) = 2n\rho^2$.

Simulations are carried out by randomly generating 10^5 sample covariance matrices under bivariate untruncated normality using the R-package 'mvtnorm' (Genz et al., 2020) when the population value is $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, where $\rho = 0.3$ (Table 2) and 0.7 (Table 3) are used each with n = 20 and 50. While all the 10^5 sample covariance matrices are used in Ex. 1, some of them are used for the remaining examples selecting the matrices satisfying both of the selection conditions for v_{11} and v_{22}. From the selected sample covariance matrices, simulated moments/cumulants and α are obtained.

Tables 2 and 3 show that the theoretical/simulated moments and α are close to each other indicating the accuracy of the computationally obtained values of the exact formulas. It is also found that the asymptotic value of α is reasonably close to the corresponding simulated and theoretical values, which may primarily be due to the exact means and covariance matrix of v_{11} and v_{22}. All the values of the means in the tables are found to be close to ρ irrespective of truncation in various ways. While it is known that in the untruncated case, the asymptotic bias of *r* is of order $O(n^{-1}) = -\rho(1-\rho^2)n^{-1}$ (Ghosh, 1966, Equation (3); Muirhead, 1982, Section 5.1.3, Equation (21)), the tables suggest similar results for *r* under variance truncation. However, Table 2 also shows that the theoretical (simulated) bias of *r* in Ex. 6 is .0042 (.0039) when *n* = 20 while .0045 (.0044) when *n* = 50, which are larger than those when *n* = 20 although their absolute values are smaller than or comparable to the values of untruncated Ex.1 i.e., -.0068 (-.0073) when *n* = 20 and -.0027 (-.0030) when *n* = 50. Note that the corresponding asymptotic values obtained from the formula shown above give -.0137 when *n* = 20 and -.0055 when *n* = 50, which are twice as large as those of the exact values. These findings suggest the non-negligible higher-order asymptotic bias.

The values of SD in the tables show the influence of variance truncation. That is, the SDs under lower (Ex. 2 to 4) or double (Ex. 5 to 7) truncation tend to be smaller than those under untruncated (Ex. 1) or inner-truncated (Ex. 8 to 11) cases. The SDs of both lower truncation for v_{11} and v_{22} (Ex. 3) are smaller than those of lower and upper truncation for v_{11} and v_{22} (Ex. 4), respectively, indicating that when sample variances are large, the corresponding *r* tends to be stable with small SD's. The values of sk are negative with the relatively large absolute values when $\rho = 0.7$ and/or *n* =20. The values of kt when $\rho = 0.3$ are approximately zero while they are mostly positive when $\rho = 0.7$. It is of interest to see that the largest kt when $\rho = 0.7$ is found in untruncated Ex. 1 and the smallest kt in Ex. 11 under both inner-truncation.

4. The multivariate null distribution

In this section, the null distribution in the p-variate case is considered. Let $r_{ij} = v_{ij} / \sqrt{v_{ii}v_{jj}}$, $\mathbf{R} = \{r_{ij}\}$ and $\rho_{ij} = \sigma_{ij} / \sqrt{\sigma_{ii}\sigma_{jj}}$ $(i, j = 1,..., p)$. Define I_{ik} $(i = 1,..., p; k = 1,..., K_i)$ with **A** and **B** as in the bivariate case with the assumption $\sigma_{ii} = 1$ $(i = 1,..., p)$. For the p-variate case, let $\mathbf{u} = (v_{11},..., v_{pp}, \mathbf{r}^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{r} = (r_{12}, r_{13},..., r_{p,p-1})^{\mathrm{T}}$, $\mathbf{\Sigma} = \mathbf{P} = \{\rho_{ij}\} = \mathbf{I}_p$ and $\mathbf{I}_p$ be the $p \times p$ identity matrix.

Lemma 2. *Suppose that* $v_{11},..., v_{pp}$ *are truncated as defined above. Employ the change of variables* $\mathbf{V} = \{v_{ij}\}$ *with* $v_{ij} = \sqrt{v_{ii}v_{jj}}\, r_{ij}$, *where* $v_{11},..., v_{pp}$ *are unchanged. Then, the pdf of the joint distribution of* $\mathbf{u} = (v_{11},..., v_{pp}, \mathbf{r}^{\mathrm{T}})^{\mathrm{T}}$ *is*

$$w_p(\mathbf{u} \mid \mathbf{\Sigma} = \mathbf{I}_p, n; \mathbf{A}, \mathbf{B}) = w_p(\mathbf{u} \mid n; \mathbf{A}, \mathbf{B})$$
$$= \frac{\exp\{-(v_{11} + ... + v_{pp}) / 2\} \mid \mathbf{R} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(n-2)/2}}{2^{np/2} \Gamma_p(n/2) F_{\mathbf{A}}^{\mathbf{B}}},$$

where

$$0 < F_{\mathbf{A}}^{\mathbf{B}} \equiv F_{\mathbf{A}(0)}^{\mathbf{B}} = \prod_{i=1}^{p} \sum_{j=1}^{K_i} \left\{ F_\Gamma\left(\frac{b_{ij}}{2} \mid \frac{n}{2}\right) - F_\Gamma\left(\frac{a_{ij}}{2} \mid \frac{n}{2}\right) \right\} \le 1.$$

Theorem 3. *Suppose that* $\mathbf{u} \sim \mathrm{W}_p(\mathbf{\Sigma} = \mathbf{I}_p, n; \mathbf{A}, \mathbf{B}) = \mathrm{W}_p(n; \mathbf{A}, \mathbf{B})$. *The joint pdf of* **r** *is*

$$f(\mathbf{r} \mid n; \mathbf{A}, \mathbf{B}) = f(\mathbf{r} \mid n) = \frac{\Gamma(n/2)^p \mid \mathbf{R} \mid^{(n-p-1)/2}}{\Gamma_p(n/2)},$$

which also holds under the elliptical distribution.

The result $f(\mathbf{r} \mid n)$ without truncation is known (Muirhead, 1982, Theorem 5.1.3).

Result 3. Theorem 3 gives the following formula:

$$\int_{\mathbf{R}>0} \mid \mathbf{R} \mid^{(n-p-1)/2} \mathrm{d}\mathbf{R} = \int_{\mathbf{R}>0} \mid \mathbf{R} \mid^{(n-p-1)/2} \mathrm{d}\mathbf{r} = \Gamma_p(n/2) / \Gamma(n/2)^p \;\; (n > p-1),$$

which is said to be the standardized multivariate gamma function in this paper due to $r_{ii} = 1 (i = 1,..., p)$.

The equation in Result 3 should mathematically hold irrespective of truncation. Though we consider the cases of $p \ge 2$, **R** becomes 1 when $p = 1$ and the above integral is defined to be 1, if necessary, which is consistent with the unit value of the right-hand side of the above equation. The above equality gives another integral expression of the multivariate gamma function

$$\Gamma_p(n/2) = \Gamma(n/2)^p \int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2} \mathrm{d}\mathbf{r}\ .$$

While it is known that

$$\Gamma_p(n/2) = \int_{\mathbf{V}>0} \exp\{-\mathrm{tr}(\mathbf{V})\} |\mathbf{V}|^{(n-p-1)/2} \mathrm{d}\mathbf{V} \quad (n > p-1)\ ,$$

which holds for real-valued $n > p-1$ (Anderson, 2003, Corollary 7.2.4), the alternative integral expression given by Result 3 is simpler than or comparable to the above one. Result 3 can also be derived as follows. Noting that $\mathrm{d}\mathbf{V}$ is interpreted as $\mathrm{d}\{\mathrm{v}(\mathbf{V})\}$ and recalling the Jacobian $\mathrm{d}\{\mathrm{v}(\mathbf{V})\} = (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$ in the proof of Lemma 2, we have

$$\begin{aligned}
\Gamma_p(n/2) &= \int_{\mathbf{V}>0} \exp\{-\mathrm{tr}(\mathbf{V})\} |\mathbf{V}|^{(n-p-1)/2} \mathrm{d}\mathbf{V} \\
&= \int_{\mathbf{V}>0} \exp\{-\mathrm{tr}(\mathbf{V})\} |\mathbf{V}|^{(n-p-1)/2} \mathrm{d}\{\mathrm{v}(\mathbf{V})\} \\
&= \int_{\mathbf{V}>0} \exp\{-(v_{11} + \ldots + v_{pp})\} |\mathbf{R}|^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(n-p-1+p-1)/2} \mathrm{d}\mathbf{u} \\
&= \int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2}\ \mathrm{d}\mathbf{r} \prod\nolimits_{i=1}^{p} \int_0^\infty \exp(-v_{ii}) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii} \\
&= \Gamma(n/2)^p \int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2}\ \mathrm{d}\mathbf{r} \\
&= \Gamma(n/2)^p \int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2}\ \mathrm{d}\mathbf{R}.
\end{aligned}$$

The above derivation extends integer $n \ge p$ to real-valued $n > p-1$. When the range of $v_{11},\ldots,v_{pp}$ is restricted as in Theorem 3, the above derivation also gives the following.

