A stochastic derivation of the surface area of the (n-1)-sphere

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A stochastic derivation of the surface area of the (*n*-1)-sphere

Abstract. The surface area of the (n-1)-dimensional sphere in the n-dimensional Euclidean space is typically obtained by induction with a similar derivation for the volume of the n-dimensional ball in the same space. Huber (1982) obtained the volume of the ball using the gamma function without induction, followed by the surface area of the corresponding sphere. In this note, the surface area is first derived from the multivariate normal distribution using the polar coordinate system and the corresponding chi distribution for the radius. The volume of the ball is given by integration.

Keywords: Jacobians, multivariate normal distribution, chi distribution, probability density function (pdf), n-ball.
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1. INTRODUCTION Huber [1, p. 301] stated that "Beginning students of analysis are often presented with a simple inductive derivation of n-sphere volume formulas" i.e., $V_n(r) = r^n \pi^{n/2} / \Gamma\{1+(n/2)\}$ for the n-dimensional ball of radius r in the n-dimensional Euclidian space, which is equal to the volume inside the (n-1)-dimensional sphere in the same space, whose surface area is given by $S_{n-1}(r) = 2\pi^{n/2} r^{n-1} / \Gamma(n/2)$, where $\Gamma(\cdot)$ is the gamma function. Huber first derived $V_n(r)$ using properties of $\Gamma(\cdot)$, followed by $S_{n-1}(r) = \mathrm{d}V_n(r)/\mathrm{d}r$. In this note, $S_{n-1}(r)$ is first obtained by the probability density functions (pdf's) of the chi and n-variate normal distributions.

2. DERIVATION OF THE SURFACE AREA OF A SPHERE AND THE VOLUME OF THE CORRESPONDING BALL The pdf of the n-variate standard normal using the usual n Cartesian coordinates $\mathbf{x} = (x_1,\ldots,x_n)^{\mathrm{T}}$ $(-\infty < x_i < \infty,\ i = 1,\ldots,n)$ is $\phi_n(\mathbf{x}) = \exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x}/2)/(2\pi)^{n/2}$. When the polar coordinate system with radial coordinate r $(0 \le r < \infty)$ and angular coordinates $\boldsymbol{\theta} = (\theta_1,..,\theta_{n-1})^{\mathrm{T}}$ $(0 \le \theta_i < \pi,\ i = 1,\ldots,n-2;\ 0 \le \theta_{n-1} < 2\pi)$ is employed, the pdf becomes

$$\psi_n(r,\boldsymbol{\theta}) = \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} J\{\mathbf{x} \to (r,\boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\},$$

where $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$; $J\{\mathbf{x} \to (r,\boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \det\{\mathrm{d}\,\mathbf{x}/\mathrm{d}(r,\boldsymbol{\theta}^{\mathrm{T}})\} = r^{n-1}g(\boldsymbol{\theta})$ is the Jacobian; and $g(\boldsymbol{\theta})$ is a known function of $\boldsymbol{\theta}$. When r is given, the marginal density at r is obtained by

$$\begin{aligned}\psi_n(r) &= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} g(\boldsymbol{\theta}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1} \\ &= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} S_{n-1}(r).\end{aligned}$$

It is known that the variable S^* corresponding to $s \equiv r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is chi-square distributed with n degrees of freedom (df) whose pdf is given by $\dfrac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)}$. Then, the distribution of the square root of S^* is said to be chi distributed with n df. The pdf of the chi at r is given by

$$f_\chi(r\,|\,n) = \frac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)}\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{r^{n-2}\exp(-r^2/2)}{2^{n/2}\Gamma(n/2)}2r = \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)}.$$

Since $\psi_n(r) = f_\chi(r\,|\,n)$, we have

$$\frac{\exp(-r^2/2)}{(2\pi)^{n/2}}S_{n-1}(r) = \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)}$$

yielding the required surface area $S_{n-1}(r) = 2\pi^{n/2}r^{n-1}/\Gamma(n/2)$. The corresponding volume is derived by

$$\begin{aligned} V_n(r) &= \int_0^r S_{n-1}(t)\,\mathrm{d}t = \int_0^r \{2\pi^{n/2}t^{n-1}/\Gamma(n/2)\}\,\mathrm{d}t \\ &= 2\pi^{n/2}r^n/\{n\Gamma(n/2)\} = \pi^{n/2}r^n\{(n/2)\Gamma(n/2)\} \\ &= \pi^{n/2}r^n\Gamma\{(n/2)+1\}. \end{aligned}$$

3. REMARKS The vector $\mathbf{x}$ is typically transformed to r and $\boldsymbol{\theta}$ as follows

$$\begin{aligned} x_i &= r\left(\prod\nolimits_{j=1}^{i-1}\sin\theta_j\right)\cos\theta_i \;\; (i = 1,\ldots,n-1), \\ x_n &= r\prod\nolimits_{j=1}^{n-1}\sin\theta_j. \end{aligned}$$

Then, it can be shown that the Jacobian is written as

$$J\{\mathbf{x} \to (r,\boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = r^{n-1}g(\boldsymbol{\theta}) = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2},$$

which can also be used to have the surface area

$$\begin{aligned} S_{n-1}(r) &= \int_0^\pi\cdots\int_0^\pi\int_0^{2\pi} r^{n-1}g(\boldsymbol{\theta})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-2}\mathrm{d}\theta_{n-1} \\ &= r^{n-1}\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi} r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-2}\mathrm{d}\theta_{n-1} \end{aligned}$$

though this integral is somewhat tedious to obtain. Conversely, the result in the previous section gives an integral formula

$$\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-2}\mathrm{d}\theta_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$$

or equivalently

$$\int_0^\pi\cdots\int_0^\pi\sin^m\theta_1\sin^{m-1}\theta_2\cdots\sin\theta_m\mathrm{d}\theta_1\cdots\mathrm{d}\theta_m = \pi^{m/2}/\Gamma\{(m/2)+1\}\;\;(m = 1,2,\ldots),$$

where the former gives the surface area of the (n-1)-sphere with unit radius i.e., $S_{n-1}(1)$.

For instance, when n = 2, we have the circumference $\int_0^{2\pi}\mathrm{d}\theta_{n-1} = 2\pi$ of a circle with unit radius. It is also found that the latter gives the volume of the m-ball with unit radius $V_m(1)$.

Reference

[1] Huber, G. (1982). Gamma function derivation of n-sphere volumes. *American Mathematical Monthly.* 89 (5): 301-302.