The density of the sample correlations under elliptical symmetry with or without the truncated variance-ratio

Preprint · April 2022

DOI: 10.13140/RG.2.2.35609.08807

1 author:

Haruhiko Ogasawara

Otaru University of Commerce

The density of the sample correlations under elliptical symmetry with or without the truncated variance-ratio

April 22, 2022

Haruhiko Ogasawara*

*Otaru University of Commerce, 3-5-21 Midori, Otaru 047-8501 Japan; Email: emt-hogasa@emt.otaru-uc.ac.jp; Web: http://www.res.otaru-uc.ac.jp/~emt-hogasa/.

The density of the sample correlations under elliptical symmetry with or without the truncated variance-ratio

Abstract: An expression of the joint probability density function (pdf) of the sample correlation matrix when increasing the number of variables sequentially under multivariate normality is obtained, which also holds for the elliptical distributions (elliptical symmetry). The pdf of the sample correlation coefficient, when the associated sample variance-ratio is variously truncated, is given under bivariate elliptical symmetry with some moments. It is derived that the joint pdf of the sample variance-ratios and correlation matrix under multivariate elliptical symmetry with possible truncation for the variance-ratios is unchanged irrespective of distinct elliptical distributions. A general condition for transformed sample variances and covariances to have unchanged pdf under elliptical symmetry is given.

Keywords: stripe truncation, Wishart distribution, elliptical distribution, weighted hypergeometric function, intra correlation.

1. Introduction

The distribution of the sample correlation coefficient denoted by R under bivariate normality was given by Fisher (1915, p. 516) using a differential expression with the bivariate Wishart density. Currently, there are many expressions of the distribution (see Johnson, Kotz & Balakrishnan, 1995, Chapter 32, Section 2). It has been known that the distribution of the sample correlation matrix under null multi-variate normality or uncorrelated normality is also robust under elliptical symmetry (Muirhead, 1982, Theorem 5.1.3). Fisher (1962) discussed the joint distribution of the sample correlation matrix under non-null normality especially under the tri-variate case. Anderson and Fang (1990, Theorem 4), and Ali and Joarder (1991) showed that the distribution of the sample correlation matrix holds under elliptical symmetry i.e., the elliptically contoured distribution. Joarder and Ali (1992, Theorem 3.1) gave the probability density function (pdf) of the sample correlation matrix under non-null multivariate normality, which holds under elliptical symmetry.

When a single Wishart observation based on N independent normal vectors is truncated due to e.g., the high or low values of the associated sample variances, the distribution of R was derived by Ogasawara (2022), which does not hold under elliptical symmetry since the distribution of the sample variances depends on the fourth cumulants of the associated N random vectors. However, Ogasawara (2022) also showed that the null distribution of the sample correlation matrix under normality with truncated associated sample-variances is equal to that without truncation and consequently holds under null elliptical symmetry.

Recently, Joarder (2013, Theorem 5.1) showed that the distribution of the sample variance-ratio is the same over the distributions under non-null bivariate elliptical symmetry. Note that there are two sample variances for a single correlation coefficient while we have a single variance-ratio or its reciprocal. Consequently, it is convenient to deal with the variance ratio rather than two sample variances for truncation of Wishart observations. Omar, Joarder and Riaz (2015) gave applications of the variance ratio in quality control.

The first purpose of this paper is to have an explicit sequential expression by increasing the number of variables for the distribution of the sample correlation matrix under non-null elliptical symmetry without truncation, which is different from the corresponding expression by Joarder (2013). The second purpose is to derive the distribution of R under bivariate elliptical symmetry when the sample variance-ratio is truncated in various ways. The joint distribution of the sample variance-ratios and correlation matrix under multivariate elliptical symmetry is also derived. The remainder of this paper is organized as follows. In Section 2, the result for the first purpose is given, followed by associated remarks in Section 3. The results for the second purpose are obtained in Section 4.

2. The density of the sample correlation matrix without truncation under elliptical symmetry

Suppose that a $p \times p$ positive definite random matrix $\mathbf{V} = \{v_{ij}\} > 0$ is Wishart distributed as denoted by $\mathbf{V} \sim \mathrm{W}_p(\mathbf{\Sigma}, n)$ with n degrees of freedom, where $\mathbf{\Sigma} > 0$ is a scale matrix. Then, pdf of the Wishart distribution is given by

$$w_p(\mathbf{V} \mid \mathbf{\Sigma}, n) = \frac{\exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\} \mid \mathbf{V} \mid^{(n-p-1)/2}}{2^{np/2} \mid \mathbf{\Sigma} \mid^{n/2} \Gamma_p(n/2)}$$

where $\Gamma_p(t) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(t - \frac{i-1}{2}\right)$ is the multivariate gamma function (Anderson, 2003, Definition 7.2.1, Subsection 7.2, Equation (19); see also DLMF, 2021, Section 35.3, https://dlmf.nist.gov/35.3); and $\mathbf{V}$ is used also as a realization of $\mathbf{V}$ for simplicity of notation.

Consider the change of variables:

$$\mathbf{V} = \{v_{ij}\} = \mathbf{DRD},$$

where $\mathbf{D} = \mathrm{diag}(v_{11}^{1/2}, \ldots, v_{pp}^{1/2})$ and $\mathbf{R} = \{r_{ij}\}$ with $r_{ij} = v_{ij} / (v_{ii} v_{jj})^{1/2}$ and $r_{ii} = 1 \; (i, j = 1, \ldots, p)$ is the sample correlation matrix. Let $\mathbf{v} = \mathrm{v}(\mathbf{V}) = (v_{11}, v_{21}, \ldots, v_{pp})^{\mathrm{T}}$ be a $\{p(p+1)/2\} \times 1$ vector, where $\mathrm{v}(\cdot)$ is the vectorizing operator taking the non-duplicated elements of a symmetric matrix in parentheses. Similarly, define a $\{p(p-1)/2\} \times 1$ vector $\mathbf{r} = \mathrm{vb}(\mathbf{R}) = (r_{21}, r_{31}, \ldots, r_{p-1,p})^{\mathrm{T}}$ with $\mathrm{vb}(\cdot)$ being the vectorizing operator taking the sub- or infra-diagonal elements i.e., those below the main diagonals of a square matrix. The change of variables is seen as that from $\mathbf{v}$ to $\mathbf{u} = (v_{11}, \ldots, v_{pp}, \mathbf{r}^{\mathrm{T}})^{\mathrm{T}}$ with unchanged $v_{11}, \ldots, v_{pp}$. It is reasonable to denote the distribution of $\mathbf{u}$ by $\mathbf{u} \sim \mathrm{W}_p(\mathbf{\Sigma}, n)$ after introducing an appropriate Jacobian with the support $\mathbf{R} > 0$ as well as $0 < v_{ii} < \infty$ $(i = 1,..., p)$. Let $\mathbf{\Sigma} = \{\sigma_{ij}\} = \mathrm{diag}(\sigma_{11}^{1/2},...,\sigma_{pp}^{1/2})\mathbf{P}\mathrm{diag}(\sigma_{11}^{1/2},...,\sigma_{pp}^{1/2})$, where $\mathbf{P} = \{\rho_{ij}\}$ with $\rho_{ii} = 1$ $(i, j = 1,..., p)$ is the population correlation matrix. Since r_{ij} is scale-free, it can be shown that the distribution of **r** or **R** does not depend on the sizes of σ_{ii}'s. Consequently, assume that $\mathbf{\Sigma} = \mathbf{P} > 0$ using unit population variances without loss of generality as long as the distribution of **r** is of interest. Then, we employ the notation $\mathbf{u} \sim \mathrm{W}_p(\mathbf{P}, n) = \mathrm{W}_p(\boldsymbol{\rho}, n)$, where $\boldsymbol{\rho} = \mathrm{vb}(\mathbf{P})$.

Lemma 1. *Let* $\mathbf{u} = (v_{11},...,v_{pp},\mathbf{r}^{\mathrm{T}})^{\mathrm{T}} \sim \mathrm{W}_p(\boldsymbol{\rho}, n)$ *with* $n > 1$ *and* $\mathbf{P} > 0$. *Then, an integral expression of the marginal joint pdf of* **r** *or* **R** *is*

$$f_p(\mathbf{r} \mid \boldsymbol{\rho}, n) = f_p(\mathbf{R} \mid \boldsymbol{\rho}, n)$$
$$= C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,...,p)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2} \right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii},$$

where $1 / C_p = 2^{np/2} \mid \mathbf{P} \mid^{n/2} \Gamma_p(n/2)$ *is the normalizing constant for the Wishart shown earlier when* $\mathbf{\Sigma} = \mathbf{P}$, $\mathbf{P}^{-1} = \{\rho^{ij}\}$; $g_i = g_i(\rho_{i,i+1},...,\rho_{ip}, r_{i,i+1},...,r_{ip}, v_{i+1,i+1},...,v_{pp})$ $= \sum_{j=i+1}^{p} \rho^{ij} r_{ij} v_{jj}^{1/2}$ $(i = 1,..., p)$, $g_p \equiv 0$; $\int_{\mathbf{0}}^{\infty}(\cdot) = \int_0^{\infty} \cdots \int_0^{\infty}(\cdot)$; *and* $\prod^{i(1,...,p)}(\cdot)$ *is a special product symbol with the i-th factor to be located in the i-th position from left in the product.*

Proof. Since the Jacobian is $\mathrm{d}\mathbf{v} = (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}\mathbf{u}$, the pdf of $\mathbf{u}$ is written as

$$w_p(\mathbf{u} \mid \boldsymbol{\rho}, n) = C_p \exp\{-\mathrm{tr}(\mathbf{P}^{-1}\mathbf{DRD})/2\} \mid \mathbf{DRD} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2}.$$

Then, the marginal pdf of **r** or **R** is given as follows.

$$f_p(\mathbf{r} \mid \boldsymbol{\rho}, n)$$
$$= C_p \int_0^{\infty} \cdots \int_0^{\infty} \exp\{-\mathrm{tr}(\mathbf{P}^{-1}\mathbf{DRD})/2\} \mid \mathbf{DRD} \mid^{(n-p-1)/2} (v_{11} \cdots v_{pp})^{(p-1)/2} \mathrm{d}v_{11} \cdots \mathrm{d}v_{pp}$$
$$= C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp\left(-\sum_{i=1}^{p} \rho^{ii} v_{ii}/2\right) \exp\left(-\sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \rho^{ij} r_{ij} v_{ii}^{1/2} v_{jj}^{1/2}\right)$$
$$\times (v_{11} \cdots v_{pp})^{(n-2)/2} \mathrm{d}v_{11} \cdots \mathrm{d}v_{pp}$$

$$= C_p \,|\, \mathbf{R} \,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp(-\rho^{11} v_{11} / 2) \exp\left(-v_{11}^{1/2} \sum\nolimits_{j=2}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2}\right) v_{11}^{(n-2)/2} \mathrm{d}v_{11}$$
$$\times \exp(-\rho^{22} v_{22} / 2) \exp\left(-v_{22}^{1/2} \sum\nolimits_{j=3}^{p} \rho^{2j} r_{2j} v_{jj}^{1/2}\right) v_{22}^{(n-2)/2} \mathrm{d}v_{22}$$
$$\vdots$$
$$\times \exp(-\rho^{p-1,p-1} v_{p-1,p-1} / 2) \exp\left(-v_{p-1,p-1}^{1/2} \rho^{p-1,p} r_{p-1,p} v_{pp}^{1/2}\right) v_{p-1,p-1}^{(n-2)/2} \mathrm{d}v_{p-1,p-1}$$
$$\times \exp(-\rho^{pp} v_{pp} / 2) v_{pp}^{(n-2)/2} \mathrm{d}v_{pp}$$

$$= C_p \,|\, \mathbf{R} \,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \exp(-\rho^{11} v_{11} / 2) \exp(-g_1 v_{11}^{1/2}) v_{11}^{(n-2)/2} \mathrm{d}v_{11}$$
$$\times \exp(-\rho^{22} v_{22} / 2) \exp(-g_2 v_{22}^{1/2}) v_{22}^{(n-2)/2} \mathrm{d}v_{22}$$
$$\vdots$$
$$\times \exp(-\rho^{p-1,p-1} v_{p-1,p-1} / 2) \exp(-g_{p-1} v_{p-1,p-1}^{1/2}) v_{p-1,p-1}^{(n-2)/2} \mathrm{d}v_{p-1,p-1}$$
$$\times \exp(-\rho^{pp} v_{pp} / 2) v_{pp}^{(n-2)/2} \mathrm{d}v_{pp}$$

$$= C_p \,|\, \mathbf{R} \,|^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,\ldots,p)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii} \,,$$

which gives the requited result. Q.E.D.

Theorem 1. *When* $p = 2$ *to* 4, *the marginal joint pdf's of* $\mathbf{r}$ *under the elliptical distribution are*

(i) $p = 2$

$$f_2(r_{12} \,|\, \rho_{12}, n) = \frac{2^{n-2}(1-\rho_{12}^2)^{n/2}(1-r_{12}^2)^{(n-3)/2}}{\pi\,\Gamma(n-1)} \sum_{k_1=0}^{\infty} \Gamma\left(\frac{n+k_1}{2}\right)^2 \frac{(2\rho_{12} r_{12})^{k_1}}{k_1!} \,,$$

where $\Gamma(\cdot)^2 = \{\Gamma(\cdot)\}^2$, *which is well documented* (*e.g., Anderson*, 1958, 2003, *Theorem* 4.2.2; *Muirhead*, 1982, *Equation* (11), *Section* 5.1.3).

(ii) $p = 3$

$$f_3(r_{12}, r_{13}, r_{23} \,|\, \rho_{12}, \rho_{13}, \rho_{23}, n)$$
$$= \frac{(1 + 2r_{12}r_{13}r_{23} - r_{12}^2 - r_{13}^2 - r_{23}^2)^{(n-4)/2}}{2^{3n/2}(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)^{n/2}\,\Gamma_3(n/2)}$$
$$\times \sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_1+k_2-m_{11}}{2}\right)(-1)^{k_1+k_2}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_1+k_2-m_{11})/2}}$$
$$\times \frac{(\rho^{12} r_{12})^{m_{11}}(\rho^{13} r_{13})^{k_1-m_{11}}(\rho^{23} r_{23})^{k_2}}{m_{11}!(k_1-m_{11})!k_2!} \,,$$

where $\sum_{k_1,k_2=0}^{\infty}(\cdot)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(\cdot)$.

(iii) $p=4$

$$f_4(r_{12},r_{13},\ldots,r_{34}\mid\rho_{12},\rho_{13},\ldots,\rho_{34},n)$$
$$=\frac{|\mathbf{R}|^{(n-5)/2}}{2^{2n}\,|\mathbf{P}|^{n/2}\,\Gamma_4(n/2)}\sum_{k_1,k_2,k_3=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}(-1)^{k_1+k_2+k_3}$$
$$\times\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)\Gamma\left(\frac{n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}(\rho^{44}/2)^{(n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22})/2}}$$
$$\times\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{14}r_{14})^{k_1-m_{11}-m_{21}}(\rho^{23}r_{23})^{m_{22}}(\rho^{24}r_{24})^{k_2-m_{22}}(\rho^{34}r_{34})^{k_3}}{m_{11}!\,m_{21}!\,(k_1-m_{11}-m_{21})!\,m_{22}!\,(k_2-m_{22})!\,k_3!}.$$

Proof. It is known that the pdf of **r** under normality holds under the elliptical distribution (for the bivariate case, see Ali & Joarder, 1991, Theorem; for the *p*-variate case, Joarder & Ali, 1992, Theorem 3.1). Then, the pdf's are derived under normality.