Result 4. When the support of $v_{11},\ldots,v_{pp}$ is constrained to $\bigcap_{i=1}^{p} \bigcup_{k=1}^{K_i} \{v_{ii} \in I_{ik}\}$, the multivariate gamma function defined under truncation becomes

$$\begin{aligned}
\Gamma_p(n/2 \mid \mathbf{A},\mathbf{B}) &= F_{\mathbf{A}}^{\mathbf{B}} \Gamma_p(n/2) \\
&= F_{\mathbf{A}}^{\mathbf{B}} \int_{\mathbf{V}>0} \exp\{-\mathrm{tr}(\mathbf{V})\} |\mathbf{V}|^{(n-p-1)/2} \mathrm{d}\mathbf{V}\ , \\
&= F_{\mathbf{A}}^{\mathbf{B}} \Gamma(n/2)^p \int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2} \mathrm{d}\mathbf{R}.
\end{aligned}$$

When $p = 2$ in Results 3 and 4, the standardized bivariate gamma function becomes

$$\int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2} \mathrm{d}\mathbf{R} = \int_{-1}^{1} (1-r^2)^{(n-3)/2} \mathrm{d}r$$
$$= \int_0^1 r^{2(-1/2)} (1-r^2)^{(n-3)/2} \mathrm{d}r^2 = \mathrm{B}\left(\frac{1}{2}, \frac{n-1}{2}\right),$$

which yields the pdf of $r = r_{12}$ under normality:

$$f(r \mid n, \mathbf{A}, \mathbf{B}) = f(r \mid n) = (1-r^2)^{(n-3)/2} / \mathrm{B}\left(\frac{1}{2}, \frac{n-1}{2}\right)$$

without using the *t*-distribution with $n-1$ degrees of freedom as mentioned in the introductory section. The normalizer of the above pdf is also obtained from Result 3 using the bivariate gamma function as

$$\int_{\mathbf{R}>0} |\mathbf{R}|^{(n-p-1)/2} \mathrm{d}\mathbf{R} = \Gamma_2(n/2) / \Gamma(n/2)^2$$
$$= \pi^{1/2} \Gamma\left(\frac{n-1}{2}\right) / \Gamma\left(\frac{n}{2}\right) = \mathrm{B}\left(\frac{1}{2}, \frac{n-1}{2}\right).$$

Since the pdf of $\mathbf{r}$ is unchanged after variance truncation, we have the moments of $|\mathbf{R}|$ or the scatter coefficient.

Corollary 2. *Under the same condition as in Theorem 3, we have*

$$\mathrm{E}(|\mathbf{R}|^k \mid n, \mathbf{A}, \mathbf{B}) = \mathrm{E}(|\mathbf{R}|^k \mid n) = \frac{\Gamma(n/2)^p \Gamma_p\{(n+2k)/2\}}{\Gamma\{(n+2k)/2\}^p \Gamma_p(n/2)} \ (k > 0).$$

In Corollary 2, $\mathrm{E}(|\mathbf{R}|^k \mid n)$ is known (Muirhead, 1982, Equation (9), Section 5.1.2).

5. Discussions

(a) The elliptical distribution: In the multivariate null distribution of **R**, it was shown that the pdf of **R** under normality without truncation is robust under the elliptical distribution with or without variance truncation. However, the robustness does not hold in the non-null distribution with variance truncation although it is known that the non-null distribution of **R** without truncation is the same under the corresponding elliptical distribution as noted in the proof of Theorem 3. The non-robust property under variance truncation is due to the different distributions of sample variances (and covariances) under elliptical distributions with distinct fourth cumulants. That is, observations are variance-truncated in different ways depending on the fourth cumulants even if the same sets of the lower and upper limits of the intervals for selection i.e., **A** and **B** are used. For instance, when observations are multivariate *t*-distributed (see e.g., Kotz & Nadarajah, Equation (1.1)) with single or double tail variance-truncation, observations are truncated more often than under normality since the variance of a sample variance is larger than the normal counterpart. Note that when inner truncation is used in this case, observations are truncated less often than under normality.

(b) The variance ratio: It is known that as in the case of **R**, the pdf of the ratio of sample variances in the bivariate elliptical distributions is the same when the scale matrix is the same (Joarder, 2013, Theorem 5.1). Then, if the variance ratio is used for truncation, it is expected that the pdf of **R** under normality holds in the corresponding elliptical distribution. Since the variance ratio as well as the variances can be used e.g., in quality control (Omar, Joarder & Riaz, 2015), the distribution of **R** with variance or variance-ratio truncation may be useful in practice.

(c) The Euler transform: Recall that in the bivariate untruncated moments of *r* shown in Section 2, the Euler transform gave simplified results including the Gauss hypergeometric functions. It is tempting to use this formula in the weighted Gauss series. However, it seems that generally

$$ {}_2F_{1w_k}(a,b;\ c;\ x; w_{(k)}) \neq (1-x)^{c-a-b}\ {}_2F_{1w_k}(c-a, c-b;\ c;\ x; w_{(k)}) \,. $$

There are several derivations for the Euler transform ${}_2F_1(a,b;\ c;\ x)$ $= (1-x)^{c-a-b}\ {}_2F_1(c-a, c-b;\ c;\ x)$ (Erdélyi, 1953; Section 2.1.4, Equations (21) & (23); Rainville, 1960, Chapter 4, Theorem 21). Among them Erdélyi's derivation is to use the Euler integral:

$$ {}_2F_1(a,b;\ c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,\mathrm{d}t \,, $$

which is a special case of the Euler integral formula for the higher-order generalized hypergeometric functions (see Slater, 1966, Section 4.1; DLMF, 2021, Equation 16.5.2, https://dlmf.nist.gov/16.5.E2) and can be extended to the corresponding weighted functions:

$$ {}_{p+1}F_{q+1,w_k}(a_1,...,a_p,a_0;b_1,...,b_q,b_0;x;w_{(k)}) \equiv \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k (a_0)_k x^k w_{(k)}}{(b_1)_k \cdots (b_q)_k (b_0)_k k!} $$
$$ = \frac{\Gamma(b_0)}{\Gamma(a_0)\Gamma(b_0-a_0)} \int_0^1 t^{a_0-1}(1-t)^{b_0-a_0-1} {}_pF_{qw_k}(a_1,...,a_p;b_1,...,b_q;\ tx;\ w_{(k)})\mathrm{d}t $$
$$ (p,q=0,1,...;\ 0 \le q \le p+1;\ 0 < a_0 < b_0;\ 0 \le |x| < 1;\ 0 < w_{(k)} \le 1), $$

which is new and can be derived by writing ${}_pF_{qw_k}(\cdot)$ in the integrand as an infinite series followed by term by term beta integration as in the usual unweighted case. While the above Euler integral holds in the weighted generalized hypergeometric functions, the Euler transform for the Gauss series uses the closed formula ${}_1F_0(a;\ ;tx) = (1-tx)^{-a}$ when $p = 1$ and $q = 0$ with the change of variable $s = (1-t)/(1-tx)$ in e.g., Erdélyi's derivation. However, in the weighed case of ${}_1F_{0w_k}(a;\ ;x;\ w_{(k)})$ it is difficult to obtain the corresponding closed expression or similar ones. This yields the difficulty of obtaining the simplification of the weighted Gauss series.

(d) The asymptotically constant truncated probability: In the numerical illustrations, the truncated probability $\alpha = {}_1F_{0w_k}(n/2;\ ;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})(1-\rho^2)^{n/2}$ (the reduced probability due to truncation) with its simulated and asymptotic values is shown, where the asymptotic value is shown since it is easily obtained by the cdf of the asymptotic bivariate normal distribution for v_{ii} or $s_{ii} = v_{ii}/n\ (i=1,2)$ though some numerical computation is generally required for the cdf as for hypergeometric functions. The asymptotic α's in Tables 2 and 3 are the same irrespective of the size of *n*, which is due to the construction of the selection (truncation) points $\mathrm{CL} = n - \sqrt{2n} = \mathrm{E}(v_{ii}) - \sqrt{\mathrm{var}(v_{ii})}$ and $\mathrm{CR} = n + \sqrt{2n} = \mathrm{E}(v_{ii}) + \sqrt{\mathrm{var}(v_{ii})}\ \ (i=1,2)$. These values are employed for ease of comparison among different *n*'s.

