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Abstract. The probability density function (pdf) of the chi-square distribution with *n* degrees of freedom is derived using the pdf's of the multivariate distribution and its transformation under the polar coordinate system. The marginal density of the radial variable corresponding to the chi distribution is obtained by integrating the angular variables, which gives the factor of the known surface area of the (*n*-1)-dimensional sphere. The pdf of the chi-square is obtained by the square transformation of the chi distribution.

Keywords: sphere, Jacobian, multivariate normal distribution, chi distribution, probability density function (pdf).

1. Introduction

The chi-square distributed variable denoted by S^* with n degrees of freedom (df) is defined as the sum of the squares of n independent standard normals $S^* = \mathbf{X}^{\mathrm{T}}\mathbf{X}$, where $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ and $X_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1)\ (i = 1, \ldots, n)$. The probability density function (pdf) of the chi-square distribution with 1 df is given in two steps of variable transformation $X_1 \to Y_1 (=| X_1 |) \to S^* (= Y_1^2)$. The first step gives the half-normally distributed variable Y_1 whose pdf at y_1 is $\sqrt{2/\pi}\exp(-y_1^2/2)\ (0 \le y_1 < \infty)$. The second step yields the pdf of S^* at $s (= y_1^2)$ with the Jacobian $\mathrm{d}y_1/\mathrm{d}s = \mathrm{d}\sqrt{s}/\mathrm{d}s = 1/(2\sqrt{s})$ as

$$\begin{aligned}&\sqrt{2/\pi}\exp(-s/2)/(2\sqrt{s}) = \exp(-s/2)/(\sqrt{2s\pi})\\ &= s^{(1/2)-1}\exp(-s/2)/\{2^{1/2}\Gamma(1/2)\}\ (0 \le s < \infty),\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. The above result is equal to the pdf of the gamma distribution with the same shape and rate parameters 1/2. The chi-square with n df is typically given by the closed property that the sum of independent gammas with equal rate parameters is gamma with the shape parameter being the sum of the n shape parameters. The closed property can directly be derived using beta integral or indirectly using the moment generating functions for n independent variables. Then, we have the pdf of the chi-square with n df as the gamma with the shape parameter n/2 and unchanged rate parameter 1/2 as

$$f_{\chi^2}(s \mid n) = \frac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)}\ (0 \le s < \infty, n = 1, 2, \ldots).$$

2. A geometric derivation of the chi-square density

Recall that in the case of 1 df, we have the one-to-one correspondence in the variable transformation from X_1 to S^*. For the general case with n df, consider the variable transformation from **X** to those in the polar coordinate system, which typically takes the following relationship when $\mathbf{X} = \mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$:

$$x_i = r\left(\prod_{j=1}^{i-1}\sin\theta_j\right)\cos\theta_i \ (i=1,...,n-1), x_n = r\prod_{j=1}^{n-1}\sin\theta_j$$
$$(0 \le r < \infty;\ 0 \le \theta_i < \pi,\ i=1,...,n-2;\ 0 \le \theta_{n-1} < 2\pi),$$

where r and $\boldsymbol{\theta} = (\theta_1,..,\theta_{n-1})^{\mathrm{T}}$ are radial and angular coordinates, respectively with $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$. Noting that the pdf of $\mathbf{X} = \mathbf{x}$ is $\phi_n(\mathbf{x}) = \exp(-\mathbf{x}^{\mathrm{T}}\mathbf{x}/2)/(2\pi)^{n/2}$, the pdf in the polar coordinate system is written by

$$\frac{\exp(-r^2/2)}{(2\pi)^{n/2}} J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\},$$

where $J\{\mathbf{x} \to (r, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \det\{\mathrm{d}\,\mathbf{x}/\mathrm{d}(r, \boldsymbol{\theta}^{\mathrm{T}})\} = r^{n-1}g(\boldsymbol{\theta})$ is the Jacobian and $g(\boldsymbol{\theta})$ is a function of $\boldsymbol{\theta}$. The marginal density of the radial variable at r is given from the above:

$$\begin{aligned}
&\frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} r^{n-1} g(\boldsymbol{\theta}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-2} \mathrm{d}\theta_{n-1} \\
&= \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} S_{n-1}(r) = \frac{\exp(-r^2/2)}{(2\pi)^{n/2}} \frac{2\pi^{n/2} r^{n-1}}{\Gamma(n/2)} \\
&= \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)},
\end{aligned}$$

where $S_{n-1}(r) = 2\pi^{n/2} r^{n-1}/\Gamma(n/2)$ is the known formula for the surface area of the (n-1)-dimensional sphere in the n-dimensional Euclidian space (see e.g., Huber, 1982). Since $r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$, the above density is that of the square root of the chi-square or the chi distribution with n df denoted by $f_{\chi}(r\,|\,n) = r^{n-1}\exp(-r^2/2)/\{2^{(n/2)-1}\Gamma(n/2)\}$. Then, the pdf of the chi-square distributed variable S^* at $s = r^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is obtained as

$$f_{\chi^2}(s\,|\,n) = \frac{r^{n-1}\exp(-r^2/2)}{2^{(n/2)-1}\Gamma(n/2)} \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{s^{(n-1)/2}\exp(-s/2)}{2^{(n/2)-1}\Gamma(n/2)} \frac{1}{2\sqrt{s}} = \frac{s^{(n/2)-1}\exp(-s/2)}{2^{n/2}\Gamma(n/2)},$$

which is the required density.

3. Remarks

In the previous section the chi distribution was first obtained since the variable corresponds to the radial variable in the polar coordinate system. A direct transformation $\mathbf{x} \to \{s(=r^2), \boldsymbol{\theta}^{\mathrm{T}}\}^{\mathrm{T}}$ can also be used when desired. However, for this transformation the Jacobian becomes

$$J\{\mathbf{x} \to (s, \boldsymbol{\theta}^{\mathrm{T}})^{\mathrm{T}}\} = \left(\det \frac{\mathrm{d}\,\mathbf{x}}{\mathrm{d}(r, \boldsymbol{\theta}^{\mathrm{T}})} \right) \frac{\mathrm{d}\,r}{\mathrm{d}\,s} = r^{n-1} g(\boldsymbol{\theta}) \frac{1}{2\sqrt{s}} \,.$$

When the above result is integrated over $\boldsymbol{\theta}$, we have $S_{n-1}(r) / (2\sqrt{s})$, which also gives the pdf of the chi-square. While the two methods seem to be comparable, the name of “chi-square” indicates the important role of the chi distribution corresponding to the radial variable.

Reference

Huber, G. (1982). Gamma function derivation of *n*-sphere volumes. *American Mathematical Monthly, 89,* 301-302.