(i) $p=2$: In Lemma 1, the first integral with respect to v_{11} is given as

$$\int_0^{\infty}\exp\left(-\frac{\rho^{11}v_{11}}{2}\right)\exp(-g_1v_{11}^{1/2})v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$
$$=\sum_{k_1=0}^{\infty}\int_0^{\infty}\exp\left(-\frac{\rho^{11}v_{11}}{2}\right)\frac{(-g_1)^{k_1}v_{11}^{(n+k_1-2)/2}}{k_1!}\mathrm{d}v_{11}$$
$$=\sum_{k_1=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-g_1)^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}.$$

Using this result, the integral in Lemma 1 up to v_{22} becomes

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,2)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_iv_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$
$$=\int_0^{\infty}\sum_{k_1=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-g_1)^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}\exp\left(-\frac{\rho^{22}v_{22}}{2}-g_2v_{22}^{1/2}\right)v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$
$$=\int_0^{\infty}\sum_{k_1,k_2=0}^{\infty}\frac{\Gamma\{(n+k_1)/2\}(-\sum_{j=2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2}k_1!}\exp\left(-\frac{\rho^{22}v_{22}}{2}\right)\frac{(-g_2)^{k_2}v_{22}^{k_2/2}}{k_2!}v_{22}^{(n-2)/2}\mathrm{d}v_{22}$$

$$= \sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} \frac{\Gamma\{(n+k_1)/2\}}{(\rho^{11}/2)^{(n+k_1)/2} k_1! k_2!} (-g_2)^{k_2}$$

$$\times (-1)^{k_1} \binom{k_1}{m_{11}} \left(\sum\nolimits_{j=3}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2}\right)^{k_1-m_{11}} \int_0^{\infty} (\rho^{12} r_{12} v_{22}^{1/2})^{m_{11}} \exp\left(-\frac{\rho^{22} v_{22}}{2}\right) v_{22}^{(n+k_2-2)/2} \mathrm{d}v_{22}$$

$$= \sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)(-1)^{k_1+k_2} (\rho^{12} r_{12})^{m_{11}} \left(\sum\nolimits_{j=3}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2}\right)^{k_1-m_{11}}}{(\rho^{11}/2)^{(n+k_1)/2} (\rho^{22}/2)^{(n+k_2+m_{11})/2} (k_1-m_{11})! k_2! m_{11}!} g_2^{k_2}.$$

In the above result, when $p = 2$, we have $\left(\sum\nolimits_{j=3}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2}\right)^{k_1-m_{11}} = 0^{k_1-m_{11}}$ and $g_2^{k_2} = 0^{k_2}$, which are defined to be 1 only when $k_1 - m_{11} = 0$ and $k_2 = 0$ otherwise 0 due to the properties of the binominal and exponential expansions, respectively. Consequently, when $p = 2$, the above integral becomes simplified:

$$\int_0^{\infty} \prod^{i(1,2)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii} = \sum_{k_1=0}^{\infty} \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_1}{2}\right)(-1)^{k_1} (\rho^{12} r_{12})^{k_1}}{(\rho^{11}/2)^{(n+k_1)/2} (\rho^{22}/2)^{(n+k_1)/2} k_1!}$$

$$= 2^n (1-\rho_{12}^2)^n \sum_{k_1=0}^{\infty} \Gamma\left(\frac{n+k_1}{2}\right)^2 \frac{(2\rho_{12} r_{12})^{k_1}}{k_1!}.$$

Using the above expression we obtain

$$f_2(r_{12} \mid \rho_{12}, n) = C_2 (1-r_{12}^2)^{(n-3)/2} \int_0^{\infty} \prod^{i(1,2)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii}$$

$$= \frac{(1-r_{12}^2)^{(n-3)/2}}{2^n (1-\rho_{12}^2)^{n/2} \Gamma_2(n/2)} 2^n (1-\rho_{12}^2)^n \sum_{k_1=0}^{\infty} \Gamma\left(\frac{n+k_1}{2}\right)^2 \frac{(2\rho_{12} r_{12})^{k_1}}{k_1!}$$

$$= \frac{2^{n-2} (1-\rho_{12}^2)^{n/2} (1-r_{12}^2)^{(n-3)/2}}{\pi \Gamma(n-1)} \sum_{k_1=0}^{\infty} \Gamma\left(\frac{n+k_1}{2}\right)^2 \frac{(2\rho_{12} r_{12})^{k_1}}{k_1!}.$$

In the above result, $\Gamma_2(n/2) = \pi^{1/2} \Gamma\{(n-1)/2\} \Gamma(n/2) = 2^{-(n-2)} \pi \Gamma(n-1)$ is used with the Legendre duplication formula $\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\{z + (1/2)\}$ (see Erdélyi, 1953a, Section 1.2, Equation (15); DLMF, 2021, Equation 5.5.5, https://dlmf.nist.gov/5.5.E5).

(ii) $p = 3$: The integral in Lemma 1 up to v_{33} is obtained as

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,2,3)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)(-1)^{k_1+k_2}(\rho^{12}r_{12})^{m_{11}}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(k_1-m_{11})!k_2!m_{11}!}$$

$$\times\int_0^{\infty}g_2^{k_2}(\textstyle\sum_{j=3}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}}\exp\left(-\frac{\rho^{33}v_{33}}{2}-g_3 v_{33}^{1/2}\right)v_{33}^{(n-2)/2}\mathrm{d}v_{33}$$

$$=\sum_{k_1,k_2,k_3=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)(-1)^{k_1+k_2+k_3}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}}$$

$$\times\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}(\sum_{j=4}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-m_{21}}(\sum_{j=4}^{p}\rho^{2j}r_{2j}v_{jj}^{1/2})^{k_2-m_{22}}}{(k_1-m_{11}-m_{21})!(k_2-m_{22})!m_{11}!m_{21}!m_{22}!}g_3^{k_3}.$$

As before, when $p = 3$, the above integral vanishes unless $k_1-m_{11}-m_{21}=0$, $k_2-m_{22}=0$ and $k_3=0$, which gives

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,2,3)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\sum_{k_1,k_2=0}^{\infty}\sum_{m_{11}=0}^{k_1}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_1+k_2-m_{11}}{2}\right)(-1)^{k_1+k_2}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_1+k_2-m_{11})/2}}$$

$$\times\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{k_1-m_{11}}(\rho^{23}r_{23})^{k_2}}{m_{11}!(k_1-m_{11})!k_2!},$$

yielding the required result.

(iii) $p = 4$: The integral up to v_{44} is derived as

$$\int_{\mathbf{0}}^{\infty}\prod^{i(1,\ldots,4)}\exp\left(-\frac{\rho^{ii}v_{ii}}{2}-g_i v_{ii}^{1/2}\right)v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}$$

$$=\sum_{k_1,k_2,k_3=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)(-1)^{k_1+k_2+k_3}}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}}$$

$$\times\int_0^{\infty}\frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}(\sum_{j=4}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-m_{21}}(\sum_{j=4}^{p}\rho^{2j}r_{2j}v_{jj}^{1/2})^{k_2-m_{22}}}{(k_1-m_{11}-m_{21})!(k_2-m_{22})!m_{11}!m_{21}!m_{22}!}g_3^{k_3}$$

$$\times\exp\left(-\frac{\rho^{44}v_{44}}{2}-g_4 v_{44}^{1/2}\right)v_{44}^{(n-2)/2}\mathrm{d}v_{44}$$

$$
= \sum_{k_1,\ldots,k_4=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} \sum_{m_{31}=0}^{k_1-m_{11}-m_{21}} \sum_{m_{32}=0}^{k_2-m_{22}} \sum_{m_{33}=0}^{k_3} (-1)^{k_1+\ldots+k_4}
$$

$$
\times \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)\Gamma\left(\frac{n+k_4+m_{31}+m_{32}+m_{33}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}(\rho^{44}/2)^{(n+k_4+m_{31}+m_{32}+m_{33})/2}}
$$

$$
\times(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}(\rho^{14}r_{14})^{m_{31}}(\rho^{24}r_{24})^{m_{32}}(\rho^{34}r_{34})^{m_{33}}
$$

$$
\times \frac{(\sum_{j=5}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-m_{21}-m_{31}}(\sum_{j=5}^{p}\rho^{2j}r_{2j}v_{jj}^{1/2})^{k_2-m_{22}-m_{32}}(\sum_{j=5}^{p}\rho^{3j}r_{3j}v_{jj}^{1/2})^{k_3-m_{33}}}{(k_1-m_{11}-m_{21}-m_{31})!(k_2-m_{22}-m_{32})!(k_3-m_{33})!m_{11}!m_{21}!\cdots m_{33}!} g_4^{k_4}.
$$

As before when $p = 4$, the above integral vanishes unless $k_1 - m_{11} - m_{21} - m_{31} = 0$, $k_2 - m_{22} - m_{32} = 0$, $k_3 - m_{33} = 0$ and $k_4 = 0$, which gives

$$
\int_0^\infty \prod^{i(1,\ldots,4)} \exp\left(-\frac{\rho^{ii}v_{ii}}{2} - g_i v_{ii}^{1/2}\right) v_{ii}^{(n-2)/2}\mathrm{d}v_{ii}
$$

$$
= \sum_{k_1,k_2,k_3=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} (-1)^{k_1+k_2+k_3}
$$

$$
\times \frac{\Gamma\left(\frac{n+k_1}{2}\right)\Gamma\left(\frac{n+k_2+m_{11}}{2}\right)\Gamma\left(\frac{n+k_3+m_{21}+m_{22}}{2}\right)\Gamma\left(\frac{n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2}(\rho^{22}/2)^{(n+k_2+m_{11})/2}(\rho^{33}/2)^{(n+k_3+m_{21}+m_{22})/2}(\rho^{44}/2)^{(n+k_1+k_2+k_3-m_{11}-m_{21}-m_{22})/2}}
$$

$$
\times \frac{(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{14}r_{14})^{k_1-m_{11}-m_{21}}(\rho^{23}r_{23})^{m_{22}}(\rho^{24}r_{24})^{k_2-m_{22}}(\rho^{34}r_{34})^{k_3}}{m_{11}!m_{21}!(k_1-m_{11}-m_{21})!m_{22}!(k_2-m_{22})!k_3!}.
$$

Then, the required result follows. Q.E.D.

Theorem 2. *The marginal joint pdf of* **r** *under the p-variate elliptical distribution is*

$$
f_p(\mathbf{r}\,|\,\boldsymbol{\rho},n)
$$

$$
= C_p\,|\,\mathbf{R}\,|^{(n-p-1)/2}
$$

$$
\times \sum_{k_1,\ldots,k_{p-1}=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \cdots \sum_{m_{p-2,1}=0}^{k_1-m_{11}\cdots-m_{p-3,1}} \sum_{m_{22}=0}^{k_2} \cdots \sum_{m_{p-2,2}=0}^{k_2-m_{22}\cdots-m_{p-3,2}} \cdots \sum_{m_{p-2.p-2}=0}^{k_{p-2}} (-1)^{k_1+\ldots+k_{p-1}}
$$

$$
\times\left[\prod_{i=1}^{p-1}\Gamma\left(\frac{n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1}}{2}\right)/(\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1})/2}\right]
$$

$$
\times\Gamma\left(\frac{n+k_1+\ldots+k_{p-1}-m_{11}-\ldots.-m_{p-2,1}\ldots-m_{p-2,p-2}}{2}\right)/(\rho^{pp}/2)^{(n+k_1+\ldots+k_{i-1}-m_{11}-\ldots-m_{p-2,1}-\ldots-m_{p-2,p-2})/2}
$$

$$
\times\prod_{i=1}^{p-1}\left\{\prod_{j=i+1}^{p-1}\frac{(\rho^{ij}r_{ij})^{m_{j-1,i}}}{m_{j-1,i}!}\right\}\frac{(\rho^{ip}r_{ip})^{k_i-m_{ii}-\ldots-m_{p-2,i}}}{(k_i-m_{ii}-\ldots-m_{p-2,i})!},
$$

where $m_{0j} = 0$ $(j = 0,1)$.

Proof. The derivation is shown by induction. The results of the integral up to v_{44} are given in Theorem 1. Assume that the integral up to q in Lemma 1 holds as shown below. Then, the result up to $v_{q+1,q+1}$ is given as follows:

$$\sum_{k_1,\ldots,k_q=0}^{\infty}\sum_{m_{11}=0}^{k_1}\sum_{m_{21}=0}^{k_1-m_{11}}\sum_{m_{22}=0}^{k_2}\cdots\sum_{m_{q-1,1}=0}^{k_1-m_{11}\cdots-m_{q-2,1}}\sum_{m_{q-1,2}=0}^{k_2-m_{22}\cdots-m_{q-2,2}}\cdots\sum_{m_{q-1.q-1}=0}^{k_{q-1}}(-1)^{k_1+\ldots+k_q}$$
$$\times\left[\prod_{i=1}^{q}\Gamma\left(\frac{n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1}}{2}\right)/(\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1})/2}\right]$$
$$\times(\rho^{12}r_{12})^{m_{11}}(\rho^{13}r_{13})^{m_{21}}(\rho^{23}r_{23})^{m_{22}}\cdots(\rho^{1q}r_{1q})^{m_{q-1,1}}\cdots(\rho^{q-1,q}r_{q-1,q})^{m_{q-1,q-1}}$$
$$\times\int_0^{\infty}\frac{(\sum_{j=q+1}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}\cdots-m_{q-1,1}}\cdots(\sum_{j=q+1}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}}}{(k_1-m_{11}\cdots-m_{q-1,1})!\cdots(k_{q-1}-m_{q-1.q-1})!m_{11}!m_{21}!\cdots m_{q-1,q-1}!}g_q^{k_q}$$
$$\times\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}-g_{q+1}v_{q+1,q+1}^{1/2}\right)v_{q+1,q+1}^{(n-2)/2}\mathrm{d}v_{q+1,q+1}.$$

The numerator of the above integrand is expanded as

$$(\sum_{j=q+1}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-,,,-m_{q-1,1}}\cdots(\sum_{j=q+1}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}}$$
$$=\sum_{m_{q1}=0}^{k_1-m_{11}-\ldots-m_{q-1,1}}\binom{k_1-m_{11}-\ldots-m_{q-1,1}}{m_{q1}}(\rho^{1,q+1}r_{1,q+1})^{m_{q1}}v_{q+1,q+1}^{m_{q1}/2}(\sum_{j=q+2}^{p}\rho^{1j}r_{1j}v_{jj}^{1/2})^{k_1-m_{11}-\ldots-m_{q1}}$$
$$\cdots\sum_{m_{q,q-1}=0}^{k_{q-1}}\binom{k_{q-1}-m_{q-1.q-1}}{m_{q,q-1}}(\rho^{q-1,q+1}r_{q-1,q+1})^{m_{q,q-1}}v_{q+1,q+1}^{m_{q,q-1}/2}(\sum_{j=q+2}^{p}\rho^{q-1,j}r_{q-1,j}v_{jj}^{1/2})^{k_{q-1}-m_{q-1.q-1}-m_{q,q-1}}.$$

The factor $g_q^{k_q}$ in the integrand becomes

$$g_q^{k_q}=(\sum_{j=q+1}^{p}\rho^{qj}r_{qj}v_{jj}^{1/2})^{k_q}$$
$$=\sum_{m_{qq}=0}^{k_q}\binom{k_q}{m_{qq}}(\rho^{q,q+1}r_{q,q+1})^{m_{qq}}v_{q+1,q+1}^{m_{qq}/2}(\sum_{j=q+2}^{p}\rho^{qj}r_{qj}v_{jj}^{1/2})^{k_q-m_{qq}}.$$

The remaining factor in the integrand is also expanded:

$$\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}-g_{q+1}v_{q+1,q+1}^{1/2}\right)v_{q+1,q+1}^{(n-2)/2}$$
$$=\exp\left(-\frac{\rho^{q+1,q+1}v_{q+1,q+1}}{2}\right)\sum_{k_{q+1}=0}^{\infty}\frac{(-1)^{k_{q+1}}g_{q+1}^{k_{q+1}}v_{q+1,q+1}^{(n+k_{q+1}-2)/2}}{k_{q+1}!}.$$