Note that the fixed selection points without using *n* can be well used for the exact moments and α. For convenience, consider the sample variances $s_{ii}\ (i=1,2)$. When $\sigma_{ii}=1$, it holds that $\mathrm{E}(s_{ii})=1\ (i=1,2)$ under arbitrary distributions as long as they exist. Let the upper (right) selection point for s_{ii} be given as $\mathrm{CR}=1+c\ (0<c<\infty)$ under single truncation. When *n* goes to infinity, we have the limiting untruncated case with $\alpha = 1$ since $\mathrm{CR} = 1+c = \mathrm{E}(s_{ii}) + c\sqrt{n/2}\sqrt{\mathrm{var}(s_{ii})}$ and $\Pr(s_{ii} \ge \mathrm{CR}) \le 2c^{-2}n^{-1}$ by the Chebyshev inequality.

In Tables 2 and 3, the exact α slightly varies over different n's. However, it can be shown that when n goes to infinity, the limiting exact α becomes equal to the asymptotic α. The α when using selection (truncation) points satisfying this property (the asymptotically constant truncated probability) is important when we consider the asymptotic moments of r (see the next subsection (e)) since the asymptotic moments under truncation e.g., in the above case with $\mathrm{CR} = 1 + c$ reduces to the usual asymptotic moments without truncation.

(e) The small bias: In the section for numerical illustrations, 11 examples are shown. These cases can also be seen as distinct estimators (trimmed or truncated estimators) of ρ using truncated sample variances. Though it is not the purpose of this paper to propose the best truncated estimator, it is of interest to compare the 11 estimators in terms of the total error index e.g., the (root) mean square error (RMSE). As Tables 2 and 3 show, the biases are small, the MSEs are mostly explained by the variances. The best case when $\rho = 0.3$ (0.7) and $n = 20$ is Ex. 6 with RMSE = .1998 (.1020) (not shown in the tables). When $\rho =$ 0.3 (0.7) and $n = 50$, the best one is Ex. 7 with RMSE = .1256 (.0632). These cases use double or lower-tail truncations. The largest RMSEs in the four conditions are given by Ex. 11 with both variances inner-truncated.

In the numerical illustrations, the biases of r under variance truncation are shown to be as small as that without truncation. This finding suggests the following.

Conjecture 1. Let v_{11} and v_{22} be truncated such that the asymptotically constant truncated probability holds. Then, we have

$$\mathrm{E}(r \mid \rho, n; \mathbf{A}, \mathbf{B}) = \frac{2}{n}(n/2)_{1/2}^2 \rho \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+2)/2;\ \rho^2;\ F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}})}$$
$$= \rho\{1 + O(n^{-1})\}.$$

This conjecture is partially supported by the expansion of the factor before ρ on the right-hand side of the first equation:

$$\frac{2}{n}(n/2)_{1/2}^2 = \frac{2}{n}\frac{\Gamma\{(n+1)/2\}^2}{\Gamma(n/2)^2} = \frac{2}{n}\left(\sqrt{\frac{n}{2}}\right)^2 \left\{1 + n^{-1}\frac{1}{2}\left(\frac{1}{2} - 1\right) + O(n^{-2})\right\}^2$$
$$= 1 - \frac{n^{-1}}{2} + O(n^{-2}),$$

where $\Gamma(z+\alpha)/\Gamma(z+\beta) = z^{\alpha+\beta}\{1 + (1/2)z^{-1}(\alpha - \beta)(\alpha + \beta - 1) + O(z^{-2})\}$ (Erdélyi, 1953,

p. 47, Equation (4)) is used.

Another partial support is given by the expansion of the factor after ρ without truncation:

$$\frac{{}_2F_{1w_k}\{(n+1)/2,(n+1)/2;(n+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\ ;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$=\frac{{}_2F_1\{(n+1)/2,(n+1)/2;(n+2)/2;\ \rho^2\}}{{}_1F_0(n/2;\ ;\rho^2)}=\frac{(1-\rho^2)^{-n/2}\,{}_2F_1\{1/2,1/2;(n+2)/2;\ \rho^2\}}{(1-\rho^2)^{-n/2}}$$

$$={}_2F_1\{1/2,1/2;(n+2)/2;\ \rho^2\}=\sum_{k=0}^{\infty}\frac{(1/2)_k^2\rho^{2k}}{\{(n+2)/2\}_k k!}=1+\frac{(1/4)\rho^2}{(n+2)/2}+O(n^{-2})$$

$$=1+\frac{\rho^2 n^{-1}}{2}+O(n^{-2}),$$

where the Euler transform is used. The above expansions give the asymptotic bias without truncation:

$$\begin{aligned}\mathrm{E}(r\,|\,\rho)&=\left\{1-\frac{n^{-1}}{2}+O(n^{-2})\right\}\left\{1+\frac{\rho^2 n^{-1}}{2}+O(n^{-2})\right\}\rho\\&=\rho-\frac{\rho(1-\rho^2)n^{-1}}{2}+O(n^{-2})\end{aligned}.$$

as mentioned earlier.

(f) The non-recursive and recursive moments: In Lemma 1, the non-recursive and recursive moments are shown. As addressed in Remark 1, the former includes ${}_3F_{2w_k}(\cdot)$ while the latter ${}_2F_{1w_k}(\cdot)$, which gives an advantage of the latter. However, if we use the latter for higher-order moments, we may suffer from the propagating errors (see the warning by Azzalini et al., 2020, http://azzalini.stat.unipd.it/SW/Pkg-mnormt/mnormt-manual.pdf, p. 16). In the recursive methods of the moments of orders higher than e.g., the tens for the truncated normal vectors (e.g., Fisher, 1931, Equation (13); Kan & Robotti, 2017, Theorem 1), it is known that the results tend to suffer from subtract cancellation errors (Pollack & Shauly-Aharonov, 2019; Ogasawara, 2021a, b).

Although ${}_2F_{1w_k}(\cdot)$ looks much simpler than the corresponding ${}_3F_{2w_k}(\cdot)$, in the examples for illustration, the required numbers of terms for convergence in the functions are almost the same. Further, it is almost equal to that of ${}_1F_{0w_k}(\cdot)$ which is required for the computation of the normalizer of the pdf and $\alpha = {}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})(1-\rho^2)^{n/2}$. Note that the added computation of ${}_3F_{2w_k}(\cdot)$ over ${}_2F_{1w_k}(\cdot)$ is an added (log) rising factorial each in the numerator and denominator in every term of the series, which seems to be rather trivial as long as the computation for the numerical examples is concerned. As addressed earlier, while ${}_2F_1(\cdot)$ was used by Ghosh (1966) and Muirhead (1982), ${}_3F_2(\cdot)$ was used by Romero-Padilla (2016, Equations (18) to (20)).

(g) The modified and non-modified hypergeometric functions: In Lemma 1 and Theorem 2, the formulas with the series expressions similar to hypergeometric functions (the modified hypergeometric functions) are shown followed by the hypergeometric expressions. The modified functions are given for their simplicity. The user cpu times required for the computation of the theoretical and asymptotic values in Tables 2 and 3 by the modified (simplified) and non-modified hypergeometric functions are 1.75 and 3.10 seconds, respectively using Intel(R) Core(TM) i7-6700 CPU @ 3.40GHz, showing an advantage of the modified functions due to the reduction of the number of evaluating (log) gamma functions (factorials). An advantage of the non-modified functions for non-recursive and recursive moments is that a general function for ${}_pF_{qw_k}(\cdot)$ $(p,q=0,1,...;\,p\le q+1)$ can be easily coded in a computer language since all the rising factorials in the series take the common form of $(\cdot)_k$ in the *k*-th term (k = 0,1,...).

(h) The moment generating function (mgf): The moments of *r* in Theorem 2 are given without using the mgf. The mgf is derived by expanding e^{tr} and taking the beta integral term by term. After some algebra we have the following expression:

$$\mathrm{M}_r(t) = \frac{1}{{}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})}\sum_{m,l=0}^{\infty}\left\{\frac{\rho t\{(n+1)/2\}_l^2(3/2)_{m+l}}{(3/2)_m(3/2)_l\{(n+2)/2\}_{m+l}}\frac{(n/2)_{1/2}}{\{(n+1)/2\}_{1/2}}F^{\mathbf{B}}_{\mathbf{A}(2l+1)} + \frac{(n/2)_l^2(1/2)_{m+l}}{(1/2)_m(1/2)_l(n/2)_{m+l}}F^{\mathbf{B}}_{\mathbf{A}(2l)}\right\}\frac{\rho^{2l}(t/2)^{2m}}{m!\,l!},$$

which is found to be 1 when t = 0 as expected using $(t/2)^0 = 1$. It is also found that when differentiated with respect to *t* and evaluated at *t* = 0, we have the result algebraically equal to the mean in Theorem 2. The above expression of the mgf is a scaled sum of two weighted bivariate hypergeometric functions similar to the Appell series when unweighted i.e.,

$F_{\mathbf{A}(l)}^{\mathbf{B}} = 1$ (see Bayley, 1935/1972, Chapter 9; Slater, 1966, Chapter 8; Zwillinger, 2015, Section 9.18).