Then, the integral becomes

$$\int_0^{\infty} \frac{(\sum_{j=q+1}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2})^{k_1-m_{11}-\dots-m_{q-1,1}} \cdots (\sum_{j=q+1}^{p} \rho^{q-1,j} r_{q-1,j} v_{jj}^{1/2})^{k_{q-1}}}{(k_1-m_{11}-\dots-m_{q-1,1})! \cdots (k_{q-1}-m_{q-1.q-1})! m_{11}! m_{21}! \cdots m_{q-1,q-1}!} g_q^{k_q}$$

$$\times \exp\left(-\frac{\rho^{q+1,q+1} v_{q+1,q+1}}{2} - g_{q+1} v_{q+1,q+1}^{1/2}\right) v_{q+1,q+1}^{(n-2)/2} \mathrm{d} v_{q+1,q+1}$$

$$= \sum_{k_{q+1}=0}^{\infty} \sum_{m_{q1}=0}^{k_1-m_{11}\cdots-m_{q-1,1}} \cdots \sum_{m_{q,q-1}=0}^{k_{q-1}} \sum_{m_{qq}=0}^{k_q} (-1)^{k_{q+1}} \binom{k_1-m_{11}-\dots-m_{q-1,1}}{m_{q1}} \cdots \binom{k_{q-1}-m_{q-1,q-1}}{m_{q,q-1}} \binom{k_q}{m_{qq}}$$

$$\times (\rho^{1,q+1} r_{1,q+1})^{m_{q1}} \cdots (\rho^{q-1,q+1} r_{q-1,q+1})^{m_{q,q-1}} (\rho^{q,q+1} r_{q,q+1})^{m_{qq}}$$

$$\times \int_0^{\infty} \exp\left(-\frac{\rho^{q+1,q+1} v_{q+1,q+1}}{2}\right) v_{q+1,q+1}^{(n+k_{q+1}+m_{q1}+\cdots+m_{qq}-2)/2} \mathrm{d} v_{q+1,q+1}$$

$$\times (\sum_{j=q+2}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2})^{k_1-m_{11}-\dots-m_{q1}} \cdots (\sum_{j=q+2}^{p} \rho^{1j} r_{1j} v_{jj}^{1/2})^{k_{q-1}-m_{q-1,q-1}-m_{q,q-1}} (\sum_{j=q+2}^{p} \rho^{qj} r_{qj} v_{jj}^{1/2})^{k_q-m_{qq}}$$

$$\times g_{q+1}^{k_{q+1}}.$$

Note that in the above result, the integral becomes

$$\int_0^{\infty} \exp\left(-\frac{\rho^{q+1,q+1} v_{q+1,q+1}}{2}\right) v_{q+1,q+1}^{(n+k_{q+1}+m_{q1}+\dots+m_{qq}-2)/2} \mathrm{d} v_{q+1,q+1}$$

$$= \Gamma\left(\frac{n+k_{q+1}+m_{q1}+\dots+m_{qq}}{2}\right) / (\rho^{q+1,q+1}/2)^{(n+k_{q+1}+m_{q1}+\dots+m_{qq})/2}$$

and when $q+1=p$, the whole above result vanishes unless $k_1-m_{11}\dots-m_{q1}=0,\cdots,$ $k_{q-1}-m_{q-1,q-1}-m_{q,q-1}=0$, $k_q-m_{qq}=0$ and $k_{q+1}=0$. Among the expanded factorials for $\binom{k_1-m_{11}-\dots-m_{q-1,1}}{m_{q1}} \cdots \binom{k_{q-1}-m_{q-1,q-1}}{m_{q,q-1}} \binom{k_q}{m_{qq}}$, those of the numerator are canceled and the half set of the factorials in the denominator become 0! = 1 due to $k_1-m_{11}-\dots-m_{q1}=0$, $\dots, k_{q-1}-m_{q-1,q-1}-m_{q,q-1}=0$ and $k_q-m_{qq}=0$ leaving only $m_{q1}!\cdots m_{q,q-1}! m_{qq}!$. Then, we have

$$
\begin{aligned}
& f_p(\mathbf{r} \mid \boldsymbol{\rho}, n) \\
& = C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \\
& \times \sum_{k_1,\ldots,k_p=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \sum_{m_{22}=0}^{k_2} \cdots \sum_{m_{p-1,1}=0}^{k_1-m_{11}-\ldots-m_{p-2,1}} \sum_{m_{p-1,2}=0}^{k_2-m_{22}-\ldots-m_{p-2,2}} \cdots \sum_{m_{p-1.p-1}=0}^{k_{p-1}} (-1)^{k_1+\ldots+k_p} \\
& \times \left[\prod_{i=1}^{p} \Gamma\left(\frac{n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1}}{2} \right) / (\rho^{ii}/2)^{(n+k_i+m_{i-1,1}+\ldots+m_{i-1,i-1})/2} \right] \\
& \times \frac{(\rho^{12} r_{12})^{m_{11}} (\rho^{13} r_{13})^{m_{21}} (\rho^{23} r_{23})^{m_{22}} \cdots (\rho^{1p} r_{1p})^{m_{p-1,1}} \cdots (\rho^{p-1,p} r_{p-1.p})^{k_{p-1}}}{m_{11}! m_{21}! \cdots m_{p-1,p-2}! k_{p-1}!} \\
& = C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \\
& \times \sum_{k_1,\ldots,k_{p-1}=0}^{\infty} \sum_{m_{11}=0}^{k_1} \sum_{m_{21}=0}^{k_1-m_{11}} \cdots \sum_{m_{p-2,1}=0}^{k_1-m_{11}-\ldots-m_{p-3,1}} \sum_{m_{22}=0}^{k_2} \cdots \sum_{m_{p-2,2}=0}^{k_2-m_{22}-\ldots-m_{p-3,2}} \cdots \sum_{m_{p-2.p-2}=0}^{k_{p-2}} (-1)^{k_1+\ldots+k_{p-1}} \\
& \times \frac{\Gamma\left(\frac{n+k_1}{2}\right) \Gamma\left(\frac{n+k_2+m_{11}}{2}\right) \cdots \Gamma\left(\frac{n+k_1+\ldots+k_{p-1}-m_{11}-\ldots-m_{p-2,1}-\ldots-m_{p-2,p-2}}{2}\right)}{(\rho^{11}/2)^{(n+k_1)/2} (\rho^{22}/2)^{(n+k_2+m_{11})/2} \cdots (\rho^{pp}/2)^{(n+k_1+\ldots+k_{p-1}-m_{11}-\ldots-m_{p-2,1}-\ldots-m_{p-2,p-2})/2}} \\
& \times \frac{(\rho^{12} r_{12})^{m_{11}} \cdots (\rho^{1p} r_{1p})^{k_1-m_{11}-\ldots-m_{p-2,1}} (\rho^{23} r_{23})^{m_{22}} \cdots (\rho^{2p} r_{24})^{k_2-m_{22}-\ldots-m_{p-2,2}} \cdots (\rho^{p-1.p} r_{p-1.p})^{k_{p-1}}}{m_{11}! \cdots (k_1-m_{11}-\ldots-m_{p-2,1})! m_{22}! \cdots (k_2-m_{22}-\ldots-m_{p-2,2})! \cdots k_{p-1}!},
\end{aligned}
$$

which gives the required result. Q.E.D.

The results of Theorem 1 can be seen as corollaries of Theorem 2. However, to have the relationship to the known result in the bivariate case and the initial condition(s) of the induction and to illustrate the sequential structure explicitly in the cases with small p's of practical importance, they are given as a theorem prior to Theorem 2 for the general p-variate cases.

3. Remarks on Theorems 1 and 2

The results of Theorems 1 and 2 show that they are given by $(p-1)$-fold infinite series along with their associated $\{(p-1)(p-2)/2\}$-fold nested series, which soon become excessively complicated when p becomes large. Joarder and Ali (1992, pp. 1958-1962) gave the results corresponding to those in Theorem 2 using different methods and expressions. Their results use a single infinite series with nested series and looks simpler than those in Theorem 2. However, two sets of expressions are equivalent or comparable in that our expressions can be summarized in a single infinite series when necessary. For instance, when $p = 3$, define $k_{12} \equiv k_1 + k_2$. Then we have

$$\sum_{k_1,k_2=0}^{\infty} \sum_{m_{11}=0}^{k_1} t(k_1, k_2, k_1 + k_2, m_{11}) = \sum_{k_{12}=0}^{\infty} \sum_{k_1=0}^{k_{12}} \sum_{m_{11}=0}^{k_1} t(k_1, k_{12} - k_1, k_{12}, m_{11}),$$

where $t(k_1, k_2, k_{12}, m_{11})$ is a function of the four arguments when $k_{12} = k_1 + k_2$ is temporarily seen as an independent variable.

Similar series expansions are available by change of variables in integration. Recall that the integral expression in Lemma 1 with respect to v_{ii} is also given by $v_i \equiv v_{ii}^{1/2}\ (i = 1, \ldots, p)$:

$$\begin{aligned} & f_p(\mathbf{r} \mid \boldsymbol{\rho}, n) \\ &= C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,\ldots,p)} \exp\left(-\frac{\rho^{ii} v_{ii}}{2} - g_i v_{ii}^{1/2} \right) v_{ii}^{(n-2)/2} \mathrm{d}v_{ii} \\ &= C_p \mid \mathbf{R} \mid^{(n-p-1)/2} \int_{\mathbf{0}}^{\infty} \prod^{i(1,\ldots,p)} 2 \exp\left(-\frac{\rho^{ii} v_i^2}{2} - g_i v_i \right) v_i^{n-1} \mathrm{d}v_i, \end{aligned}$$

where the first integral with respect to v_1 becomes

$$\int_0^{\infty} \exp\left(-\frac{\rho^{11}}{2} v_1^2 - g_1 v_1 \right) v_1^{n-1} \mathrm{d}v_1 = (\rho^{11})^{-n/2} \Gamma(n) e^{g_1^2/(4\rho^{11})} D_{-n}(g_1 / \sqrt{\rho^{11}})$$

(Erdélyi, 1954, Section 6.3, Equation (13); Zwillinger, 2015, Section 3.462, Equation 1), where $D_{-n}(\cdot)$ is the parabolic cylinder function using the traditional Whittaker notation (Erdélyi, 1953b, Chapter 8; Magnus, Oberhettinger & Soni, 1966, Chapter VIII; Zwillinger, 2015, Sections 9.24-9.25; DLMF, 2021, Chapter 12), whose series expression is given by

$$D_\nu(z) = 2^{\nu/2} e^{-z^2/4} \left\{ \frac{\sqrt{\pi}}{\Gamma\{(1-\nu)/2\}} {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2} \right) - \frac{\sqrt{2\pi}\, z}{\Gamma(-\nu/2)} {}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; \frac{z^2}{2} \right) \right\},$$

where

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}$$

is the Kummer confluent hypergeometric function (Winkelbauer, 2014, Equation (6); Zwillinger, 2015, Section 9.210, Equation 1; DLMF, 2021, Chapter 13); and $(a)_k = a(a+1)\cdots(a+k-1)$ is the rising or ascending factorial using the Pochhammer symbol when k is a non-negative integer with $(a)_0 = 1\ (a \neq 0)$ otherwise $(a)_k = \Gamma(a+k)/\Gamma(a)\ (a>0, k>0)$.

Using the series expression, we obtain

$$\int_0^\infty \exp\left(-\frac{\rho^{11}}{2}v_1^2 - g_1 v_1\right) v_1^{n-1} \mathrm{d}v_1$$
$$= (\rho^{11})^{-n/2}\Gamma(n) e^{g_1^2/(4\rho^{11})} D_{-n}(g_1/\sqrt{\rho^{11}})$$
$$= (2\rho^{11})^{-n/2}\Gamma(n)$$
$$\times\left\{\frac{\sqrt{\pi}}{\Gamma\{(n+1)/2\}}\, {}_1F_1\left(\frac{n}{2};\ \frac{1}{2};\ \frac{g_1^2}{2\rho^{11}}\right) - \frac{\sqrt{2\pi}g_1/\sqrt{\rho^{11}}}{\Gamma(n/2)}\, {}_1F_1\left(\frac{n+1}{2};\ \frac{3}{2};\ \frac{g_1^2}{2\rho^{11}}\right)\right\}.$$

It is intriguing to use this expression since $D_{-n}(\cdot)$ and ${}_1F_1(\cdot)$ are known functions with positive variable $g_1^2/(2\rho^{11})$ in ${}_1F_1(\cdot)$, expecting stable convergence. However, $g_1 = \sum_{j=2}^p \rho^{1j} r_{1j} v_{jj}^{1/2} = \sum_{j=2}^p \rho^{1j} r_{1j} v_j$ is a function of $v_i\ (i = 2,\ldots,p)$, which will be variables for integration in the subsequent stages. Consequently, term-by-term integration in ${}_1F_1(\cdot)$ will again be required unless some simplification is derived.

The positive variable used in ${}_1F_1(\cdot)$ for the integral with respect to v_i's can similarly be employed using v_{ii}'s, when desired, by considering even and odd terms as

$$\int_0^\infty \exp(-\rho^{11}v_{11}/2)\exp(-g_1 v_{11}^{1/2}) v_{11}^{(n-2)/2}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^\infty \int_0^\infty \exp(-\rho^{11}v_{11}/2)\frac{(-g_1)^{k_1} v_{11}^{(n+k_1-2)/2}}{k_1!}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^\infty \int_0^\infty \exp(-\rho^{11}v_{11}/2)\left\{\frac{g_1^{2k_1} v_{11}^{(n+2k_1-2)/2}}{(2k_1)!} - \frac{g_1^{2k_1+1} v_{11}^{(n+2k_1-1)/2}}{(2k_1+1)!}\right\}\mathrm{d}v_{11}$$
$$= \sum_{k_1=0}^\infty \left\{\frac{\Gamma\{(n+2k_1)/2\} g_1^{2k_1}}{(\rho^{11}/2)^{(n+2k_1)/2}(2k_1)!} - \frac{\Gamma\{(n+2k_1+1)/2\} g_1^{2k_1+1}}{(\rho^{11}/2)^{(n+2k_1+1)/2}(2k_1+1)!}\right\}$$

$$=\left(\frac{2}{\rho^{11}}\right)^{n/2}\sum_{k_1=0}^{\infty}\left\{\frac{\sqrt{\pi}\Gamma\{(n+2k_1)/2\}(g_1/\sqrt{2\rho^{11}})^{2k_1}}{\Gamma\{(2k_1+1)/2\}k_1!}\right.$$

$$\left.-\frac{\sqrt{\pi}\Gamma\{(n+2k_1+1)/2\}(g_1/\sqrt{2\rho^{11}})^{2k_1+1}}{\Gamma\{(2k_1+3)/2\}k_1!}\right\}$$

$$=\left(\frac{2}{\rho^{11}}\right)^{n/2}\sqrt{\pi}\sum_{k_1=0}^{\infty}\Big[\{(2k_1+1)/2\}_{(n-1)/2}(g_1/\sqrt{2\rho^{11}})^{2k_1}$$

$$-\{(2k_1+3)/2\}_{(n-2)/2}(g_1/\sqrt{2\rho^{11}})^{2k_1+1}\Big]/k_1!.$$

Closed form formulas for the product moments of **r** or raw moments of $|\mathbf{R}|$ (the scatter coefficient) are not given as applications of the main results, which are open problems. The integrals in Theorem 2 are given sequentially in element-wise methods yielding multiple series. Recently, the Holonomic gradient method to have the Fisher-Bingham integral was developed (see Nakayama et al., 2011), which is seen as a multivariate counterpart of the scaled parabolic cylinder function using an integral expression. This method has been applied to the distributions of statistics associated with the Wishart (Hashiguchi et al, 2013; Shimizu & Hashiguchi, 2019). Mura et al. (2019) showed the Holonomic property of the distribution of the sample correlation coefficient in the bivariate Wishart based on the formula of Hotelling (1953, Equation (25)) using the Gauss hypergeometric series (Abramowitz & Stegun, 1964/2014, Section 15.1; Zwillinger, 2015, Sections 9.1; Hankin, 2016; DLMF, 2021, Section 15.2 (i)). This finding suggests a similar property in the p-variate case and an efficient method to have the infinite series with a matrix argument (see also Pham-Gia & Choulakian, 2014; Pham-Gia & Thanh, 2016).

4. The density of the sample correlation coefficient with the truncated variance-ratio under elliptical symmetry

Let $(X_{1j}, X_{2j})^{\mathrm{T}} \overset{\text{i.i.d.}}{\sim} \mathrm{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ $(j=1,\ldots,N)$ with $\boldsymbol{\Sigma}=\begin{pmatrix}\sigma_{11} & \sigma_1\sigma_2\rho \\ \sigma_2\sigma_1\rho & \sigma_{22}\end{pmatrix}$ and

$\sigma_i=\sigma_{ii}^{1/2}$ $(i=1, 2)$. Define $\bar{H}=\dfrac{\sum_{j=1}^{N}(X_{1j}-\bar{X}_1)^2/\sigma_{11}}{\sum_{j=1}^{N}(X_{2j}-\bar{X}_2)^2/\sigma_{22}}$ with $\bar{X}_i=\sum_{j=1}^{N}X_{ij}/N$ $(i=1, 2)$,

which is the ratio of the scaled or standardized sample variances or the ratio of two correlated chi-squared variables each with $n = N - 1$ degrees of freedom. The corresponding unscaled ratio is denoted by $H = \frac{\sum_{i=1}^{N}(X_{1j} - \bar{X}_1)^2}{\sum_{i=1}^{N}(X_{2j} - \bar{X}_2)^2} = \bar{H}\sigma_{11} / \sigma_{22}$.