Appendix

Proof of Theorem 1. The pdf of $\mathbf{u}$ is written as

$$w_2(\mathbf{u} \mid \rho, n; \mathbf{A}, \mathbf{B}) = C_2 \exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-3)/2},$$

where $\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} v_{11} & \sqrt{v_{11}v_{22}}\,r \\ \sqrt{v_{22}v_{11}}\,r & v_{22} \end{pmatrix}$ and C_2 is the normalizing constant satisfying

$$\begin{aligned} 1/C_2 &= \int_{-1}^{1} \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} \int_{a_{1i}}^{b_{1i}} \int_{a_{2j}}^{b_{2j}} \exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-3)/2} \sqrt{v_{11}v_{22}}\,\mathrm{d}v_{11}\mathrm{d}v_{22}\mathrm{d}r \\ &\equiv \int_{-1}^{1} \int_{\mathbf{A}}^{\mathbf{B}} \exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-3)/2} \sqrt{v_{11}v_{22}}\,\mathrm{d}\mathbf{u}. \end{aligned}$$

Expanding $\exp\{\cdot\}$, we have

$$\begin{aligned} &\int_{\mathbf{A}}^{\mathbf{B}} \exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-3)/2} \sqrt{v_{11}v_{22}}\,\mathrm{d}v_{11}\mathrm{d}v_{22} \\ &= \int_{\mathbf{A}}^{\mathbf{B}} \exp\left\{-\frac{v_{11} + v_{22} - 2\rho\sqrt{v_{11}v_{22}}\,r}{2(1-\rho^2)}\right\} (v_{11}v_{22})^{(n-2)/2} (1-r^2)^{(n-3)/2} \mathrm{d}v_{11}\mathrm{d}v_{22} \\ &= \int_{\mathbf{A}}^{\mathbf{B}} \exp\left\{-\frac{v_{11} + v_{22}}{2(1-\rho^2)}\right\} \exp\left(\frac{\rho\sqrt{v_{11}v_{22}}\,r}{1-\rho^2}\right) (v_{11}v_{22})^{(n-2)/2} (1-r^2)^{(n-3)/2} \mathrm{d}v_{11}\mathrm{d}v_{22} \\ &= \sum_{l=0}^{\infty} \left\{\frac{2\rho r}{2(1-\rho^2)}\right\}^l \int_{\mathbf{A}}^{\mathbf{B}} \exp\left\{-\frac{v_{11} + v_{22}}{2(1-\rho^2)}\right\} \frac{1}{l!} (v_{11}v_{22})^{(n+l-2)/2} (1-r^2)^{(n-3)/2} \mathrm{d}v_{11}\mathrm{d}v_{22} \\ &= \{2(1-\rho^2)\}^n \sum_{l=0}^{\infty} \frac{(2\rho)^l}{l!} r^l (1-r^2)^{(n-3)/2} \\ &\qquad \times \prod_{i=1}^{2} \sum_{j=1}^{K_i} \left[\gamma\left\{\frac{b_{ij}}{2(1-\rho^2)} \mid \frac{n+l}{2}\right\} - \gamma\left\{\frac{a_{ij}}{2(1-\rho^2)} \mid \frac{n+l}{2}\right\}\right], \end{aligned}$$

where $\gamma(x \mid \alpha) = \int_0^x t^{\alpha-1} \exp(-t)\,\mathrm{d}t$ is the lower incomplete gamma function with $\gamma(\infty \mid \alpha) = \Gamma(\alpha)$. When integrating the above result with respect to *r*, noting that the integral becomes zero if *l* is odd and using $l = 2k$ $(k = 0, 1, ...)$, we have

$$\int_{-1}^{1} r^l (1-r^2)^{(n-3)/2} \mathrm{d}r = \int_0^1 r^{2\{k-(1/2)\}} (1-r^2)^{(n-3)/2} \mathrm{d}r^2 = \mathrm{B}\left(\frac{2k+1}{2}, \frac{n-1}{2}\right),$$

where $\mathrm{B}(\cdot\,,\,\cdot)$ is the beta function. Consequently, the normalizing constant becomes

$$1/C_2 = \{2(1-\rho^2)\}^n \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)!} \mathrm{B}\left(\frac{2k+1}{2}, \frac{n-1}{2}\right)$$
$$\times \prod_{i=1}^{2} \sum_{j=1}^{K_i} \left[\gamma\left\{ \frac{b_{ij}}{2(1-\rho^2)} \,|\, \frac{n+2k}{2} \right\} - \gamma\left\{ \frac{a_{ij}}{2(1-\rho^2)} \,|\, \frac{n+2k}{2} \right\} \right]$$
$$= \{2(1-\rho^2)\}^n \Gamma\left(\frac{n-1}{2}\right) \pi^{1/2}$$
$$\times \sum_{k=0}^{\infty} \frac{\rho^{2k}}{k!} \Gamma\left(\frac{n+2k}{2}\right) \prod_{i=1}^{2} \sum_{j=1}^{K_i} \left[F_\Gamma\left\{ \frac{b_{ij}}{2(1-\rho^2)} \,|\, \frac{n+2k}{2} \right\} - F_\Gamma\left\{ \frac{a_{ij}}{2(1-\rho^2)} \,|\, \frac{n+2k}{2} \right\} \right],$$

where

$$\frac{(2\rho)^{2k}}{(2k)!} \mathrm{B}\left(\frac{2k+1}{2}, \frac{n-1}{2}\right) = \frac{\rho^{2k}}{k!} \pi^{1/2} \Gamma\left(\frac{n-1}{2}\right) / \Gamma\left(\frac{n+2k}{2}\right)$$

is used. Canceling the common factor $\{2(1-\rho^2)\}^n$ and redefining C_2 after this cancellation with the definition of the rising factorial and the Legendre duplication formula $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\{z+(1/2)\}$ (see Erdélyi, 1953, Section 1.2, Equation (15); DLMF, 2021, Equation 5.5.5, https://dlmf.nist.gov/5.5.E5), we have

$$1/C_2 = \Gamma\left(\frac{n-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \frac{\rho^{2k}}{k!} \Gamma\left(\frac{n+2k}{2}\right) F_{\mathbf{A}(2k)}^{\mathbf{B}}$$
$$= \Gamma\left(\frac{n-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \rho^{2k} (k+1)_{(n-2)/2} F_{\mathbf{A}(2k)}^{\mathbf{B}}$$
$$= \Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{n}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \frac{\rho^{2k}}{k!} \Gamma\left(\frac{n+2k}{2}\right) \Gamma\left(\frac{n}{2}\right)^{-1} F_{\mathbf{A}(2k)}^{\mathbf{B}}$$
$$= 2^{2-n} \pi\, \Gamma(n-1) \sum_{k=0}^{\infty} \frac{\rho^{2k}(n/2)_k}{k!} F_{\mathbf{A}(2k)}^{\mathbf{B}}$$
$$\equiv 2^{2-n} \pi\, \Gamma(n-1)\, {}_1F_{0w_k}(n/2;\,;\rho^2; F_{\mathbf{A}(2k)}^{\mathbf{B}}),$$

which gives the required expressions.

Proof of Corollary 1. For the untruncated case, in the proof of Theorem 1 we have

$$\prod_{i=1}^{2} \sum_{j=1}^{K_i} \left[\gamma\left\{ \frac{b_{ij}}{2(1-\rho^2)} \,|\, \frac{n+l}{2} \right\} - \gamma\left\{ \frac{a_{ij}}{2(1-\rho^2)} \,|\, \frac{n+l}{2} \right\} \right] = \Gamma\left(\frac{n+l}{2}\right)^2$$

and

$$1/C_2 = \Gamma\left(\frac{n-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\rho^{2k}(k+1)_{(n-2)/2},$$

which give the required result.

Proof of Lemma 1. Using the pdf given by Theorem 1, we obtain

(i) $\mathrm{E}\{r^{2i+1}(1-r^2)^j \mid i, j = 0,1,...;\rho, n; \mathbf{A}, \mathbf{B}\}$

$$= \int_{-1}^{1} C_2 \sum_{l=0}^{\infty} r^{2i+1}(1-r^2)^j \frac{(2\rho r)^l}{l!}(1-r^2)^{(n-3)/2}\Gamma\left(\frac{n+l}{2}\right)^2 F_{\mathbf{A}(l)}^{\mathbf{B}}\mathrm{d}r$$

$$= C_2 \sum_{l=0}^{\infty}\Gamma\left(\frac{n+l}{2}\right)^2 F_{\mathbf{A}(l)}^{\mathbf{B}}\frac{(2\rho)^l}{l!}\int_{-1}^{1} r^{l+2i+1}(1-r^2)^{(n+2j-3)/2}\mathrm{d}r$$

$$= C_2 \sum_{k=0}^{\infty}\Gamma\left(\frac{n+2k+1}{2}\right)^2 F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\frac{(2\rho)^{2k+1}}{(2k+1)!}\int_{-1}^{1} r^{2(k+i+1)}(1-r^2)^{(n+2j-3)/2}\mathrm{d}r$$

$$= C_2 \sum_{k=0}^{\infty}\Gamma\left(\frac{n+2k+1}{2}\right)^2 F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\frac{(2\rho)^{2k+1}}{(2k+1)!}\int_{0}^{1} r^{2\{k+i+(1/2)\}}(1-r^2)^{(n+2j-3)/2}\mathrm{d}r^2$$

$$= C_2 \sum_{k=0}^{\infty}\Gamma\left(\frac{n+2k+1}{2}\right)^2 F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\frac{(2\rho)^{2k+1}}{(2k+1)!}\mathrm{B}\left(\frac{2k+2i+3}{2}, \frac{n+2j-1}{2}\right)$$

$$= C_2\Gamma\left(\frac{n+2j-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\Gamma\left(\frac{n+2k+1}{2}\right)^2\Gamma\left(\frac{2k+2i+3}{2}\right)$$
$$\times\Gamma\left(\frac{n+2k+2i+2j+2}{2}\right)^{-1}\Gamma\left(\frac{2k+3}{2}\right)^{-1} F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\frac{\rho^{2k+1}}{k!}$$

$$= C_2\Gamma\left(\frac{n+2j-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\frac{\{(2k+3)/2\}_i(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{(2i+2j+1)/2}}F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\rho^{2k+1}$$

$$= \frac{\{(n-1)/2\}_j\sum_{k=0}^{\infty}\rho^{2k+1}\frac{\{(2k+3)/2\}_i(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{(2i+2j+1)/2}}F_{\mathbf{A}(2k+1)}^{\mathbf{B}}}{\sum_{k=0}^{\infty}\rho^{2k}(k+1)_{(n-2)/2}F_{\mathbf{A}(2k)}^{\mathbf{B}}}$$

$$= D_2\{(n-1)/2\}_j\sum_{k=0}^{\infty}\rho^{2k+1}\frac{\{(2k+3)/2\}_i(k+1)_{(n-1)/2}}{\{(n+2k+1)/2\}_{(2i+2j+1)/2}}F_{\mathbf{A}(2k+1)}^{\mathbf{B}}.$$

The alternative expression is given by the left-hand side of the third equation from the last:

$\mathrm{E}\{r^{2i+1}(1-r^2)^j \mid i = 0,1,...;\ j = 0,1,...;\rho, n; \mathbf{A}, \mathbf{B}\}$

$$= C_2\Gamma\left(\frac{n+2j-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\Gamma\left(\frac{n+2k+1}{2}\right)^2\Gamma\left(\frac{2k+2i+3}{2}\right)$$
$$\times\Gamma\left(\frac{n+2k+2i+2j+2}{2}\right)^{-1}\Gamma\left(\frac{2k+3}{2}\right)^{-1} F_{\mathbf{A}(2k+1)}^{\mathbf{B}}\frac{\rho^{2k+1}}{k!}$$

$$= C_2\Gamma\left(\frac{n+2j-1}{2}\right)\pi^{1/2}\sum_{k=0}^{\infty}\{(n+1)/2\}_k^2\Gamma\left(\frac{n+1}{2}\right)^2\{(2i+3)/2\}_k\Gamma\left(\frac{2i+3}{2}\right)$$

$$\times\{(n+2i+2j+2)/2\}_k^{-1}\Gamma\left(\frac{n+2i+2j+2}{2}\right)^{-1}$$

$$\times(3/2)_k^{-1}\Gamma\left(\frac{3}{2}\right)^{-1}F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\frac{\rho^{2k+1}}{k!}$$

$$= C_2\Gamma\left(\frac{n+2j-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\pi^{1/2}\frac{(3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n+2j-1)/2}}\rho$$

$$\times{}_3F_{2w_k}\{(n+1)/2,(n+1)/2,(2i+3)/2;(n+2i+2j+2)/2,\ 3/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}$$

$$= \Gamma\left(\frac{n+2j-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\pi^{1/2}\frac{(3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n+2j-1)/2}}\rho$$

$$\times\frac{{}_3F_{2w_k}\{(n+1)/2,(n+1)/2,(2i+3)/2;(n+2i+2j+2)/2,\ 3/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{2^{2-n}\pi\,\Gamma(n-1)\,{}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \Gamma\left(\frac{n+2j-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)^{-1}\Gamma\left(\frac{n}{2}\right)^{-1}\frac{(3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n+2j-1)/2}}\rho$$

$$\times\frac{{}_3F_{2w_k}\{(n+1)/2,(n+1)/2,(2i+3)/2;(n+2i+2j+2)/2,\ 3/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \frac{\{(n-1)/2\}_j(n/2)_{1/2}(3/2)_{(n-2)/2}}{\{(2i+3)/2\}_{(n+2j-1)/2}}\rho$$

$$\times\frac{{}_3F_{2w_k}\{(n+1)/2,(n+1)/2,(2i+3)/2;(n+2i+2j+2)/2,\ 3/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})},$$

where $\{\cdot\}_k^m = [\{\cdot\}_k]^m$ and the second last result is given by the Legendre duplication formula.

The result of $\mathrm{E}\{r(1-r^2)^j \mid j=0,1,...;\rho,n;\mathbf{A},\mathbf{B}\}$ is given from above with $i = 0$ and ${}_3F_{2w_k}\{\cdot\}$ replaced by ${}_2F_{1w_k}\{\cdot\}$.

(ii) $\mathrm{E}\{r^{2i}(1-r^2)^j \mid i,j=0,1,...;\rho,n;\mathbf{A},\mathbf{B}\}$

$$= \int_{-1}^{1} C_2\sum_{l=0}^{\infty} r^{2i}(1-r^2)^j\frac{(2\rho r)^l}{l!}(1-r^2)^{(n-3)/2}\Gamma\left(\frac{n+l}{2}\right)^2 F^{\mathbf{B}}_{\mathbf{A}(l)}\mathrm{d}r$$

$$= C_2\sum_{l=0}^{\infty}\Gamma\left(\frac{n+l}{2}\right)^2 F^{\mathbf{B}}_{\mathbf{A}(l)}\frac{(2\rho)^l}{l!}\int_{-1}^{1} r^{l+2i}(1-r^2)^{(n+2j-3)/2}\mathrm{d}r$$

$$= C_2 \sum_{k=0}^{\infty} \Gamma\left(\frac{n+2k}{2}\right)^2 F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{(2\rho)^{2k}}{(2k)!} \int_{-1}^{1} r^{2(k+i)} (1-r^2)^{(n+2j-3)/2} \mathrm{d}r$$

$$= C_2 \sum_{k=0}^{\infty} \Gamma\left(\frac{n+2k}{2}\right)^2 F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{(2\rho)^{2k}}{(2k)!} \int_{0}^{1} r^{2\{k+i-(1/2)\}} (1-r^2)^{(n+2j-3)/2} \mathrm{d}r^2$$

$$= C_2 \sum_{k=0}^{\infty} \Gamma\left(\frac{n+2k}{2}\right)^2 F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{(2\rho)^{2k}}{(2k)!} \mathrm{B}\left(\frac{2k+2i+1}{2}, \frac{n+2j-1}{2}\right)$$

$$= C_2 \Gamma\left(\frac{n+2j-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \Gamma\left(\frac{n+2k}{2}\right)^2 \Gamma\left(\frac{2k+2i+1}{2}\right)$$

$$\times \Gamma\left(\frac{n+2k+2i+2j}{2}\right)^{-1} \Gamma\left(\frac{2k+1}{2}\right)^{-1} F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{\rho^{2k}}{k!}$$

$$= C_2 \Gamma\left(\frac{n+2j-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \frac{\{(2k+1)/2\}_i (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_{i+j}} F_{\mathbf{A}(2k)}^{\mathbf{B}} \rho^{2k}$$

$$= \frac{\{(n-1)/2\}_j \sum_{k=0}^{\infty} \frac{\{(2k+1)/2\}_i (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_{i+j}} F_{\mathbf{A}(2k)}^{\mathbf{B}} \rho^{2k}}{\sum_{k=0}^{\infty} \rho^{2k} (k+1)_{(n-2)/2} F_{\mathbf{A}(2k)}^{\mathbf{B}}}$$

$$= D_2 \{(n-1)/2\}_j \sum_{k=0}^{\infty} \frac{\{(2k+1)/2\}_i (k+1)_{(n-2)/2}}{\{(n+2k)/2\}_{i+j}} F_{\mathbf{A}(2k)}^{\mathbf{B}} \rho^{2k}.$$