We consider $\bar{H}$ whose realization is denoted by $\bar{h}$. Suppose that an observation is truncated/selected such that only when $\bigcup_{k=1}^{K}\{\bar{h} \in I_k\}$, the observation is selected otherwise truncated, where I_k's are non-overlapping intervals satisfying

$$I_k = [a_k, b_k) \text{ with } I_k = [a_k, b_k),\ 0 \le a_1 < b_1 < ... < a_K < b_K \le \infty\ (k = 1, ..., K)$$

and *K* being the number of the intervals. When $K = 1$ with $0 = a_1 < b_1 < \infty$ or $0 < a_1 < b_1 = \infty$, $\bar{H}$ is singly upper- or lower-truncated, respectively while when $K = 1$ with $0 < a_1 < b_1 < \infty$, $\bar{H}$ is doubly truncated. When $K = 2$ with $0 = a_1 < b_1 < a_2 < b_2 = \infty$, $\bar{H}$ is inner-truncated with the two tails being selected. These cases occur e.g., when the observation is discarded if $\bar{H}$ is too small or too large due to its possible irregularity. When truncation does not occur irrespective of the value of $\bar{H}$, $K = 1$ with $0 = a_1 < b_1 = \infty$ is employed. This truncation is similar to stripe truncation for univariate cases (Ogasawara, 2021a). Then, $\bar{H}$ is said to be stripely truncated in this paper.

Theorem 3. *Suppose that the scaled variance-ratio* $\bar{H}$ *is stripely truncated as above. Then, the pdf of the sample correlation coefficient under elliptical symmetry is*

$$f_R(r \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}) = C_R(1 - r^2)^{(n-3)/2}\ {}_1F_{0\mathrm{W}}\left\{n;\ ; 2\rho r; \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n,\ j), j = 0, 1, ...\right\},$$

where

$$C_R = \frac{2^{n-2}(n-1)\mathrm{B}(n/2, n/2)}{\pi\ {}_1F_{0\mathrm{W}}\left\{(n+1)/2;; 4\rho^2; \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n,\ 2j), j = 0, 1, ...\right\}}$$

is the normalizing constant; ${}_1F_{0\mathrm{W}}\{c;; z; w(\cdot)\} = \sum_{j=0}^{\infty}(c)_j z^j w(d^* j) / j!$ *is a weighted hypergeometric function with* d^* *being a constant or the weighted negative binomial expansion when the weight for the j-th term is* $w(d_j j) = \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, d^* j)$ (*if the weight is* 1, ${}_1F_{0\mathrm{W}}(c;; x; 1)$ *becomes the usual hypergeometric function or the negative binomial expansion* ${}_1F_0(c;; x) = \sum_{j=0}^{\infty}(c)_j x^j / j!$);

$$\mathrm{B}_{\mathbf{a}/(\mathbf{1-a})}^{\mathbf{b}/(\mathbf{1-b})}(n,d^* j)=\sum_{k=1}^{K}\left\{\mathrm{B}\left(\frac{b_k}{1+b_k},\frac{n+d^* j}{2},\frac{n+d^* j}{2}\right)-\mathrm{B}\left(\frac{a_k}{1+a_k},\frac{n+d^* j}{2},\frac{n+d^* j}{2}\right)\right\}$$

and $\mathrm{B}(x,p,q)=\int_0^x t^{p-1}(1-t)^{q-1}\mathrm{d}t$ *is the incomplete beta function.*

Proof. It is known that the pdf of $\bar{H}$ under normality holds under elliptical symmetry (Joarder, 2013, Theorem 5.1) as in the case of the sample correlation coefficient *R* as mentioned earlier (Ali & Joarder, 1991, Theorem). Although the joint distribution of $\bar{H}$ and *R* under normality has not explicitly been shown to hold under elliptical symmetry, an intermediate result of Theorem 5.1 of Joarder (2013, Equation (5.3)) shows that the joint pdf under elliptical symmetry does not depend on specific forms of elliptical distributions, which indicates the robustness of the pdf under normality. Then, we can use the bivariate Wishart distribution without loss of generality. The Wishart density when p = 2 with n degrees of freedom is

$$w_2(\mathbf{V}\mid\mathbf{\Sigma},n)=\frac{\exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\}\mid\mathbf{V}\mid^{(n-p-1)/2}}{2^{np/2}\mid\mathbf{\Sigma}\mid^{n/2}\Gamma_p(n/2)}$$
$$=\frac{\exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\}\mid\mathbf{V}\mid^{(n-3)/2}}{2^{n}\mid\mathbf{\Sigma}\mid^{n/2}\Gamma_2(n/2)}$$
$$=\frac{\exp\{-\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{V})/2\}\mid\mathbf{V}\mid^{(n-3)/2}}{4\pi\mid\mathbf{\Sigma}\mid^{n/2}\Gamma(n-1)}.$$

We use the change of (mathematical) variables from $\mathbf{v}=(v_{11},v_{22},v_{12})^{\mathrm{T}}$ to $\mathbf{u}=(u_{11},u_{22},r)^{\mathrm{T}}$,where $\mathbf{v}=(u_{11}\sigma_{11},u_{22}\sigma_{22},u_1u_2r\sigma_1\sigma_2)^{\mathrm{T}}$ with the Jacobian $\mathrm{d}\mathbf{v}=\sigma_{11}^{3/2}\sigma_{22}^{3/2}u_1u_2\mathrm{d}\mathbf{u}$ and $u_i=u_{ii}^{1/2}$ $(i=1,\ 2)$. Then, the pdf at **u** becomes

$$w_2(\mathbf{u}\mid\sigma_{11},\sigma_{22},\rho,n)=\frac{(\sigma_{11}\sigma_{22}u_{11}u_{22})^{(n-3)/2}(1-r^2)^{(n-3)/2}\sigma_{11}^{3/2}\sigma_{22}^{3/2}u_1u_2}{4\pi\mid\mathbf{\Sigma}\mid^{n/2}\Gamma(n-1)}$$
$$\times\exp\left\{-\frac{u_{11}+u_{22}-2\rho ru_1u_2}{2(1-\rho^2)}\right\}$$
$$=\frac{(u_{11}u_{22})^{(n-2)/2}(1-r^2)^{(n-3)/2}}{4\pi(1-\rho^2)^{n/2}\Gamma(n-1)}\exp\left\{-\frac{u_{11}+u_{22}-2\rho ru_1u_2}{2(1-\rho^2)}\right\},$$

which is algebraically equal to the known one (see e.g., Joarder, 2013, Equation (4.2)).

Employ the variable transformation from u_{11} to $\bar{h}=u_{11}/u_{22}$ with unchanged u_{22} and

r, where the Jacobian is $\mathrm{d}u_{11} = u_{22}\mathrm{d}\bar{h}$. Then, the pdf of $\bar{\mathbf{H}} = (\bar{H}, R)^{\mathrm{T}}$ at $\bar{\mathbf{h}} = (\bar{h}, r)^{\mathrm{T}}$ becomes

$$w_2(\bar{\mathbf{h}} \mid \sigma_{11}, \sigma_{22}, \rho, n) = \frac{\bar{h}^{(n-2)/2}(1-r^2)^{(n-3)/2}}{4\pi(1-\rho^2)^{n/2}\Gamma(n-1)} \int_0^\infty u_{22}^{n-1} \exp\left\{-u_{22}\frac{1+\bar{h}-2\rho r\bar{h}^{1/2}}{2(1-\rho^2)}\right\}\mathrm{d}u_{22}.$$

Using the transformation $z = u_{22}\dfrac{1+\bar{h}-2\rho r\bar{h}^{1/2}}{1-\rho^2}$ with the Jacobian

$\mathrm{d}u_{22} = \dfrac{1-\rho^2}{1+\bar{h}-2\rho r\bar{h}^{1/2}}\mathrm{d}z$, we have

$$\begin{aligned} w_2(\bar{\mathbf{h}} \mid \sigma_{11}, \sigma_{22}, \rho, n) &= \frac{\bar{h}^{(n-2)/2}(1-r^2)^{(n-3)/2}(1-\rho^2)^n}{4\pi(1-\rho^2)^{n/2}\Gamma(n-1)(1+\bar{h})^n}\left(1-\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^{-n}\int_0^\infty z^{n-1}\exp(-z/2)\mathrm{d}z \\ &= \frac{2^{n-2}(1-\rho^2)^{n/2}(n-1)(1-r^2)^{(n-3)/2}\bar{h}^{(n-2)/2}}{\pi(1+\bar{h})^n}\left(1-\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^{-n}, \end{aligned}$$

which is the joint pdf of $\bar{H}$ and R without truncation

Since $\bar{H}$ is stripely truncated by assumption, the normalizing constant is derived by integrating the above pdf with respect to $\bar{h}$ and r under truncation for $\bar{H}$. Define

$$w_2^*(\bar{\mathbf{h}}) = \frac{(1-r^2)^{(n-3)/2}\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left(1-\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^{-n}.$$

We start with integrating r using the binomial expansion:

$$\int_{-1}^1 w_2^*(\bar{\mathbf{h}})\,\mathrm{d}r = \frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\sum_{k=0}^\infty \frac{(n)_k}{k!}\left(\frac{2\rho\bar{h}^{1/2}}{1+\bar{h}}\right)^k \int_{-1}^1 r^k(1-r^2)^{(n-3)/2}\mathrm{d}r.$$

Since the above integrand is an odd function, the odd terms vanish. Then, noting that

$$\int_{-1}^1 r^{2k}(1-r^2)^{(n-3)/2}\mathrm{d}r = \int_0^1 r^{2\{k-(1/2)\}}(1-r^2)^{(n-3)/2}\mathrm{d}r^2 = \mathrm{B}\left(\frac{2k+1}{2}, \frac{n-1}{2}\right),$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function, we obtain

$$\int_{-1}^{1} w_2^*(\bar{\mathbf{h}})\,\mathrm{d}r = \frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\sum_{k=0}^{\infty}\frac{(n)_{2k}}{(2k)!}\left(\frac{2\rho\bar{h}^{1/2}}{1+\bar{h}}\right)^{2k}\mathrm{B}\left(\frac{2k+1}{2},\frac{n-1}{2}\right)$$
$$=\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\sum_{k=0}^{\infty}\frac{\Gamma(n+2k)/\Gamma(n)}{2^{2k}\pi^{-1/2}k!\Gamma\{(2k+1)/2\}}\left(\frac{2\rho\bar{h}^{1/2}}{1+\bar{h}}\right)^{2k}\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n+2k}{2}\right)$$
$$=\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\sum_{k=0}^{\infty}\frac{2^{n+2k-1}\pi^{-1/2}\Gamma\{(n+2k+1)/2\}}{2^{2k}\pi^{-1/2}k!\Gamma(n)}\left(\frac{2\rho\bar{h}^{1/2}}{1+\bar{h}}\right)^{2k}\Gamma\left(\frac{n-1}{2}\right)$$
$$=\frac{2^{n-1}\Gamma\{(n-1)/2\}\Gamma\{(n+1)/2\}}{\Gamma(n)}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\sum_{k=0}^{\infty}\frac{\Gamma\{(n+2k+1)/2\}}{\Gamma\{(n+1)/2\}k!}\left\{\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^k$$
$$=\frac{2\Gamma\{(n+1)/2\}}{\pi^{-1/2}\Gamma(n/2)(n-1)}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2},$$

where the Legendre duplication formula and inverse binomial expansion are used. Then, the pdf of untruncated $\bar{H}$ becomes

$$f_{\bar{H}}(\bar{h}\mid\sigma_{11},\sigma_{22},\rho,n)=\frac{2^{n-2}(1-\rho^2)^{n/2}(n-1)}{\pi}\int_{-1}^{1} w_2^*(\bar{\mathbf{h}})\,\mathrm{d}r$$
$$=\frac{2^{n-2}(1-\rho^2)^{n/2}(n-1)}{\pi}\frac{2\Gamma\{(n+1)/2\}}{\pi^{-1/2}\Gamma(n/2)(n-1)}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2}$$
$$=\frac{(1-\rho^2)^{n/2}2^{n-1}\Gamma\{(n+1)/2\}}{\pi^{1/2}\Gamma(n/2)}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2}$$
$$=\frac{(1-\rho^2)^{n/2}}{\mathrm{B}(n/2,n/2)}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2},$$

where $\frac{2^{n-1}\Gamma\{(n+1)/2\}}{\pi^{1/2}\Gamma(n/2)}=\frac{2^{n-1}\Gamma(n/2)\Gamma\{(n+1)/2\}}{\pi^{1/2}\Gamma(n/2)^2}=\frac{\Gamma(n)}{\Gamma(n/2)^2}=1/\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)$ is used with $\Gamma(\cdot)^2=\{\Gamma(\cdot)\}^2$.

One of the remaining tasks is to have the integral $f_{\bar{H}}(\bar{h}\mid\sigma_{11},\sigma_{22},\rho,n)$ with respect to $\bar{h}$ under stripe truncation. Define $\int_{\mathbf{a}}^{\mathbf{b}}(\cdot)\mathrm{d}\bar{h}\equiv\sum_{k=1}^{K}\int_{a_k}^{b_k}(\cdot)\mathrm{d}\bar{h}$ with $\mathbf{a}=(a_1,\ldots,a_K)^{\mathrm{T}}$ and $\mathbf{b}=(b_1,\ldots,b_K)^{\mathrm{T}}$. Then, we have

$$\int_{\mathbf{a}}^{\mathbf{b}} f_{\bar{H}}(\bar{h}\mid\sigma_{11},\sigma_{22},\rho,n)\mathrm{d}\bar{h}=\frac{(1-\rho^2)^{n/2}}{\mathrm{B}(n/2,n/2)}\int_{\mathbf{a}}^{\mathbf{b}}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2}\mathrm{d}\bar{h}.$$

The above integral becomes

$$\int_{\mathbf{a}}^{\mathbf{b}} \frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left\{1-\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^{-(n+1)/2} \mathrm{d}\bar{h} = \sum_{j=0}^{\infty}\int_{\mathbf{a}}^{\mathbf{b}} \frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n} \frac{\{(n+1)/2\}_j}{j!}\left\{\frac{4\rho^2\bar{h}}{(1+\bar{h})^2}\right\}^j \mathrm{d}\bar{h}$$

$$= \sum_{j=0}^{\infty}\frac{\{(n+1)/2\}_j(4\rho^2)^j}{j!}\int_{\mathbf{a}}^{\mathbf{b}}\frac{\bar{h}^{(n+2j-2)/2}}{(1+\bar{h})^{n+2j}}\mathrm{d}\bar{h}$$

$$= \sum_{j=0}^{\infty}\frac{\{(n+1)/2\}_j(4\rho^2)^j}{j!}\sum_{k=1}^{K}\left\{\mathrm{B}\left(\frac{b_k}{1+b_k}\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)-\mathrm{B}\left(\frac{a_k}{1+a_k}\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)\right\}$$

$$= \sum_{j=0}^{\infty}\frac{\{(n+1)/2\}_j(4\rho^2)^j}{j!}\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,2j),$$

where the incomplete beta function $\mathrm{B}(x\,|\,p,q)=\int_0^x t^{p-1}(1-t)^{q-1}\mathrm{d}t=\int_0^{x/(1-x)}\frac{t^{p-1}}{(1+t)^{p+q}}\mathrm{d}t$ is used. Then, the reciprocal of the normalizing constant is

$$\int_{\mathbf{a}}^{\mathbf{b}} f_{\bar{H}}(\bar{h}\,|\,\sigma_{11},\sigma_{22},\rho,n)\mathrm{d}\bar{h}$$

$$= \frac{(1-\rho^2)^{n/2}}{\mathrm{B}(n/2,n/2)}\sum_{j=0}^{\infty}\frac{\{(n+1)/2\}_j(4\rho^2)^j}{j!}\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,2j)$$

$$= \frac{(1-\rho^2)^{n/2}}{\mathrm{B}(n/2,n/2)}\,{}_1F_{0\mathrm{W}}\left\{(n+1)/2;;4\rho^2;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\,2j),\,j=0,1,...\right\}.$$

The remaining task is to have

$$\int_{\mathbf{a}}^{\mathbf{b}} w_2(\bar{\mathbf{h}}\,|\,\sigma_{11},\sigma_{22},\rho,n)\mathrm{d}\bar{h} = \frac{2^{n-2}(1-\rho^2)^{n/2}(n-1)}{\pi}\int_{\mathbf{a}}^{\mathbf{b}} w_2^*(\bar{\mathbf{h}})\,\mathrm{d}\bar{h}\,,$$

where recall that $w_2^*(\bar{\mathbf{h}})=\frac{(1-r^2)^{(n-3)/2}\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left(1-\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^{-n}$. The integral $\int_{\mathbf{a}}^{\mathbf{b}} w_2^*(\bar{\mathbf{h}})\,\mathrm{d}\bar{h}$ is obtained similarly as before.

$$\int_{\mathbf{a}}^{\mathbf{b}} w_2^*(\bar{\mathbf{h}})\mathrm{d}\bar{h} = \int_{\mathbf{a}}^{\mathbf{b}}\frac{(1-r^2)^{(n-3)/2}\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left(1-\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^{-n}\mathrm{d}\bar{h}$$

$$= (1-r^2)^{(n-3)/2}\sum_{j=0}^{\infty}\frac{(n)_j}{j!}\int_{\mathbf{a}}^{\mathbf{b}}\frac{\bar{h}^{(n-2)/2}}{(1+\bar{h})^n}\left(\frac{2\rho r\bar{h}^{1/2}}{1+\bar{h}}\right)^j\mathrm{d}\bar{h}$$

$$= (1-r^2)^{(n-3)/2}\sum_{j=0}^{\infty}\frac{(n)_j(2\rho r)^j}{j!}\int_{\mathbf{a}}^{\mathbf{b}}\frac{\bar{h}^{(n+j-2)/2}}{(1+\bar{h})^{n+j}}\mathrm{d}\bar{h}$$

$$= (1-r^2)^{(n-3)/2}\sum_{j=0}^{\infty}\frac{(n)_j(2\rho r)^j}{j!}\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,j)$$

$$= (1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\,j),\,j=0,1,...\right\},$$

which gives

$$\int_{\mathbf{a}}^{\mathbf{b}} w_2(\overline{\mathbf{h}} \mid \sigma_{11}, \sigma_{22}, \rho, n)\mathrm{d}\overline{h}$$
$$= \pi^{-1} 2^{n-2}(1-\rho^2)^{n/2}(n-1)(1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;\,;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\ j), j=0,1,\ldots\right\}.$$

Using the above results, we obtain the pdf of R with the stripely truncated variance-ratio:

$$f_R(r \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}) = \frac{\int_{\mathbf{a}}^{\mathbf{b}} w_2(\overline{\mathbf{h}} \mid \sigma_{11}, \sigma_{22}, \rho, n)\mathrm{d}\overline{h}}{\int_{\mathbf{a}}^{\mathbf{b}}\int_{-1}^{1} w_2(\overline{\mathbf{h}} \mid \sigma_{11}, \sigma_{22}, \rho, n)\mathrm{d}r\mathrm{d}\overline{h}}$$
$$= \frac{\pi^{-1} 2^{n-2}(1-\rho^2)^{n/2}(n-1)(1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;\,;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\ j), j=0,1,\ldots\right\}}{\{(1-\rho^2)^{n/2}/\mathrm{B}(n/2,n/2)\}\,{}_1F_{0\mathrm{W}}\left\{(n+1)/2;;4\rho^2;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\ 2j), j=0,1,\ldots\right\}}$$
$$= \frac{2^{n-2}(n-1)\mathrm{B}(n/2,n/2)(1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;\,;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\ j), j=0,1,\ldots\right\}}{\pi\,{}_1F_{0\mathrm{W}}\left\{(n+1)/2;;4\rho^2;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\ 2j), j=0,1,\ldots\right\}}$$
$$= C_R(1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;\,;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n, j), j=0,1,\ldots\right\},$$

which is the required result. Q.E.D.

Remark 1. Weighted hypergeometric functions similar to the weighted binomial expansion shown earlier are used by Ogasawara, (2021b, Equation (4) and (7)) when the unweighted cases are the Kummer confluent and Gauss hypergeometric functions.

Remark 2. The pdf of $\overline{H}$ i.e., $f_{\overline{H}}(\overline{h} \mid \sigma_{11}, \sigma_{22}, \rho, n)$ shown earlier under elliptical symmetry is known (Joarder, 2013, Theorem 5.1), which can also be derived from the pdf of $\overline{H}^{1/2}$ under normality (Bose, 1935, Equations (4.2) and (5.1); Finney, 1938, p. 191) using the Jacobian $\mathrm{d}\overline{h}^{1/2} = (\overline{h}^{-1/2}/2)\mathrm{d}\overline{h}$. The pleasantly simple expression of the normalizer $(1-\rho^2)^{n/2}/\mathrm{B}(n/2,n/2)$ in $f_{\overline{H}}(\overline{h} \mid \sigma_{11}, \sigma_{22}, \rho, n)$ was given by Bose (1935). Consequently, to have the normalizer for the pdf of R under the truncated variance-ratio, we can start with the known $f_{\overline{H}}(\overline{h} \mid \sigma_{11}, \sigma_{22}, \rho, n)$. However, the derivation does not seem to be well documented except Bose (1935) who used an associated result by Pearson (1925, Equation (viii)). As expected, it was found using a different self-contained method in the proof of Theorem 3 that the expression $(1-\rho^2)^{n/2}/\mathrm{B}(n/2,n/2)$ by Bose (1935) is correct.

Remark 3. The random variable $\overline{H} = \dfrac{\sum_{i=1}^{N}(X_{1j}-\overline{X}_1)^2 / \sum_{i=1}^{N}(X_{2j}-\overline{X}_1)^2}{\sigma_{11}/\sigma_{22}}$ depends on the population value $\sigma_{11/22} \equiv \sigma_{11}/\sigma_{22}$ unless $\sigma_{11} = \sigma_{22}$ as noted by Bose (1935, Equation (5.1)). So, the truncation by the variance ratio is assumed to be performed by some hypothesis about $\sigma_{11/22}$, which is weaker than those for σ_{11} and σ_{22}.

Remark 4. In the proof of Theorem 3, the reciprocal of the normalizing constant for the pdf of *R* with truncated $\bar{H}$ is given by

$$\begin{aligned}\alpha^* &\equiv \int_{\mathbf{a}}^{\mathbf{b}}\int_{-1}^{1} w_2(\overline{\mathbf{h}}\,|\,\sigma_{11},\sigma_{22},\rho,n)\mathrm{d}r\mathrm{d}\bar{h} \\ &= \frac{(1-\rho^2)^{n/2}}{\mathrm{B}(n/2,n/2)}\,{}_1F_{0\mathrm{W}}\left\{(n+1)/2;;4\rho^2;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\,2j),\,j=0,1,...\right\},\end{aligned}$$

which is the reduced probability due to truncation (the truncated probability for short).

Corollary 1. *Define* $0 \le a_k^* = a_k/(1+a_k) < 1$ *and* $0 \le b_k^* = b_k/(1+b_k) < 1$ $(k=1,...,K)$. *Suppose that the intervals for selection using* a_k^* *and* b_k^* *denoted by* $I_k^* = [a_k^*, b_k^*)$ *give the pattern such that* I_k^**'s from* 0 *to* 1/2 *are the same as those from* 1/2 *to* 1 *when reflected about* 1/2 *by exchanging the roles of selection and truncation* (*skew-symmetric truncation for short*). *The pdf of R given by Theorem 3 under the skew-symmetric truncation is equal to that without truncation*.

Proof. In the pdf

$$\begin{aligned}& f_R(r\,|\,\sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b}) \\ &= \frac{2^{n-2}(n-1)\mathrm{B}(n/2,n/2)(1-r^2)^{(n-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;;2\rho r;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\,j),\,j=0,1,...\right\}}{\pi\,{}_1F_{0\mathrm{W}}\left\{(n+1)/2;;4\rho^2;\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,\,2j),\,j=0,1,...\right\}},\end{aligned}$$

note that

$$\begin{aligned}\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,2j) &= \sum_{k=1}^{K}\left\{\mathrm{B}\left(\frac{b_k}{1+b_k}\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)-\mathrm{B}\left(\frac{a_k}{1+a_k}\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)\right\} \\ &= \sum_{k=1}^{K}\left\{\mathrm{B}\left(b_k^*\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)-\mathrm{B}\left(a_k^*\,|\,\frac{n+2j}{2},\frac{n+2j}{2}\right)\right\} \\ &\equiv \mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,2j)\end{aligned}$$

and

$$\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,j) = \sum_{k=1}^{K}\left\{\mathrm{B}\left(b_k^*\,|\,\frac{n+j}{2},\frac{n+j}{2}\right)-\mathrm{B}\left(a_k^*\,|\,\frac{n+j}{2},\frac{n+j}{2}\right)\right\} = \mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,j).$$

Since the integrand of the incomplete beta function $\mathrm{B}(x\,|\,q,q)=\int_0^x (t-t^2)^{q-1}\mathrm{d}t$ with equal parameters is symmetric about 1/2, the skew-symmetric truncation pattern gives

$$\mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,2j) = \frac{1}{2}\left\{\mathrm{B}\left(1 \,|\, \frac{n+2j}{2}, \frac{n+2j}{2}\right) - \mathrm{B}\left(0 \,|\, \frac{n+2j}{2}, \frac{n+2j}{2}\right)\right\}$$
$$= \frac{1}{2}\mathrm{B}\left(\frac{n+2j}{2}, \frac{n+2j}{2}\right) = \frac{1}{2}\mathrm{B}_0^1(n,2j)$$

and $\mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,j) = \frac{1}{2}\mathrm{B}_0^1(n,2j)$. That is, the factors $\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,j) = \mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,j)$ and $\mathrm{B}_{\mathbf{a}/(1+\mathbf{a})}^{\mathbf{b}/(1+\mathbf{b})}(n,2j) = \mathrm{B}_{\mathbf{a}^*}^{\mathbf{b}^*}(n,2j)$ in the pdf's are the halves of the corresponding ones without truncation. Since the equal proportional constants are canceled in the pdf when using the factors without truncation, we have the required result. Q.E.D.

When $K = 1$, cases with the skew-symmetric truncation pattern are given only by $I_1^* = [a_1^*, b_1^*) = [0, 1/2)$ $(I_1 = [a_1, b_1) = [0,\ 1))$ under upper truncation and $I_1^* = [1/2,\ 1)$ $(I_1 = [1,\ \infty))$ under lower truncation. When $K = 2$, the cases satisfying the pattern are obtained by $I_1^* = [0, c)$ $(I_1 = [0, c/(1-c)))$ and $I_2^* = [1-c, 1)$ $(I_2 = [(1-c)/c, \infty))$ $(0 < c \le 1/2)$. Note that these cases are subject to upper and inner truncation. The cases with exchanged truncation and selection intervals also satisfy the skew-symmetric pattern under lower and inner truncation. Two of these cases will be numerically illustrated later.

Lemma 2. *Under elliptical symmetry with possible truncation for the variance-ratio, we have*

$$\mathrm{E}\{R^{2l+1}(1-R^2)^m \,|\, \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}\}\ (l, m = 0, 1, \ldots)$$
$$= C_R \sum_{j=0}^{\infty} \frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j+1)\mathrm{B}\left(\frac{2l+2j+3}{2}, \frac{n+2m-1}{2}\right)$$

and

$$\mathrm{E}\{R^{2l}(1-R^2)^m \,|\, \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}\}\ (l, m = 0, 1, \ldots)$$
$$= C_R \sum_{j=0}^{\infty} \frac{(n)_{2j}(2\rho)^{2j}}{(2j)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j)\mathrm{B}\left(\frac{2l+2j+1}{2}, \frac{n+2m-1}{2}\right).$$

Proof. Using the pdf of R under the condition, the first set of moments is

$$
\begin{aligned}
&\mathrm{E}\{R^{2l+1}(1-R^2)^m \mid \sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b}\}\ (l,m=0,1,\ldots)\\
&=\int_{-1}^{1} r^{2l+1}(1-r^2)^m f_R(r\mid\sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b})\mathrm{d}r\\
&=C_R\int_{-1}^{1} r^{2l+1}(1-r^2)^{(n+2m-3)/2}\,{}_1F_{0\mathrm{W}}\left\{n;\,;2\rho r;\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,\ j),\ j=0,1,\ldots\right\}\mathrm{d}r\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_j(2\rho)^j}{j!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,j)\int_{-1}^{1} r^{2l+j+1}(1-r^2)^{(n+2m-3)/2}\mathrm{d}r\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j+1)\int_{-1}^{1} r^{2(l+j+1)}(1-r^2)^{(n+2m-3)/2}\mathrm{d}r\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j+1)\mathrm{B}\left(\frac{2l+2j+3}{2},\frac{n+2m-1}{2}\right).
\end{aligned}
,
$$

Similarly, for the second set of the moments, we obtain

$$
\begin{aligned}
&\mathrm{E}\{R^{2l}(1-R^2)^m \mid \sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b}\}\ (l,m=0,1,\ldots)\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_j(2\rho)^j}{j!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,j)\int_{-1}^{1} r^{2l+j}(1-r^2)^{(n+2m-3)/2}\mathrm{d}r\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j}(2\rho)^{2j}}{(2j)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j)\int_{-1}^{1} r^{2(l+j)}(1-r^2)^{(n+2m-3)/2}\mathrm{d}r\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j}(2\rho)^{2j}}{(2j)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j)\mathrm{B}\left(\frac{2l+2j+1}{2},\frac{n+2m-1}{2}\right).
\end{aligned}
$$

The above results give the required expressions. Q.E.D.

Theorem 4. *Under the same condition as in Lemma 2, we have*

$$
\begin{aligned}
&\mathrm{E}(R \mid \sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b})\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j+1)\mathrm{B}\left(\frac{2j+3}{2},\frac{n-1}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}(R^2 \mid \sigma_{11},\sigma_{22},\rho,n;\mathbf{a},\mathbf{b})\\
&=C_R\sum_{j=0}^{\infty}\frac{(n)_{2j}(2\rho)^{2j}}{(2j)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j)\mathrm{B}\left(\frac{2j+3}{2},\frac{n-1}{2}\right)\\
&=1-C_R\sum_{j=0}^{\infty}\frac{(n)_{2j}(2\rho)^{2j}}{(2j)!}\mathrm{B}^{\mathbf{b}/(\mathbf{1-b})}_{\mathbf{a}/(\mathbf{1-a})}(n,2j)\mathrm{B}\left(\frac{2j+1}{2},\frac{n+1}{2}\right),
\end{aligned}
$$

$$\mathrm{E}(R^3 \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b})$$
$$= C_R \sum_{j=0}^{\infty} \frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j+1) \mathrm{B}\left(\frac{2j+5}{2}, \frac{n-1}{2}\right)$$
$$= \mathrm{E}(R \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}) - C_R \sum_{j=0}^{\infty} \frac{(n)_{2j+1}(2\rho)^{2j+1}}{(2j+1)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j+1) \mathrm{B}\left(\frac{2j+3}{2}, \frac{n+1}{2}\right)$$

and

$$\mathrm{E}(R^4 \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b})$$
$$= C_R \sum_{j=0}^{\infty} \frac{(n)_{2j}(2\rho)^{2j}}{(2j)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j+1) \mathrm{B}\left(\frac{2j+5}{2}, \frac{n-1}{2}\right)$$
$$= \mathrm{E}(R^2 \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}) - C_R \sum_{j=0}^{\infty} \frac{(n)_{2j}(2\rho)^{2j}}{(2j)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j) \mathrm{B}\left(\frac{2j+3}{2}, \frac{n+1}{2}\right)$$
$$= -1 + 2\mathrm{E}(R^2 \mid \sigma_{11}, \sigma_{22}, \rho, n; \mathbf{a}, \mathbf{b}) + C_R \sum_{j=0}^{\infty} \frac{(n)_{2j}(2\rho)^{2j}}{(2j)!} \mathrm{B}_{\mathbf{a}/(1-\mathbf{a})}^{\mathbf{b}/(1-\mathbf{b})}(n, 2j) \mathrm{B}\left(\frac{2j+1}{2}, \frac{n+3}{2}\right).$$

Proof. Using the results of Lemma 2, the required expressions follow. Q.E.D.

5. Numerical illustrations of the moments of the sample correlation coefficient with the truncated variance-ratio

In this section, we show numerical illustrations of the theoretical and simulated moments of the sample correlation coefficient with the truncated variance-ratio. The moments subject to the truncation of the variance-ratio (truncated moments) given in Theorem 4 are exact ones. Consequently, the simulated truncated moments are unnecessary as long as the formulas are correct. However, the formulas include infinite series, which are approximated by finite ones in actual computation. Then, theoretical and corresponding simulated moments are shown in this section. Since the pdf of R is the same under arbitrary bivariate elliptical symmetry, the bivariate normal distribution is used for the simulation.

Table 1 summarizes 11 examples used in the numerical illustrations. The unit population values $\sigma_{11} = \sigma_{22} = 1$ are used throughout the illustrations. Four truncation types i.e., “untruncated”, “lower”, “double”, “inner” and “lower-inner” are employed. Since the two variables X_{1j} and X_{2j} $(j = 1, ..., N)$ are exchangeable under bivariate elliptical symmetry, single upper-truncation is not used. The pairs of symmetric limit values (3/2, 2/3) and (3/4, 4/3) of the intervals for selection are used for double and lower-inner truncation.