The alternative expression is given by the left-hand side of the third equation from the last:

$$\mathrm{E}\{r^{2i}(1-r^2)^j \mid i, j = 0, 1, \ldots; \rho, n; \mathbf{A}, \mathbf{B}\}$$

$$= C_2 \Gamma\left(\frac{n+2j-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} \Gamma\left(\frac{n+2k}{2}\right)^2 \Gamma\left(\frac{2k+2i+1}{2}\right)$$

$$\times \Gamma\left(\frac{n+2k+2i+2j}{2}\right)^{-1} \Gamma\left(\frac{2k+1}{2}\right)^{-1} F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{\rho^{2k}}{k!}$$

$$= C_2 \Gamma\left(\frac{n+2j-1}{2}\right) \pi^{1/2} \sum_{k=0}^{\infty} (n/2)_k^2 \Gamma\left(\frac{n}{2}\right)^2 \{(2i+1)/2\}_k \Gamma\left(\frac{2i+1}{2}\right)$$

$$\times \{(n+2i+2j)/2\}_k^{-1} \Gamma\left(\frac{n+2i+2j}{2}\right)^{-1} (1/2)_k^{-1} \Gamma\left(\frac{1}{2}\right)^{-1} F_{\mathbf{A}(2k)}^{\mathbf{B}} \frac{\rho^{2k}}{k!}$$

$$= C_2 \Gamma\left(\frac{n+2j-1}{2}\right) \Gamma\left(\frac{n}{2}\right)^2 \frac{1}{\{(2i+1)/2\}_{(n+2j-1)/2}}$$

$$\times {}_3F_{2w_k}\{n/2, n/2, (2i+1)/2; (n+2i+2j)/2,\ 1/2;\ \rho^2;\ F_{\mathbf{A}(2k)}^{\mathbf{B}}\}$$

$$= \Gamma\left(\frac{n+2j-1}{2}\right)\Gamma\left(\frac{n}{2}\right)^2 \frac{1}{\{(2i+1)/2\}_{(n+2j-1)/2}}$$

$$\times \frac{{}_3F_{2w_k}\{n/2, n/2, (2i+1)/2; (n+2i+2j)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{2^{2-n}\pi\,\Gamma(n-1)\,{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \Gamma\left(\frac{n+2j-1}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)^{-1} \frac{\pi^{-1/2}}{\{(2i+1)/2\}_{(n+2j-1)/2}}$$

$$\times \frac{{}_3F_{2w_k}\{n/2, n/2, (2i+1)/2; (n+2i+2j)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \frac{\{(n-1)/2\}_j (1/2)_{(n-1)/2}}{\{(2i+1)/2\}_{(n+2j-1)/2}}$$

$$\times \frac{{}_3F_{2w_k}\{n/2, n/2, (2i+1)/2; (n+2i+2j)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

The alternative expression of $\mathrm{E}\{(1-r^2)^j \mid j=0,1,...; \rho, n; \mathbf{A}, \mathbf{B}\}$ is given from above with $i = 0$ and ${}_3F_{2w_k}\{\cdot\}$ replaced by ${}_2F_{1w_k}\{\cdot\}$:

$$\mathrm{E}\{(1-r^2)^j \mid j=0,1,...; \rho, n; \mathbf{A}, \mathbf{B}\}$$

$$= \frac{\{(n-1)/2\}_j (1/2)_{(n-1)/2}}{(1/2)_{(n+2j-1)/2}} \frac{{}_2F_{1w_k}\{n/2, n/2; (n+2j)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \frac{\{(n-1)/2\}_{1/2}}{\{(n+2j-1)/2\}_{1/2}} \frac{{}_2F_{1w_k}\{n/2, n/2; (n+2j)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

Proof of Theorem 2. The expressions without hypergeometric functions are easily given from Lemma 1. The alternative expression for the mean is obtained by Lemma 1 (i):

$$\mathrm{E}\{r = r(1-r^2)^j \mid j=0; \rho, n; \mathbf{A}, \mathbf{B}\}$$

$$= \frac{(n/2)_{1/2}(3/2)_{(n-2)/2}}{(3/2)_{(n-1)/2}} \rho \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

$$= \frac{2}{n}(n/2)^2_{1/2}\rho \frac{{}_2F_{1w_k}\{(n+1)/2, (n+1)/2; (n+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k+1)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

The expressions using hypergeometric functions for the moment of the second power are obtained by Lemma 1 (ii):

$$\mathrm{E}\{r^2 = r^{2i}(1-r^2)^j \mid i=1,\ j=0; \rho, n; \mathbf{A}, \mathbf{B}\}$$
$$= \frac{(1/2)_{(n-1)/2}}{(3/2)_{(n-1)/2}} \frac{{}_3F_{2w_k}\{n/2,\ n/2, 3/2;\ (n+2)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$
$$= n^{-1} \frac{{}_3F_{2w_k}\{n/2,\ n/2, 3/2;\ (n+2)/2,\ 1/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$

and

$$\mathrm{E}\{1-r^2 = (1-r^2)^j \mid j=1; \rho, n; \mathbf{A}, \mathbf{B}\}$$
$$= \frac{\{(n-1)/2\}_{1/2}}{\{(n+1)/2\}_{1/2}} \frac{{}_2F_{1w_k}\{n/2,\ n/2;\ (n+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}$$
$$= (1-n^{-1}) \frac{{}_2F_{1w_k}\{n/2,\ n/2;\ (n+2)/2;\ \rho^2;\ F^{\mathbf{B}}_{\mathbf{A}(2k)}\}}{{}_1F_{0w_k}(n/2;\ ; \rho^2; F^{\mathbf{B}}_{\mathbf{A}(2k)})}.$$

In the expressions of the moment of the fourth power, the coefficients in the terms of hypergeometric functions are given by Lemma 1 (ii) with

$$\frac{(1/2)_{(n-1)/2}}{(5/2)_{(n-1)/2}} = \frac{(3/2)(1/2)}{\{(n+2)/2\}n/2} = \frac{3}{(n+2)n} \quad \text{and} \quad \frac{\{(n-1)/2\}_{1/2}}{\{(n+3)/2\}_{1/2}} = \frac{(n+1)(n-1)}{(n+2)n}.$$

Proof of Lemma 2. The pdf of $\mathbf{u}$ is written as

$$w_p(\mathbf{u} \mid n; \mathbf{A}, \mathbf{B}) = C_p \exp\{-\mathrm{tr}(\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2},$$

where $\mathbf{V}$ is assumed to be expressed by $\mathbf{u}$; $\mathrm{d}\{\mathrm{v}(\mathbf{V})\} = (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$ is the Jacobian; $\mathrm{v}(\cdot)$ is the vectorizing operator taking the non-duplicated elements of a matrix in parentheses; and C_p is the normalizing constant satisfying

$$1/C_p = \int_{\mathbf{R}>0} \sum_{i_1=1}^{K_1} \cdots \sum_{i_p=1}^{K_p} \int_{a_{i_1}}^{b_{i_1}} \cdots \int_{a_{i_p}}^{b_{i_p}} \exp\{-\mathrm{tr}(\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$$
$$\equiv \int_{\mathbf{R}>0} \int_{\mathbf{A}}^{\mathbf{B}} \exp\{-\mathrm{tr}(\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$$
$$= \int_{\mathbf{R}>0} \int_{\mathbf{A}}^{\mathbf{B}} \exp\{-\mathrm{tr}(\mathbf{V})/2\} \mid \mathbf{R} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(n-2)/2} \mathrm{d}\mathbf{u}$$
$$= 2^{np/2} \int_{\mathbf{R}>0} \mid \mathbf{R} \mid^{(n-p-1)/2} \mathrm{d}\mathbf{r} \prod_{i=1}^{p} \sum_{j=1}^{K_i} \left\{ \gamma\left(\frac{b_{ij}}{2} \mid \frac{n}{2}\right) - \gamma\left(\frac{a_{ij}}{2} \mid \frac{n}{2}\right) \right\}.$$

In the last result, when without truncation, the constant product becomes

$$\prod_{i=1}^{p} \sum_{j=1}^{K_i} \left\{ \gamma\left(\frac{b_{ij}}{2} \mid \frac{n}{2}\right) - \gamma\left(\frac{a_{ij}}{2} \mid \frac{n}{2}\right) \right\} = \Gamma\left(\frac{n}{2}\right)^p.$$

Consequently $1/C_p$ under the above truncation is found to be decreased to

$$\left[\prod_{i=1}^{p}\sum_{j=1}^{K_i}\left\{\gamma\left(\frac{b_{ij}}{2}|\frac{n}{2}\right)-\gamma\left(\frac{a_{ij}}{2}|\frac{n}{2}\right)\right\}\right]/\Gamma\left(\frac{n}{2}\right)^p=\prod_{i=1}^{p}\sum_{j=1}^{K_i}\left\{F_\Gamma\left(\frac{b_{ij}}{2}|\frac{n}{2}\right)-F_\Gamma\left(\frac{a_{ij}}{2}|\frac{n}{2}\right)\right\}=F_{\mathbf{A}}^{\mathbf{B}}$$

times the corresponding $1/C_p$ without truncation. Since the reciprocal of the normalizer of the Wishart distribution without truncation is $2^{np/2}\,|\,\mathbf{\Sigma}\,|^{n/2}\,\Gamma_p(n/2)=2^{np/2}\Gamma_p(n/2)$ as shown earlier, which is unchanged irrespective of variable transformations for $v_{ij}(1\le i<j\le p)$. Then, we have

$$1/C_p=2^{np/2}\Gamma_p(n/2)F_{\mathbf{A}}^{\mathbf{B}},$$

which gives the required result.