Ex. (Example) 1 without truncation is for comparison. Ex. 4 and 11 with the truncated probability $\alpha^* = 1/2$ satisfy the pattern of the skew-symmetric truncation defined in Corollary 1 when K = 1 and 2, respectively, giving the same pdf's as the corresponding untruncated cases.

Tables 2 and 3 give the simulated and theoretical values of the mean, variance, skewness and excess kurtosis of R when ρ = 0.3 and 0.7, respectively each with n = 20 and 50. The asymptotic value of α^* is given using

$$\text{avar}(\bar{H}) = \text{avar}(v_{11} / v_{22}) = (1,-1)\text{acov}\{(v_{11}, v_{22})^{\text{T}} / n\}(1,-1)^{\text{T}} = 4(1-\rho^2)n^{-1}$$

under normality with the usual normal approximation:

$$\alpha^* \doteq \sum_{k=1}^{K} \left(\Phi[(b_k - 1) / \{\text{avar}(\bar{H})\}^{1/2}] - \Phi[(a_k - 1) / \{\text{avar}(\bar{H})\}^{1/2}] \right)$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) for the standard normal. When $a_1 = 0$ i.e., with no lower-tail truncation, $a_1 = -\infty$ is used for the normal approximation, which gives slightly better results. The simulations are performed using randomly generated 10^5 sets of $N = n + 1$ observations under bivariate normality. From the 10^5 sets, $\bar{H}$'s satisfying the selection patterns are chosen. The proportions of $\bar{H}$'s to the generated 10^5 sets give the simulated α^*'s in the tables. The simulated moments/cumulants in the tables are given by the selected $\bar{H}$'s.

The simulated and the corresponding theoretical values in Table 2 are reasonably close to each other. Some of the asymptotic values of α^* show relatively poor approximations as seen in Ex. 11. It is found that the simulated and theoretical values of the mean under various types of truncation are rather close to those without truncation. The small absolute values of the negative biases are found, where the smallest simulated/theoretical biases are given by Ex. 8 under double truncation though the differences are small. It is of interest to find that the untruncated case does not give the best results. The SD's are found to be similar among the 11 examples with the smallest theoretical SD's given again by Ex. 8. The negative similar sk's and small absolute values of kt's are found for the 11 examples. Note also that the boldfaced theoretical values of the moments and cumulants in Ex. 4 and 11 are equal to those of Ex. 1 without truncation, which is expected from Corollary 1.

Table 3 give the results when ρ = 0.7. Tendencies similar to those in Table 2 are found in Table 3 except that the absolute values of sk's and kt's are much larger than those in Table 2, which is expected since the upper bound of R is 1, which tends to be closer to the sample values when ρ = 0.7 than ρ = 0.3. It is of interest to find the smallest simulated/theoretical biases and SD's are again found in Ex. 8. The substantial discrepancy of the simulated and theoretical kt's i.e., .7983 and .5535, respectively are seen when n = 50 in Ex. 6, which is partially explained by the small proportion (2.48%) of sets of observations used for simulated values among 10^5 generated sets.

Remark 5. The results of the numerical illustrations are impressive in two points. Firstly, the truncated moments are almost the same as the untruncated values, which is found even in the extreme case of Ex. 6 when ρ = 0.7 and n = 50 with as small as α^* = 0.0243 i.e., selecting only the small upper-tail of the distribution of $\bar{H}$. This finding yields the conjecture shown in the next section. The second point is based on the small differences of the moments among the various truncations. The smallest bias and SD are given by Ex. 8 with double truncation. When the variance-ratio is far off the unit value, the sample variances can be outliers. It is reassuring to find that when excluding such relatively irregular cases, the distribution of R is improved though the amount of improvement is small.

6. A conjecture on the asymptotic moments of the sample correlation coefficient with the truncation of the variance-ratio

So far, the values of a_k and b_k $(k = 1,\ldots,K)$ which do not depend on n have been used. This is reasonable as long as a fixed n is considered as in the numerical illustrations. However, when the asymptotic behavior of R with n being increased is considered, this gives degenerate or meaningless results. This is because the cdf $\Phi[(b_k - 1)/\{\mathrm{avar}(\bar{H})\}^{1/2}]$ as used earlier in the asymptotic α^* goes to 1 or 0 depending on $b_k > 1$ or $b_k < 1$, respectively when $n \to \infty$ since $\mathrm{avar}(\bar{H}) = O(n^{-1})$. When $b_k = 1$, the cdf = 1/2 is unchanged. Similar results for a_k are obtained. Then, when $n \to \infty$, we have limiting untruncated or always-truncating degenerate cases including the cases of the unchanged cdf

= 1/2 yielding the untruncated case due to Corollary 1.

As in Ogasawara (2022) , $\bar{b}_k = b_k\{\text{avar}(\bar{H})\}^{1/2} + 1$ and $\bar{a}_k = a_k\{\text{avar}(\bar{H})\}^{1/2} + 1$ $(k = 1,...,K)$, where b_k and a_k do not depend on *n*, may be used when the asymptotic behavior of *R* is considered in the case of $\sigma_{11}/\sigma_{22} = 1$. In this case, $\Phi[(\bar{b}_k - 1)/\{\text{avar}(\bar{H})\}^{1/2}] = \Phi(b_k)$ is unchanged irrespective of *n* and similarly for a_k. Then, it can be shown that the asymptotic α^* when using $\bar{b}_k$ and $\bar{a}_k$ approaches the exact α^* when $n \to \infty$. Ogasawara (2022) called this asymptotic α^* as the asymptotically constant truncated probability.

It is known that the asymptotic bias of *R* under normality (and consequently under elliptical symmetry) without truncation is of order $O(n^{-1})$ (Ghosh, 1966, Equation (3); Muirhead, 1982, Section 5.1.3, Equation (21)). When stripe truncation each for two sample variances is considered under bivariate normality, Ogasawara (2022) conjectured that the asymptotic bias of *R* under the condition of the asymptotically constant truncated probability is also of order $O(n^{-1})$ due to simulated and partially theoretical results. In the case of the variance ratio, considering the existence of the truncated cases with the same pdf's as those of the untruncated cases shown in Corollary 1 and the numerical results given earlier, we have the following conjecture.

Conjecture 1. The asymptotic bias, variance, third and fourth cumulants for *R* of orders $O(n^{-1})$, $O(n^{-1})$, $O(n^{-2})$ and $O(n^{-3})$, respectively with the stripely-truncated variance-ratio under elliptical symmetry satisfying the condition of the asymptotically constant truncated probability are the same as those in the untruncated case.

7. Some extended results for the joint pdf of the transformed sample variances and covariances under elliptical symmetry

In this section, we show that the joint pdf of the sample variance-ratios and correlation matrix under multivariate normality is robust under elliptical symmetry. Suppose that *N* random *p*-dimensional vectors $\mathbf{X}_i (i = 1,...,N)$ with $\mathbf{X}(p \times N) = (\mathbf{X}_1,...,\mathbf{X}_N)$ follow the elliptical distribution with the joint pdf

$$f_{\mathbf{X}}(\mathbf{x}) = K_{N,p}\,|\,\mathbf{\Lambda}\,|^{N/2}\; g\left\{\sum\nolimits_{i=1}^{N}(\mathbf{x}_i-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{\Lambda}^{-1}(\mathbf{x}_i-\boldsymbol{\mu})\right\}\ \text{ with }\ \mathbf{x}(p\times N) = (\mathbf{x}_1,\ldots,\mathbf{x}_N)\,,$$

where $K_{N,p}$ is the normalizing constant; $\mathbf{\Lambda}$ is a scale matrix; and $\boldsymbol{\mu}$ is a location parameter vector. Let $\mathbf{V}^* = \{V_{ij}\} = \sum\nolimits_{k=1}^{N}(\mathbf{X}_k-\overline{\mathbf{X}})(\mathbf{X}_k-\overline{\mathbf{X}})^{\mathrm{T}}$ with $\overline{\mathbf{X}} = \sum\nolimits_{k=1}^{N}\mathbf{X}_k / N$. Define the $(p-1)\times 1$ vector of the unscaled variance-ratios

$$\mathbf{H} = (H_1,\ldots,H_{p-1})^{\mathrm{T}}\ \text{ with }\ H_i = V_{ii}/V_{pp}\ (i = 1,\ldots,p-1)\,.$$

and the random correlation matrix

$$\mathbf{R}^* = \{R_{ij}\}\ \text{ with }\ R_{ij} = V_{ij}/(V_{ii}V_{jj})^{1/2}\ (i,j = 1,\ldots,p)\,.$$

Similarly, define the scaled or standardized sample variance-ratios

$$\overline{\mathbf{H}} \equiv (\overline{H}_1,\ldots,\overline{H}_{p-1})^{\mathrm{T}}\ \text{ with }\ \overline{H}_i = \frac{H_i}{\sigma_{ii/pp}} = \frac{\sum\nolimits_{k=1}^{N}(X_{ik}-\overline{X}_i)^2/\sigma_{ii}}{\sum\nolimits_{k=1}^{N}(X_{pk}-\overline{X}_p)^2/\sigma_{pp}}\ \text{ and}$$

$$\sigma_{ii/pp} \equiv \sigma_{ii}/\sigma_{pp}(i = 1,\ldots,p-1)\,.$$

Theorem 5. (i) *The joint distribution of* $\mathbf{H}$ *and* $\mathbf{R}^*$ (*or* $\mathbf{r}^* \equiv \mathrm{vb}(\mathbf{R}^*)$) *does not depend on the forms of* $g(\cdot)$ *or the normalizing constants in elliptical distributions, whose joint pdf of* $\mathbf{H}$ *and* $\mathbf{R}^*$ *at* $\mathbf{h}$ *and* $\mathbf{r}$, *respectively is*

$$f_{(\mathbf{H}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\mathbf{h}^{\mathrm{T}},\mathbf{r}^{T})\} = \frac{\Gamma(np/2)}{\Gamma_p(n/2)}|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod\nolimits_{i=1}^{p-1}h_i^{(n-2)/2}\right)\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h)\}^{-np/2}\,.$$

where $\mathbf{R}_h \equiv \mathrm{diag}(\mathbf{h}^{1/2},1)\mathbf{R}\,\mathrm{diag}(\mathbf{h}^{1/2},1)$ *and* $\mathrm{diag}(\mathbf{h}^{1/2},1) \equiv \mathrm{diag}(h_1^{1/2},\ldots,h_{p-1}^{1/2},1)$.

(ii) *The joint pdf of* $\overline{\mathbf{H}}$ *and* $\mathbf{R}^*$ *at* $\overline{\mathbf{h}}$ *and* $\mathbf{r}$, *respectively is*

$$f_{(\overline{\mathbf{H}}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\overline{\mathbf{h}}^{\mathrm{T}},\mathbf{r}^{T})\} = \frac{\Gamma(np/2)\,|\,\mathbf{R}\,|^{(n-p-1)/2}}{\Gamma_p(n/2)\,|\,\mathbf{P}\,|^{n/2}}\left(\prod\nolimits_{i=1}^{p-1}\overline{h}_i^{(n-2)/2}\right)\{\mathrm{tr}(\mathbf{P}^{-1}\mathbf{R}_{\overline{h}})\}^{-np/2}\,,$$

where $\mathbf{P}$ *is the population correlation matrix*;

$$\mathbf{R}_{\overline{h}} \equiv \mathrm{diag}(\overline{\mathbf{h}}^{1/2},1)\mathbf{R}\,\mathrm{diag}(\overline{\mathbf{h}}^{1/2},1)\ \textit{ and }\ \mathrm{diag}(\overline{\mathbf{h}}^{1/2},1) \equiv \mathrm{diag}(\overline{h}_1^{1/2},\ldots,\overline{h}_{p-1}^{1/2},1)\,.$$

The pdf $f_{(\overline{\mathbf{H}}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\overline{\mathbf{h}}^{\mathrm{T}},\mathbf{r}^{T})\}$ *holds under arbitrary elliptical symmetry as in the case of* $f_{(\mathbf{H}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\mathbf{h}^{\mathrm{T}},\mathbf{r}^{T})\}$ *and is scale-free.*

Proof. Let $n = N-1$ as before. Then, Sutradhar and Ali (1998, Theorem 2.1) showed that the pdf of $\mathbf{V}^*$ at $\mathbf{V} = \{v_{ij}\} = \sum\nolimits_{i=1}^{N}(\mathbf{x}_i-\overline{\mathbf{x}})(\mathbf{x}_i-\overline{\mathbf{x}})^{\mathrm{T}}$ with $\overline{\mathbf{x}} = \sum\nolimits_{i=1}^{N}\mathbf{x}_i/N$ is

$$f_{\mathbf{V}^*}(\mathbf{V}) = C_{n,p}\,|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{V}\,|^{(n-p-1)/2}\; g_{n,p}\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{V})\}\,,$$

where $C_{n,p} = \pi^{np/2} K_{n,p} / \Gamma_p(n/2)$ is the normalizing constant (Sutradhar & Ali, 1998, Equation (2.6)); $\Gamma_p(n/2)$ is the p-variate gamma function defined earlier; and $g_{n,p}\{\cdot\}$ is a function which depends on the form of $g(\cdot)$ as well as n and p.

Consider the change of variables from $\mathbf{V}^*$ to $\mathbf{R}^*$ with unchanged $V_{11},...,V_{pp}$. Let $\mathbf{R}$ and $\mathbf{V} = \{v_{ij}\}$ be mathematical variables, which are also used as realizations of $\mathbf{R}^*$ and $\mathbf{V}^* = \{V_{ij}\}$, respectively. Note that the Jacobian is

$$\mathrm{d}\mathbf{V} = \mathrm{d}\{\mathrm{v}(\mathbf{V})\} = \mathrm{d}\mathbf{v} = \left(\prod_{i=1}^{p} v_{ii}^{(p-1)/2}\right)\mathrm{d}\,(v_{11},...,v_{pp},\mathbf{r}^{\mathrm{T}})^{\mathrm{T}} = \left(\prod_{i=1}^{p} v_{ii}^{(p-1)/2}\right)\mathrm{d}\mathbf{u}\,,$$

as used in Section 2, where $\mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{R} = \mathrm{d}\{\mathrm{vb}(\mathbf{R})\}$ with $\mathrm{v}(\cdot)$ and $\mathrm{vb}(\cdot)$ being vectorizing operators defined earlier. Then, we have the pdf of $\mathbf{U} = (V_{11},...,V_{pp},\mathbf{r}^{*\mathrm{T}})^{\mathrm{T}}$ with $\mathbf{r}^* \equiv \mathrm{vb}(\mathbf{R}^*)$ at $\mathbf{u}$:

$$f_{\mathbf{U}}(\mathbf{u}) = C_{n,p}\,|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod_{i=1}^{p} v_{ii}^{(n-2)/2}\right) g_{n,p}\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{DRD})\}\,,$$

where $\mathbf{D} = \mathrm{diag}(v_{11}^{1/2},...,v_{pp}^{1/2})$ as used earlier.

We employ the second change of variables from $V_{11},...,V_{p-1,p-1}$ to $\mathbf{H} = (H_1,...,H_{p-1})^{\mathrm{T}}$ with unchanged V_{pp} and $\mathbf{R}^*$ (or $\mathbf{r}^*$), where the Jacobian is $\mathrm{d}(v_{11},...,v_{p-1,p-1})^{\mathrm{T}} = v_{pp}^{p-1}\mathrm{d}(h_1,...,h_{p-1})^{\mathrm{T}} = v_{pp}^{p-1}\mathrm{d}\mathbf{h}$. Then, the pdf of $\mathbf{H}^* \equiv (\mathbf{H}^{\mathrm{T}},V_{pp},\mathbf{r}^{*\mathrm{T}})^{\mathrm{T}}$ at $\mathbf{h}^* \equiv (\mathbf{h}^{\mathrm{T}},v_{pp},\mathbf{r}^{\mathrm{T}})^{\mathrm{T}}$ becomes

$$\begin{aligned} f_{\mathbf{H}^*}(\mathbf{h}^*) &= C_{n,p}\,|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod_{i=1}^{p-1} h_i^{(n-2)/2}\right) v_{pp}^{\{(n-2)p/2\}+p-1} \\ &\quad\times g_{n,p}[v_{pp}\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathrm{diag}(\mathbf{h}^{1/2},1)\mathbf{R}\mathrm{diag}(\mathbf{h}^{1/2},1)\}] \\ &= C_{n,p}\,|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod_{i=1}^{p-1} h_i^{(n-2)/2}\right) v_{pp}^{(np-2)/2} g_{n,p}\{v_{pp}\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h)\}. \end{aligned}$$

Let $W_{p,\mathbf{R}_h^*} = V_{pp}\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h^*)$. Then, the third change of variable is from $\mathbf{H}^* \equiv (\mathbf{H}^{\mathrm{T}},V_{pp},\mathbf{r}^{*T})^{\mathrm{T}}$ to $\mathbf{H}^{**} \equiv (\mathbf{H}^{\mathrm{T}},Y_p,\mathbf{r}^{*\mathrm{T}})^{\mathrm{T}}$ with $Y_p \equiv V_{pp}\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h^*)$ and the Jacobian $\mathrm{d}v_{pp} = \{1/\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h)\}\mathrm{d}y_p$, yielding the pdf at $\mathbf{h}^{**} \equiv (\mathbf{h}^{\mathrm{T}},y_p,\mathbf{r}^{\mathrm{T}})^{\mathrm{T}}$:

$$f_{\mathbf{H}^{**}}(\mathbf{h}^{**}) = C_{n,p}\,|\,\mathbf{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod_{i=1}^{p-1} h_i^{(n-2)/2}\right)\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{R}_h)\}^{-np/2} y_p^{(np-2)/2} g_{n,p}(y_p)\,.$$

In order to have the joint distribution of $\mathbf{H} = (H_1,...,H_{p-1})^{\mathrm{T}}$ and $\mathbf{R}^*$, the above pdf is integrated with respect to y_p. It is known that

$$\pi^{np/2}K_{n,p}\int_0^\infty y_p^{(np-2)/2}g_{n,p}(y_p)\,\mathrm{d}y_p = \Gamma(np/2)$$

(Muirhead, 1982, Theorem 1.5.5, Equation (11); Joarder & Ali, 1992, Lemma 2.1). Recalling that $C_{n,p} = \pi^{np/2}K_{n,p}/\Gamma_p(n/2)$, we obtain

$$\begin{aligned}
&f_{(\mathbf{H}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\mathbf{h}^{\mathrm{T}},\mathbf{r}^{T})\} \\
&= C_{n,p}\,|\,\boldsymbol{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod\nolimits_{i=1}^{p-1} h_i^{(n-2)/2}\right)\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\mathbf{R}_h)\}^{-np/2}\int_0^\infty y_p^{(np-2)/2}g_{n,p}(y_p)\,\mathrm{d}y_p\,. \\
&= \frac{\Gamma(np/2)}{\Gamma_p(n/2)}|\,\boldsymbol{\Lambda}\,|^{-n/2}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod\nolimits_{i=1}^{p-1} h_i^{(n-2)/2}\right)\{\mathrm{tr}(\boldsymbol{\Lambda}^{-1}\mathbf{R}_h)\}^{-np/2},
\end{aligned}$$

which does not depend on the forms of $g(\cdot)$ or the normalizing constants $K_{n,p}$'s in elliptical distributions.

Next, we derive the joint pdf of the scaled or standardized variance-ratios

$$\bar{\mathbf{H}} \equiv (\bar{H}_1,\ldots,\bar{H}_{p-1})^{\mathrm{T}} \text{ with } \bar{H}_i = \frac{H_i}{\sigma_{ii/pp}} = \frac{\sum_{k=1}^{N}(X_{ik}-\bar{X}_i)^2/\sigma_{11}}{\sum_{k=1}^{N}(X_{pk}-\bar{X}_p)^2/\sigma_{pp}}\ (i=1,\ldots,p-1).$$

and correlation matrix. Using the Jacobian $\mathrm{d}\mathbf{H} = \left(\prod_{i=1}^{p-1}\sigma_{ii/pp}\right)\mathrm{d}\bar{\mathbf{H}}$, we obtain the joint pdf of $\bar{\mathbf{H}}$ and $\mathbf{R}^*$ at $\bar{\mathbf{h}}$ and $\mathbf{r}$, respectively:

$$\begin{aligned}
&f_{(\bar{\mathbf{H}}^{\mathrm{T}},\mathbf{r}^{*\mathrm{T}})}\{(\bar{\mathbf{h}}^{\mathrm{T}},\mathbf{r}^{T})\} \\
&= \frac{\Gamma(np/2)\left(\prod_{i=1}^{p-1}\sigma_{ii/pp}^{(n-2)/2}\right)\prod_{i=1}^{p-1}\sigma_{ii/pp}}{\Gamma_p(n/2)\,|\,\mathbf{P}\,|^{n/2}\prod_{i=1}^{p}\sigma_{ii}^{n/2}}|\,\mathbf{R}\,|^{(n-p-1)/2}\left(\prod\nolimits_{i=1}^{p-1}\bar{h}_i^{(n-2)/2}\right)\{\sigma_{pp}^{-1}\mathrm{tr}(\mathbf{P}^{-1}\mathbf{R}_{\bar{h}})\}^{-np/2}\,. \\
&= \frac{\Gamma(np/2)\,|\,\mathbf{R}\,|^{(n-p-1)/2}}{\Gamma_p(n/2)\,|\,\mathbf{P}\,|^{n/2}}\left(\prod\nolimits_{i=1}^{p-1}\bar{h}_i^{(n-2)/2}\right)\{\mathrm{tr}(\mathbf{P}^{-1}\mathbf{R}_{\bar{h}})\}^{-np/2},
\end{aligned}$$

which is also found to be robust under arbitrary elliptical symmetry and scale-free as expected. These results give the required expressions. Q.E.D.

In the pdf's of $\bar{\mathbf{H}}$ ($\mathbf{H}$) and $\mathbf{R}^*$ derived by Theorem 5, the vanishing $K_{n,p}$ is expected since the random part in each of the pdf is common to elliptical distributions and gives a partial support of the pdf's. Theorem 5 gives the following obvious properties.

Corollary 2. *The marginal distributions of the following random quantities hold under arbitrary elliptical symmetry*.

(i) $\mathbf{H}$, $\bar{\mathbf{H}}$ *and their functions given by integrating* $\mathbf{R}$ *over the range* $\mathbf{R} > 0$ *in the joint* *pdf's*,

(ii) $\mathbf{R}^*$ *and its functions given by integrating* $\mathbf{h}(\bar{\mathbf{h}})$ *over the range with possible truncation for* $\mathbf{H}$ ($\bar{\mathbf{H}}$).

Although it is difficult to have the closed-forms of the integrals in Corollary 2 except when $p = 2$ as obtained earlier, the result in (ii) indicates an alternative integral formula to have the pdf of $\mathbf{R}^*$ derived in Theorems 1 and 2. In this case, note that the number of variables to be integrated is smaller than those in Theorems 1 and 2 by 1 since v_{pp} is already integrated out.

Remark 6. Examples of Corollary 2 given by functions of **H** are (i) the sample intraclass correlation coefficient defined by

$\hat{\rho}_{\rm ICC} = \dfrac{\mathbf{1}_p^{\rm T}\mathbf{V}\mathbf{1}_p - {\rm tr}(\mathbf{V})}{(p-1){\rm tr}(\mathbf{V})} = \dfrac{\mathbf{1}_p^{\rm T}\mathbf{R}_h^*\mathbf{1}_p - (h_1 + ... + h_{p-1} + 1)}{(p-1)(h_1 + ... + h_{p-1} + 1)}$ with $\mathbf{1}_p$ being the $p \times 1$ vector of 1's

i.e., the ratio of the mean of the off-diagonal elements of **V** to that of the diagonal elements (see Coffman, Maydeu-Olivares & Arnau, 2008, Equation (7)), which is different from the historical definitions of Harris (1913) and Fisher (1970, Chapter VII); and (ii) the sample alpha coefficient

$$\hat{\alpha} = \frac{p}{p-1}\left\{1 - \frac{{\rm tr}(\mathbf{V})}{\mathbf{1}_p^{\rm T}\mathbf{V}\mathbf{1}_p}\right\} = \frac{p}{p-1}\frac{\mathbf{1}_p^{\rm T}\mathbf{R}_h^*\mathbf{1}_p - (h_1 + ... + h_{p-1} + 1)}{\mathbf{1}_p^{\rm T}\mathbf{R}_h^*\mathbf{1}_p} = \frac{p{\rm tr}(\mathbf{V})}{\mathbf{1}_p^{\rm T}\mathbf{V}\mathbf{1}_p}\hat{\rho}_{\rm ICC} \geq \hat{\rho}_{\rm ICC}$$

(the ratio of the mean of the off-diagonal elements of **V** to that of all of the elements, which should not be confused with the sample truncated probability $\hat{\alpha}^*$ used earlier and has been used in psychometrics, education, survey researches, accounting and so on; see Ogasawara, 2006 and the references therein; Smith, 2015, Chapter 9) and the sample standardized alpha

$$\hat{\alpha}_\rho = \frac{p}{p-1}\left(1 - \frac{p}{\mathbf{1}_p^{\rm T}\mathbf{R}^*\mathbf{1}_p}\right)$$

(see Hayashi & Kamata, 2005, Equation (3); Ogasawara, 2006, Equation (32)). The exact distributions of $\hat{\rho}_{\rm ICC}$ and $\hat{\alpha}$ under normality were derived by Kistner and Muller (2004, Theorem 1). Then, it is found that these distributions hold under arbitrary elliptical symmetry.

Consider the distribution for the random matrix $\mathbf{V}^* = \{V_{ij}\} = \sum_{k=1}^N (\mathbf{X}_k - \bar{\mathbf{X}})(\mathbf{X}_k - \bar{\mathbf{X}})^{\rm T}$

as in Theorem 5 under elliptical symmetry. Redefine the $p^* \times 1$ vector $\mathbf{U} = (U_1, ..., U_{p*})^{\mathrm{T}}$ with $p^* = p(p+1)/2$, which was used in the proof of Theorem 5, such that $\mathbf{U} = \mathbf{U}\{\mathrm{v}(\mathbf{V}^*)\} = \mathbf{U}(\mathbf{v}^*)$ is a bijective differential function of $\mathbf{V}^*$ with the Jacobian generically written by

$$\mathrm{d}\mathbf{V} = \mathrm{d}\{\mathrm{v}(\mathbf{V})\} = \mathrm{d}\mathbf{v} = J(\mathbf{u})\mathrm{d}\mathbf{u} \quad \text{with} \quad J(\mathbf{u}) > 0 .$$

Then, recalling that $f_{\mathbf{V}^*}(\mathbf{V}) = C_{n,p} \mid \mathbf{\Lambda} \mid^{-n/2} \mid \mathbf{V} \mid^{(n-p-1)/2} g_{n,p}\{\mathrm{tr}(\mathbf{\Lambda}^{-1}\mathbf{V})\}$, the pdf of $\mathbf{U}$ is given by

$$f_{\mathrm{U}}(\mathbf{u}) = C_{n,p} \mid \mathbf{\Lambda} \mid^{-n/2} \mid \mathbf{V}(\mathbf{u}) \mid^{(n-p-1)/2} J(\mathbf{u}) g_{n,p}[\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}(\mathbf{u})\}],$$

where $\mathbf{V}(\mathbf{u})$ is a matrix function of **u**. Suppose that $\mathbf{V}(\mathbf{u})$ satisfies the following condition referred to as "separability":

$$\mathbf{V}(\mathbf{u}) = u_{p*}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}),$$

where $\mathbf{u}_{(-p*)} = (u_1, ..., u_{p*-1})^{\mathrm{T}}$, $\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})$ is a function of $\mathbf{u}_{(-p*)}$ and the last element u_{p*} in **u** is chosen without loss of generality under separability. Then, we have the following general result.

Theorem 6. *The marginal joint distribution of* $\mathbf{U}_{(-p*)}$ *does not depend on the forms of* $g_{n,p}[\cdot]$ *or the normalizing constant* $K_{n,p}$ *for the original elliptically distributed* **X** *in* $C_{n,p} = \pi^{np/2} K_{n,p} / \Gamma_p(n/2)$. *The pdf is given by*

$$\begin{aligned} & f_{\mathbf{U}_{(-p^*)}}(\mathbf{u}_{(-p*)}) \\ & = \frac{\Gamma(np/2)}{\Gamma_p(n/2)} \mid \mathbf{\Lambda} \mid^{-n/2} \mid \mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}) \mid^{(n-p-1)/2} J_{(-p*)}(\mathbf{u}_{(-p*)})[\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})\}]^{-np/2}, \end{aligned}$$

where $J_{(-p*)}(\mathbf{u}_{(-p*)}) = \det\left\{\dfrac{\partial \mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})}{\partial \mathbf{u}^{\mathrm{T}}_{(-p*)}}, \mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})\right\}$ and $\mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})$ $= \mathrm{v}\{\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})\}$.

Proof. The separable condition $\mathbf{V}(\mathbf{u}) = u_{p*}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})$ is also written as $\mathbf{v}(\mathbf{u}) = u_{p*}\{\mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})\}$. Then, the Jacobian becomes

$$
\begin{aligned}
J(\mathbf{u}) &= u_{p*}^{(n-p-1)p/2} \det\left\{\frac{\partial \mathbf{v}(\mathbf{u})}{\partial \mathbf{u}^{\mathrm{T}}}\right\} \\
&= u_{p*}^{(n-p-1)p/2} u_{p*}^{p*-1} \det\left\{\frac{\partial \mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})}{\partial \mathbf{u}_{(-p*)}^{\mathrm{T}}},\ \mathbf{v}_{(-p*)}(\mathbf{u}_{(-p*)})\right\} \\
&= u_{p*}^{(np-2)/2} J_{(-p*)}(\mathbf{u}_{(-p*)}).
\end{aligned}
$$

Employ the variable transformation from U_{p*} to $Y_{p^*} = U_{p*}\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}_{(-p*)}^{*}(\mathbf{U}_{(-p*)})\}$, which gives

$$
\begin{aligned}
& f_{\mathbf{U}_{(-p^*)}}(\mathbf{u}_{(-p*)}) \\
&= C_{n,p} \,|\, \mathbf{\Lambda} \,|^{-n/2} |\, \mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}) \,|^{(n-p-1)/2}\ J_{(-p*)}(\mathbf{u}_{(-p*)}) \int_0^\infty u_{p*}^{(np-2)/2} g_{n,p}[u_{p*}\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})\}]\mathrm{d}u_{p*} \\
&= C_{n,p} \,|\, \mathbf{\Lambda} \,|^{-n/2} |\, \mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}) \,|^{(n-p-1)/2}\ J_{(-p*)}(\mathbf{u}_{(-p*)}) [\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})\}]^{-np/2} \\
&\times \int_0^\infty y_{p*}^{(np-2)/2} g_{n,p}(y_{p*})\mathrm{d}y_{p*} \\
&= \frac{\Gamma(np/2)}{\Gamma_p(n/2)} |\, \mathbf{\Lambda} \,|^{-n/2} |\, \mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}) \,|^{(n-p-1)/2}\ J_{(-p*)}(\mathbf{u}_{(-p*)}) [\mathrm{tr}\{\mathbf{\Lambda}^{-1}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)})\}]^{-np/2},
\end{aligned}
$$

where $\pi^{np/2} K_{n,p} \int_0^\infty y_{p*}^{(np-2)/2} g_{n,p}(y_{p*})\,\mathrm{d}y_{p*} = \Gamma(np/2)$ is used as in Theorem 5. The last result gives the required expression. Q.E.D.

Remark 7. Note that Theorem 5 is a special case of Theorem 6 when $\mathbf{U} = V_{pp}(\bar{H}_1, ..., \bar{H}_{p-1}, \mathbf{r}^{*\mathrm{T}})^{\mathrm{T}}$ and $U_{p*} = V_{pp}$ satisfying the separable condition:

$$
\mathbf{V}(\mathbf{u}) = u_{p*}\mathbf{V}_{(-p*)}(\mathbf{u}_{(-p*)}) = v_{pp}\mathbf{\Lambda}^{-1}\mathrm{diag}(\mathbf{h}^{1/2}, 1)\mathbf{R}\mathrm{diag}(\mathbf{h}^{1/2}, 1)
$$

with $u_{p*} = v_{pp}$.