Proof of Theorem 3. Using the result of $1/C_p$ in the proof of Lemma 2, we obtain

$$\begin{aligned}1/C_p&=\int_{\mathbf{R}>0}2^{np/2}\,|\,\mathbf{R}\,|^{(n-p-1)/2}\prod_{i=1}^{p}\sum_{j=1}^{K_i}\left\{\gamma\left(\frac{b_{ij}}{2}|\frac{n}{2}\right)-\gamma\left(\frac{a_{ij}}{2}|\frac{n}{2}\right)\right\}\mathrm{d}\mathbf{r}\\&=\int_{\mathbf{R}>0}2^{np/2}\,|\,\mathbf{R}\,|^{(n-p-1)/2}\,\Gamma(n/2)^p\,F_{\mathbf{A}}^{\mathbf{B}}\mathrm{d}\mathbf{r},\end{aligned}$$

where the integrand is $1/C_p$ times the pdf of **r**. Canceling the common factors $2^{np/2}$ and $F_{\mathbf{A}}^{\mathbf{B}}$ in the integrand and $1/C_p$, the required pdf is obtained. The equality $f(\mathbf{r}\,|\,n;\mathbf{A},\mathbf{B})=f(\mathbf{r}\,|\,n)$ is given by the vanishing $F_{\mathbf{A}}^{\mathbf{B}}$ in the expression. The robust property of the pdf against the violation of normality is due to the known equality of the pdf of **R** without truncation under the elliptical distribution (for the bivariate case, see Ali & Joarder, 1991, Theorem; for the p-variate case, Joarder & Ali, 1992, Theorem 3.1).

Declarations

The author states that there is no conflict of interest.

References

Abadir, K. M. (1999). An introduction to hypergeometric functions for economists. *Econometric Reviews, 18*, 287-330.

Abramowitz, M., & Stegun, I. A. (Eds.). (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (ninth printing). New York: Dover.

Ali, M. M., & Joarder, A. H. (1991). Distribution of the correlation coefficient for the class of bivariate elliptical models. *The Canadian Journal of Statistics, 19*, 447-452.

Anderson, T. W. (1958). *An introduction to multivariate statistical analysis*. New York: Wiley.

Anderson, T. W. (2003). *An introduction to multivariate statistical analysis* (3rd ed.). New York: Wiley.

Arismendi, J. C., & Broda, S. (2017). Multivariate elliptical truncated moments. *Journal of Multivariate Analysis, 157*, 29-44.

Azzalini, A., Genz, A., Miller, A, Wichura, M. J., Hill, G. W., & Ge, Y. (2020). *The multivariate normal and t distributions, and their truncated versions*. R package version 2.0.2. https://CRAN.R-project.org/package=mnormt.

Bayley, W. N. (1935). *Generalized hypergeometric series*. Cambridge: Cambridge University Press. Reprinted, 1972, New York: Hafner.

DLMF (2021). *NIST Digital Library of Mathematical Functions*. https://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain (Eds.).

Erdélyi, A. (Ed.). (1953). *Higher transcendental functions* (Vol. 1). New York: McGraw-Hill. https://authors.library.caltech.edu/43491/.

Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika, 10*, 507-521.

Fisher, R. A. (1931). Introduction to *Mathematical Tables*, *1*, xxvi-xxxv. British Association for the Advancement of Science. Reprinted in R. A. Fisher, *Contributions to Mathematical Statistics* (pp. 517-526) with the title "The sampling error of estimated deviates, together with other illustrations of the properties and applications of the integrals and derivatives of the normal error function" and the author's note (CMS 23.xxva). New York: Wiley, 1950.
https://digital.library.adelaide.edu.au/dspace/handle/2440/3860.

Galarza, C. E., Lin, T.-I, Wang, W.-L., & Lachos, V. H. (2021). On moments of folded and truncated multivariate Student-*t* distributions based on recurrence relations. *Metrika*, *84*, 825–850.

Genz, A., Bretz, F., Miwa, T., Mi, X., Leisch, F., Scheipl, F., & Hothorn, T. (2020). *mvtnorm: Multivariate normal and t distributions*. R package version 1.1-0. http://CRAN.R-project.org/package=mvtnorm.

Ghosh, B. K. (1966). Asymptotic expansions for the moments of the distribution of correlation coefficient. *Biometrika*, *53*, 258-262.

Hotelling, H. (1953). New light on the correlation coefficient and its transforms. (with discussion) *Journal of the Royal Statistical Society, Series B, 15*, 193-232.

Joarder, A. H. (2006). Product moments of bivariate Wishart distribution. *Journal of Probability and Statistical Science*, *4*(2), 233-243.

Joarder, A. H. (2013). Robustness of correlation coefficient and variance ratio under elliptical symmetry. *Bulletin of the Malaysian Mathematical Sciences Society*, *Series 2, 36*, 277-284. http://math.usm.my/bulletin/pdf/v36n2/v36n2p2.pdf.

Joarder, A. H., & Ali, M. M. (1992). Distribution of the correlation matrix for a class of elliptical models. *Communications in Statistics - Theory and Methods*, *21*, 1953-1964.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions* Vol. 2 (2nd ed.). New York: Wiley.

Kan, R., & Robotti, C. (2017). On moments of folded and truncated multivariate normal distributions. *Journal of Computational and Graphical Statistics, 26*, 930-934.

Kendall, M. G. (1948). *The advanced theory of statistics Vol.1* (4th ed.). London: Charles Griffin.

Kirkby, J. L., Nguyen, D., & Nguyen, D. (2021). *Moments of Student's t-distribution: A unified approach*.arXiv:1912.01607v2 [math.PR] 26 Mar 2021.

Kotz, S., Balakrishnan, N., & Johnson, N. L. (2000). *Continuous multivariate distributions* (2nd ed.). Vol. 1: Models and applications. New York: Wiley.

Kotz, S., & Nadarajah, S. (2004): Multivariate *t* distributions and their applications. Cambridge: Cambridge University Press.

Mathai, A. M. (1993). *A handbook for generalized functions for statistical and physical sciences*. Oxford: Oxford University press.

Muirhead, R. J. (1982). *Aspects of multivariate statistical theory*. New York: Wiley.

Ogasawara, H. (2021a). Unified and non-recursive formulas for moments of the normal distribution with stripe truncation. *Communications in Statistics – Theory and Methods* (online published). https://doi.org/10.1080/03610926.2020.1867742.

Ogasawara, H. (2021b). A non-recursive formula for various moments of the multivariate normal distribution with sectional truncation. *Journal of Multivariate Analysis* (online published). https://doi.org/10.1016/j.jmva.2021.104729.

Omar, M. H., Joarder, A. H., & Riaz, M. (2015). On a correlated variance ratio distribution and its industrial application. *Communications in Statistics - Theory and Methods*, *44*, 261-274.

Pollack, M., & Shauly-Aharonov, M. (2019). A double recursion for calculating moments of the truncated normal distribution and its connection to change detection. *Methodology and Computing in Applied Probability, 21*, 889-906.

Rady, E. H. A., Fergany, H. A., & Edress, A. M. (2005). An easy way to calculate the moments of sample correlation coefficients. *Communications, Faculty of Sciences, University of Ankara, Series A1 Mathematics and Statistics, 54*(1), 21-28. https://doi.org/10.1501/Commua1_0000000322.

Rainville, E. D. (1960). *Special functions*. New York: Macmillan.

R Core Team (2020). *R: A language and environment for statistical computing*. Vienna, Austria: R Foundation for Statistical Computing. URL https://www.R-project.org/.

Romero-Padilla, J. (2016). Product moments of sample variances and correlation for variables with bivariate normal distribution. *Journal of Mathematics and Statistics*, *12*(1), 12-22. https://doi.org/10.3844/jmssp.2016.12.22.

Slater, L. J. (1966). *Generalized hypergeometric functions*. Cambridge: Cambridge University Press. Reprinted, 2008.

Soper, H. E., Young, A. W., Cave, B. M., Lee, A., & Pearson, K. (1917). On the distribution of the correlation coefficient in small samples. Appendix II to the papers of "Student" and R. A. Fisher. *Biometrika*, *11*, 328-413.

Stuart, A., & Ord, J. K. (1994). *Kendall's advanced theory of statistics Vol.1 Distribution theory* (6th ed.). London: Arnold.

Tallis, G. M. (1961). The moment generating function of the truncated multi-normal distribution. *Journal of the Royal Statistical Society, B, 23*, 223-229.

Tallis, G. M. (1963). Elliptical and radial truncation in normal populations. *The Annals of Mathematical Statistics, 34*, 940-944.

Tallis, G. M. (1965). Plane truncation in normal populations. *Journal of the Royal Statistical Society, B, 27*, 301-307.

Zwillinger, D. (2015). *Table of integrals, series and products* (8th ed.). Translated from Russian by Scripta Technica Inc. Amsterdam: Elsevier. The original Russian text by I. S. Gradshteyn and I. M. Ryzhik, 1962. Moscow: Fiziko-Matematicheskoy Literatury.

Table 1. Eleven examples and their variance truncation

	Truncation type		The numbers of intervals for selection		The selection points			
					v_{11}		v_{22}	
Ex. No.	v_{11}	v_{22}	v_{11}	v_{22}	A	B	A	B
1	untrnc	untrnc	1	1	0	Inf	0	Inf
2	lower	untrnc	1	1	CL	Inf	0	Inf
3	*	lower	1	1	CL	Inf	CL	Inf
4	*	upper	1	1	CL	Inf	0	CR
5	double	untrnc	1	1	CL	CR	0	Inf
6	*	lower	1	1	CL	CR	CL	Inf
7	*	double	1	1	CL	CR	CL	CR
8	inner	untrnc	2	1	0	CL	0	Inf
					CR	Inf		
9	*	lower	2	1	0	CL	CL	Inf
					CR	Inf		
10	*	double	2	1	0	CL	CL	CR
					CR	Inf		
11	*	inner	2	2	0	CL	0	CL
					CR	Inf	CR	Inf

Note. $v_{11} = \sum_{i=1}^{n+1} (X_{1i} - \bar{X}_1)^2$, $v_{22} = \sum_{i=1}^{n+1} (X_{2i} - \bar{X}_2)^2$; untrnc = untruncated, lower = lower-tail truncated, upper = upper-tail truncated, double = doubly truncated, inner = inner-truncated; Inf $= \infty$, CL = $\mathrm{E}(v_{ii}) - \sqrt{\mathrm{var}(v_{ii})} = n - \sqrt{2n}$, CR = $\mathrm{E}(v_{ii}) + \sqrt{\mathrm{var}(v_{ii})} = n + \sqrt{2n}$ $(i = 1, 2)$; '*' indicates 'the same as above'.

Table 2. Simulated and theoretical values of the moments/cumulants of r with the truncated probability α when $\rho = 0.3$ (the number of generated (not selected) sample covariance matrices = 10^5)

	Mean		SD		sk		kt		α The truncated probability		
Ex. No.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Asy.
	$\rho = 0.3$, $n = 20$										
1	.2927	.2932	.2060	.2058	-.3694	-.3648	-.0024	-.0334	1	1	1
2	.3044	.3051	.2029	.2026	-.3764	-.3738	.0113	-.0134	.84751	.84656	.84134
3	.3163	.3167	.1995	.1992	-.3820	-.3804	.0388	.0034	.72130	.72088	.71337
4	.2916	.2926	.2032	.2029	-.3587	-.3568	-.0102	-.0302	.71102	.71032	.70283
5	.2922	.2928	.2034	.2030	-.3555	-.3575	-.0231	-.0296	.69215	.69120	.68269
6	.3039	.3042	.2001	.1998	-.3613	-.3638	.0067	-.0139	.58469	.58464	.57485
7	.2925	.2930	.2008	.2004	-.3424	-.3486	-.0135	-.0298	.47905	.47880	.46702
8	.2937	.2940	.2117	.2120	-.3979	-.3793	.0267	-.0536	.30785	.30880	.31731
9	.3063	.3069	.2083	.2087	-.4059	-.3949	.0442	-.0244	.26210	.26193	.26649
10	.2913	.2925	.2079	.2088	-.3851	-.3743	.0162	-.0425	.21208	.21240	.21567
11	.2991	.2972	.2197	.2188	-.4288	-.3924	.0320	-.0862	.09577	.09639	.10164
	$\rho = 0.3$, $n = 50$										
1	.2970	.2973	.1294	.1293	-.2413	-.2448	-.0181	-.0007	1	1	1
2	.3050	.3050	.1276	.1275	-.2492	-.2453	-.0029	.0031	.84261	.84323	.84134
3	.3120	.3124	.1260	.1258	-.2506	-.2445	-.0014	.0051	.71603	.71579	.71337
4	.2969	.2971	.1272	.1273	-.2358	-.2384	-.0007	-.0001	.70497	.70573	.70283
5	.2976	.2972	.1272	.1274	-.2482	-.2388	-.0007	.0000	.68500	.69599	.68269
6	.3044	.3045	.1259	.1256	-.2514	-.2379	.0062	.0018	.57888	.57829	.57485
7	.2971	.2973	.1258	.1256	-.2332	-.2314	.0130	-.0017	.47190	.47156	.46702
8	.2957	.2975	.1338	.1335	-.2252	-.2562	-.0684	-.0183	.31500	.31401	.31731
9	.3945	.3060	.1319	.1316	-.2287	-.2610	-.0653	-.0091	.26473	.26494	.26649
10	.2951	.2969	.1312	.1312	-.2255	-.2523	-.0740	-.0012	.21447	.21443	.21567
11	.2970	.2986	.1393	.1383	-.2271	-.2652	-.0832	-.0485	.10053	.09959	.10164

Note. SD = standard deviation, sk = skewness, kt = excess kurtosis, α (the truncated probability) = ${}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})(1-\rho^2)^{n/2}$; Sim. = simulated values, Th. = theoretical values, Asy. = asymptotic values.

Table 3. Simulated and theoretical values of the moments/cumulants of r with the truncated probability α when $\rho = 0.7$ (the number of generated (not selected) sample covariance matrices = 10^5)

	Mean		SD		sk		kt		α The truncated probability		
Ex. No.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Asy.
	$\rho = 0.7$, $n = 20$										
1	.6904	.6907	.1219	.1218	-.9532	-.9543	1.4637	1.4431	1	1	1
2	.7083	.7086	.1092	.1091	-.8414	-.8452	1.0923	1.1047	.84615	.84656	.84134
3	.7198	.7201	.1014	.1014	-.7616	-.7683	.8304	.8636	.74816	.74786	.74427
4	.6940	.6941	.1096	.1095	-.8079	-.8127	1.0321	1.0457	.69530	.69498	.68672
5	.6941	.6941	.1096	.1095	-.8101	-.8135	1.0383	1.0471	.69130	.69120	.68269
6	.7062	.7061	.1018	.1018	-.7283	-.7351	.7740	.8078	.59732	.59627	.58965
7	.6973	.6972	.1025	.1024	-.7042	-.7106	.7267	.7607	.50835	.50726	.49660
8	.6821	.6831	.1454	.1454	-.9689	-.9723	1.0600	1.0267	.30870	.30880	.31731
9	.7133	.7145	.1244	.1247	-.9829	-1.0114	1.1160	1.2152	.25029	.25029	.25170
10	.6850	.6856	.1263	.1267	-.8412	-.8700	.8121	.8996	.18448	.18394	.18609
11	.6778	.6795	.1697	.1692	-.9646	-.9599	.6287	.5710	.12422	.12485	.13122
	$\rho = 0.7$, $n = 50$										
1	.6964	.6964	.0743	.0741	-.6207	-.6002	.6780	.5901	1	1	1
2	.7073	.7073	.0679	.0679	-.5209	-.5092	.4457	.3905	.84279	.84323	.84134
3	.7144	.7142	.0641	.0643	-.4592	-.4581	.3057	.2890	.74277	.74465	.74427
4	.6979	.6978	.0670	.0670	-.5156	-.5031	.4640	.4004	.69015	.68992	.68672
5	.6979	.6978	.0669	.0670	-.5162	-.5035	.4726	.4006	.68585	.68599	.68269
6	.7052	.7049	.0632	.0635	-.4512	-.4513	.3302	.2965	.58985	.59134	.58965
7	.6993	.6990	.0629	.0632	-.4467	-.4412	.3298	.2909	.49935	.50071	.49660
8	.6930	.6933	.0881	.0875	-.6410	-.6220	.3138	.2528	.31415	.31401	.31731
9	.7130	.7129	.0770	.0770	-.6351	-.6459	.2989	.3620	.25136	.25189	.25170
10	.6949	.6945	.0764	.0762	-.5215	-.5466	.1727	.2496	.18508	.18528	.18609
11	.6903	.6915	.1025	.1017	-.6457	-.6148	-.0551	-.1423	.12907	.12873	.13122

Note. SD = standard deviation, sk = skewness, kt = excess kurtosis, α (the truncated probability) = ${}_1F_{0w_k}(n/2;\,;\rho^2;F^{\mathbf{B}}_{\mathbf{A}(2k)})(1-\rho^2)^{n/2}$; Sim. = simulated values, Th. = theoretical values, Asy. = asymptotic values.