An application of Theorem 6 other than that of Theorem 5 is given by principal component analysis (PCA):

$$
\mathbf{V}(\mathbf{U}) = U_{p*}\hat{\mathbf{W}}\mathrm{diag}(\hat{\lambda}_1 / U_{p*}, ..., \hat{\lambda}_p / U_{p*})\hat{\mathbf{W}}^{\mathrm{T}} = U_{p_+}\hat{\mathbf{W}}\mathrm{diag}(\hat{\bar{\lambda}}_1, ..., \hat{\bar{\lambda}}_p)\hat{\mathbf{W}}^{\mathrm{T}}
$$

where $U_{p*} = U_{p_+} = \mathrm{tr}(\mathbf{V})$, $\hat{\mathbf{W}}$ is the matrix of sample component weights or loadings based on the spectral decomposition $\mathbf{V} = \mathbf{V}(\hat{\mathbf{W}}, \hat{\lambda}_1, ..., \hat{\lambda}_p) = \hat{\mathbf{W}}\mathrm{diag}(\hat{\lambda}_1, ..., \hat{\lambda}_p)\hat{\mathbf{W}}^{\mathrm{T}}$, $\hat{\mathbf{W}}^{\mathrm{T}}\hat{\mathbf{W}} = \hat{\mathbf{W}}\hat{\mathbf{W}}^{\mathrm{T}} = \mathbf{I}_p$, $\mathbf{I}_p$ is the $p \times p$ identity matrix and $\hat{\bar{\lambda}}_i = \hat{\lambda}_i / U_{p*} = \hat{\lambda}_i / (\hat{\lambda}_1 + ... + \hat{\lambda}_p)$ $(i = 1, ..., p)$ are the sample relative contributions of the *p* components (for other formulations of PCA see Ogasawara, 2000). Theorem 6 shows that the joint distribution of

$\hat{\mathbf{W}}$ and $\hat{\hat{\lambda}}_i \, (i = 1, ..., p)$ in PCA holds under arbitrary elliptical symmetry.

References

Abramowitz, M., & Stegun, I. A. (Eds.). (1964). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards, Applied Mathematics Series, No. 55, U. S. Government Printing Office, Washington, D.C. Reprinted, 2014, Mansfield Centre, CT: Martino.

Ali, M. M., & Joarder, A. H. (1991). Distribution of the correlation coefficient for the class of bivariate elliptical models. *The Canadian Journal of Statistics, 19*, 447-452.

Anderson, T. W. (1958). *An introduction to multivariate statistical analysis*. New York: Wiley.

Anderson, T. W. (2003). *An introduction to multivariate statistical analysis* (3rd ed.). New York: Wiley.

Anderson, T. W., & Fang, K. T. (1990). On the theory of multivariate elliptically contoured distributions and their applications. In K. T. Fang & T. W. Anderson (Eds.), *Statistical inference in elliptically contoured and related distributions* (pp. 1-23). New York: Allerton Press.

Bose, S. (1935). On the distribution of the ratio of variances of two samples drawn from a given normal bivariate correlated population (with comments by P. C. Mahalanobis). *Sankhyā*, *2*, 65-72.

Coffman, D. L., Maydeu-Olivares, A., & Arnau, J. (2008). Asymptotic distribution free interval estimation: For an intraclass correlation coefficient with applications to longitudinal data. *Methodology: European Journal of Research Methods for the Behavioral and Social Sciences*, *4*, 4-9.

DLMF (2021). *NIST Digital Library of Mathematical Functions*. https://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain (Eds.).

Erdélyi, A. (Ed.). (1953a). *Higher transcendental functions* (Vol. 1). New York: McGraw-Hill. https://resolver.caltech.edu/CaltechAUTHORS:20140123-104529738.

Erdélyi, A. (Ed.). (1953b). *Higher transcendental functions* (Vol. 2). New York: McGraw-Hill. https://resolver.caltech.edu/CaltechAUTHORS:20140123-104529738.

Erdélyi, A. (Ed.). (1954). *Tables of integral transforms* (Vol. 1). New York: McGraw-Hill. https://resolver.caltech.edu/CaltechAUTHORS:20140123-101456353.

Finney, D. J. (1938). The distribution of the ratio of estimates of the two variances in a sample from a normal bi-variate population. *Biometrika*, *30*, 190-192.

Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. *Biometrika, 10*, 507-521.

Fisher, R. A. (1962). The simultaneous distribution of correlation coefficients. *Sankhyā, A, 24*, 1-8.

Fisher, R. A. (1970). *Statistical methods for research workers* (14th ed.). Edinburgh: Oliver & Boyd.

Ghosh, B. K. (1966). Asymptotic expansions for the moments of the distribution of correlation coefficient. *Biometrika*, *53*, 258-262.

Hankin, R. S. R. (2016). hypergeo: The Gauss hypergeometric function. R package version 1.2-13. https://CRAN.R-project.org/package=hypergeo.

Harris, J. A. (1913). On the calculation of intra-class and inter-class coefficients of correlation from class moments when the number of possible combinations is large. *Biometrika*, *9*, 446-472.

Hashiguchi, H., Numata, Y., Takayama, N., & Takemura, A. (2013). The holonomic gradient method for the distribution function of the largest root of a Wishart matrix. *Journal of Multivariate Analysis*, *117*, 296-312.

Hayashi, K., & Kamata, A. (2005). A note on the estimator of the alpha coefficient for standardized variables under normality. *Psychometrika, 70*, 579-589.

Hotelling, H. (1953). New light on the correlation coefficient and its transforms. (with discussion) *Journal of the Royal Statistical Society, Series B, 15*, 193-232.

Joarder, A. H. (2013). Robustness of correlation coefficient and variance ratio under elliptical symmetry. *Bulletin of the Malaysian Mathematical Sciences Society*, *Series 2, 36*, 277-284. http://math.usm.my/bulletin/pdf/v36n2/v36n2p2.pdf.

Joarder, A. H., & Ali, M. M. (1992). Distribution of the correlation matrix for a class of elliptical models. *Communications in Statistics - Theory and Methods*, *21*, 1953-1964.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions* Vol. 2 (2nd ed.). New York: Wiley.

Kistner, E. O., & Muller, K. E. (2004). Exact distributions of intraclass correlation and Cronbach's alpha with Gaussian data and general covariance. *Psychometrika, 69*, 459-474.

Magnus, W., Oberhettinger, F., & Soni, R. P. (1966). *Formulas and theorems for the special functions of mathematical physics* (3rd enlarged ed.). Berlin: Springer-Verlag.

Muirhead, R. J. (1982). *Aspects of multivariate statistical theory*. New York: Wiley.

Mura, H., Hashiguchi, H., Nakagawa, S., & Ono, Y. (2019). Holonomic properties and recurrence formula for the distribution of sample correlation coefficient. *SUT Journal of Mathematics*, *55*, 39-52. https://www.rs.tus.ac.jp/sutjmath/index.html.

Nakayama, H., Nishiyama, K., Noro, M., Ohara, K., Sei, T., Takayama, N., & Takemura, A. (2011). Holonomic gradient descent and its application to the Fisher–Bingham integral. *Advances in Applied Mathematics*, *47*, 639-658.

Ogasawara, H. (2000). Standard errors of the principal component loadings for unstandardized and standardized variables. *British Journal of Mathematical and Statistical Psychology, 53*, 155-174.

Ogasawara, H. (2006). Asymptotic expansion of the sample correlation coefficient under nonnormality. *Computational Statistics and Data Analysis, 50,* 891-910.

Ogasawara, H. (2021a). Unified and non-recursive formulas for moments of the normal distribution with stripe truncation. *Communications in Statistics – Theory and Methods* (online published). https://doi.org/10.1080/03610926.2020.1867742.

Ogasawara, H. (2021b). A non-recursive formula for various moments of the multivariate normal distribution with sectional truncation. *Journal of Multivariate Analysis* (online published). https://doi.org/10.1016/j.jmva.2021.104729.

Ogasawara, H. (2022). *The distribution of the sample correlation coefficient under variance-truncated normality.* Preprint at https://doi.org/10.13140/RG.2.2.19080.90883.

Omar, M. H., Joarder, A. H., & Riaz, M. (2015). On a correlated variance ratio distribution and its industrial application. *Communications in Statistics - Theory and Methods*, *44*, 261-274.

Pearson, K. (1925). Further contributions to the theory of small samples. *Biometrika*, *17*, 176-200.

Pham-Gia, T., & Choulakian, V. (2014). Distribution of the sample correlation matrix and applications. *Open Journal of Statistics*, *4,* 330-344.

Pham-Gia, T., & Thanh, D. N. (2016). Hypergeometric functions: From one scalar variable to several matrix arguments, in statistics and beyond. *Open Journal of Statistics*, *6*, 951-994.

Shimizu, K., & Hashiguchi, H. (2021). Heterogeneous hypergeometric functions with two matrix arguments and the exact distribution of the largest eigenvalue of a singular beta-Wishart matrix. *Journal of Multivariate Analysis*, *183*, 104714.

Smith, M. (2015). *Research methods in accounting* (3rd ed.). London: Sage.

Sutradhar, B. C., & Ali, M. M. (1989). A generalization of the Wishart distribution for the elliptical model and its moments for the multivariate *t* model. *Journal od Multivariate Analysis, 29*, 155-162.

Winkelbauer, A. (2014). *Moments and absolute moments of the normal distribution*. arXiv:1209.4340v2 [math.ST] 15 Jul 2014.

Zwillinger, D. (2015). *Table of integrals, series and products* (8th ed.). Translated from Russian by Scripta Technica Inc. Amsterdam: Elsevier. The original Russian text by I. S. Gradshteyn and I. M. Ryzhik, 1962. Moscow: Fiziko-Matematicheskoy Literatury.

Table 1. Eleven examples and their truncation of the variance-ratio $H\,(\bar{H})$

Example No.	Truncation type	The numbers of intervals for selection (K)	The selection points A	B
1	untruncated	1	0	Inf
2	lower	1	2/3	Inf
3	*	1	3/4	Inf
4	*	1	1	Inf
5	*	1	4/3	Inf
6	*	1	3/2	Inf
7	double	1	2/3	3/2
8	*	1	3/4	4/3
9	inner	2	0	2/3
			3/2	Inf
10	*	2	0	3/4
			4/3	Inf
11	lower-inner	2	2/3	1
			3/2	Inf

Note. $H(\bar{H}) = V_{11} / V_{22} = \sum_{j=1}^{n+1} (X_{1j} - \bar{X}_1)^2 / \sum_{j=1}^{n+1} (X_{2j} - \bar{X}_2)^2$; lower = lower-tail truncated, double = doubly truncated, inner = inner-truncated, lower-inner = lower and inner truncated, Inf $= \infty$; '*' indicates 'the same as above'.

Table 2. Simulated and theoretical values of the moments/cumulants of the sample correlation coefficient with the truncated probability α^* when $\rho = 0.3$ (the number of generated (not selected) sample covariance matrices = 10^5)

Ex. No.	Mean		SD		sk		kt		α^* The truncated probability		
	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Asy.
	$\rho = 0.3$, $n = 20$										
1	.2927	**.2932**	.2060	**.2058**	-.3694	**-.3648**	-.0024	**-.0334**	1	1	1
2	.2942	.2952	.2060	.2055	-.3725	-.3672	-.0000	-.0299	.8239	.8250	.7827
3	.2946	.2951	.2060	.2055	-.3734	-.3672	.0033	-.0300	.7468	.7468	.7211
4	.2927	**.2932**	.2061	**.2058**	-.3685	**-.3648**	-.0108	**-.0334**	.5000	.5	.5000
5	.2868	.2875	.2069	.2065	-.3485	-.3575	-.0760	-.0431	.2546	.2532	.2173
6	.2833	.2839	.2068	.2070	-.3432	-.3529	-.0765	-.0491	.1764	.1750	.1206
7	.2971	.2982	.2057	.2050	-.3806	-.3708	.0228	-.0244	.6475	.6500	.6621
8	.2987	.2990	.2054	.2049	-.3863	-.3719	.0476	-.0230	.4922	.4936	.5038
9	.2845	.2839	.2063	.2070	-.3497	-.3529	-.0439	-.0491	.3525	.3500	.3379
10	.2868	.2875	.2064	.2065	-.3535	-.3575	-.0470	-.0431	.5078	.5064	.4962
11	.2918	**.2932**	.2063	**.2058**	-.3660	**-.3648**	-.0180	**-.0334**	.5004	.5	.4033
	$\rho = 0.3$, $n = 50$										
1	.2970	**.2973**	.1294	**.1293**	-.2413	**-.2448**	-.0181	**-.0007**	1	1	1
2	.2974	.2978	.1293	.1292	-.2409	-.2453	-.0151	-.0002	.9310	.9317	.8917
3	.2977	.2980	.1291	.1292	-.2425	-.2454	-.0129	-.0000	.8550	.8551	.8229
4	.2971	**.2973**	.1292	**.1293**	-.2327	**-.2448**	-.0031	**-.0007**	.4994	.5	.5000
5	.2928	.2928	.1295	.1297	-.2399	-.2411	.0017	-.0043	.1456	.1449	.1083
6	.2903	.2895	.1293	.1300	-.2401	-.2384	.0251	-.0068	.0691	.0683	.0319
7	.2979	.2985	.1293	.1292	-.2411	-.2457	-.0184	.0003	.8619	.8634	.8597
8	.2987	.2991	.1290	.1291	-.2430	-.2462	-.0161	.0008	.7094	.7102	.7146
9	.2910	.2895	.1297	.1300	-.2426	-.2384	-.0176	-.0068	.1381	.1366	.1403
10	.2928	.2928	.1302	.1297	-.2361	-.2411	-.0238	-.0043	.2907	.2898	.2854
11	.2966	**.2973**	.1294	**.1293**	-.2488	**-.2448**	-.0216	**-.0007**	.5006	.5	.4236

Note. SD = standard deviation, sk = skewness, kt = excess kurtosis, α^* (the truncated probability) = the reduced probability due to truncation; Sim. = simulated values, Th. = theoretical values, Asy. = asymptotic values.

Table 3. Simulated and theoretical values of the moments/cumulants of the sample correlation coefficient with the truncated probability α^* when $\rho = 0.7$ (the number of generated (not selected) sample covariance matrices = 10^5)

Ex. No.	Mean		SD		sk		kt		α^* The truncated probability		
	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Sim.	Th.	Asy.
	$\rho = 0.7$, $n = 20$										
1	.6904	**.6907**	.1219	**.1218**	-.9532	**-.9543**	1.4637	**1.4431**	1	1	1
2	.6928	.6931	.1210	.1208	-.9594	-.9557	1.4953	1.4517	.8917	.8921	.8517
3	.6933	.6934	.1209	.1208	-.9654	-.9569	1.5312	1.4557	.8118	.8116	.7831
4	.6902	**.6907**	.1223	**.1218**	-.9701	**-.9543**	1.5152	**1.4431**	.5025	.5	.5000
5	.6785	.6789	.1253	.1257	-.9544	-.9356	1.5094	1.3700	.1879	.1884	.1483
6	.6722	.6708	.1275	.1281	-.9580	-.9220	1.5717	1.3193	.1079	.1079	.0587
7	.6957	.6962	.1198	.1195	-.9555	-.9559	1.4643	1.4573	.7838	.7842	.7930
8	.6978	.6979	.1192	.1189	-.9650	-.9585	1.5233	1.4679	.6239	.6232	.6348
9	.6714	.6708	.1275	.1281	-.9250	-.9220	1.3975	1.3193	.2162	.2158	.2070
10	.6783	.6789	.1253	.1257	-.9270	-.9356	1.3599	1.3700	.3761	.3768	.3652
11	.6910	**.6907**	.1214	**.1218**	-.9523	**-.9543**	1.5118	**1.4431**	.4970	.5	.4104
	$\rho = 0.7$, $n = 50$										
1	.6964	**.6964**	.0743	**.0741**	-.6207	**-.6002**	.6780	**.5901**	1	1	1
2	.6967	.6968	.0742	.0739	-.6232	-.5998	.6869	.5895	.9753	.9757	.9506
3	.6971	.6972	.0741	.0739	-.6266	-.6002	.6936	.5904	.9203	.9204	.8921
4	.6964	**.6964**	.0743	**.0741**	-.6303	**-.6002**	.6988	**.5901**	.5039	.5	.5000
5	.6874	.6868	.0773	.0760	-.5819	-.5920	.5881	.5698	.0820	.0796	.0494
6	.6800	.6794	.0793	.0775	-.6587	-.5851	.7983	.5535	.0248	.0243	.0067
7	.6972	.6972	.0740	.0738	-.6203	-.5993	.6770	.5886	.9505	.9514	.9439
8	.6980	.6982	.0737	.0736	-.6291	-.5998	.7022	.5900	.8383	.8408	.8426
9	.6805	.6794	.0780	.0775	-.5936	-.5851	.6333	.5535	.0495	.0486	.0561
10	.6876	.6868	.0765	.0760	-.5689	-.5920	.5665	.5698	.1617	.1592	.1574
11	.6962	**.6964**	.0744	**.0741**	-.6214	**-.6002**	.6924	**.5901**	.4962	.5	.4572

Note. SD = standard deviation, sk = skewness, kt = excess kurtosis, α^* (the truncated probability) = the reduced probability due to truncation; Sim. = simulated values, Th. = theoretical values, Asy. = asymptotic values.