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Preprint · April 2022

DOI: 10.13140/RG.2.2.11878.50240

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April 17, 2022

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Abstract: Moments and the moment generating function of the univariate power-gamma distribution are obtained. This distribution was proposed elsewhere when a power of a variable follows the gamma distribution. The moments yield those of the marginal distributions of the multivariate power-gamma distribution introduced in this paper. The multivariate power-gamma distribution is based on the multivariate gamma distribution when each variable follows the distribution of the sum of independent gammas with a similar pattern found in factor analysis. Then, power transformations of the variables give the multivariate power-gamma. The probability density function (pdf) of the distribution is derived using integral or series expressions. The one-factor models with some restrictions yield simplified pdf's.

Keywords: multivariate gamma, power-gamma, moments, factor analysis, one-factor model, series expression.

1. Introduction

It is well-known that the gamma distribution has a variety of special cases e.g., the exponential and chi-squares with positive integers or real values for the degrees of freedom (df) corresponding to the shape parameter (Johnson, Kotz & Balakrishnan, 1994, Chapter 17). Transformations of gamma distributed variables e.g., the inverse gamma are also used. Among them the inverse chi-square, chi and inverse chi are relatively well-known as transformations of the chi-square. Especially, the inverse chi is recently focused on due to the convenience to yield the *t*-distribution by the product of a scaled inverse chi and an independent standard normal (Kollo, Käärik & Selart, 2021; Ogasawara, 2021).

The above transformations are seen as special cases of power transformations of the gamma. The power-gamma distribution was introduced by Ogasawara (2021) as a special case of the Amoroso (1925) distribution, whose probability density function (pdf; see Crooks, 2015, Equation (1)) is

$$g_{\text{Amoroso}}(x\,|\,a,\theta,\alpha,\beta)=\frac{|\beta/\theta|}{\Gamma(\alpha)}\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1}\exp\left\{-\left(\frac{x-a}{\theta}\right)^{\beta}\right\}$$
$$(x\geq a \text{ if } \theta>0;\ x\leq a \text{ if } \theta<0;\ \beta\in\mathrm{R},\ \beta\neq 0;\ \alpha>0;\ \theta\in\mathrm{R},\ \theta\neq 0).$$

Let Y be power-gamma distributed, which is denoted by $Y\sim\text{Power-}\Gamma(\alpha,\beta,\gamma)$. Then, the pdf of $Y=y$ is

$$\begin{aligned}g_{\text{Power-}\Gamma}(y\,|\,\alpha,\beta,\gamma)&=\frac{\beta^{\alpha}y^{\gamma(\alpha-1)}|\gamma|\,y^{\gamma-1}}{\Gamma(\alpha)}\exp\left(-\beta y^{\gamma}\right)\\&=\frac{\beta^{\alpha}|\gamma|\,y^{\gamma\alpha-1}}{\Gamma(\alpha)}\exp\left(-\beta y^{\gamma}\right)\end{aligned}\tag{1}$$
$$(0<y<\infty, 0<\alpha<\infty,\ 0<\beta<\infty,\ -\infty<\gamma<\infty,\ \gamma\neq 0).$$

The power-gamma is also seen as a reparametrization of Stacy's (1962) generalized gamma, which takes the pdf using our notation:

$$g_{\text{Generalized-}\Gamma}(y\,|\,\alpha,\beta^{*},\gamma)=\frac{\gamma y^{\alpha-1}}{\beta^{*\alpha}\Gamma(\alpha/\gamma)}\exp\{-(y/\beta^{*})^{\gamma}\}\tag{2}$$
$$(0<y<\infty, 0<\alpha<\infty,\ 0<\beta^{*}<\infty,\ 0<\gamma<\infty).$$

It is seen that the power gamma has an extension of the non-zero real power over the positive power of the generalized gamma. Note that β^* in (2) is a scale parameter while $\beta(=1/\beta^*)$ in (1) is the rate parameter when $\gamma = 1$.

It is found that special cases of the power-gamma are usual gamma ($\gamma = 1$), inverse gamma ($\gamma = -1$) and members of the power chi-square subfamily e.g., chi-square ($\alpha = \nu/2,\ \beta = 1/2,\ \gamma = 1$), inverse chi-square ($\alpha = \nu/2,\ \beta = 1/2,\ \gamma = -1$), chi ($\alpha = \nu/2,\ \beta = 1/2,\ \gamma = 2$) and inverse chi ($\alpha = \nu/2,\ \beta = 1/2,\ \gamma = -2$) with ν being a typically integer-valued df. Other special cases are half-normal ($\alpha = 1/2,\ \beta = 1/(2\sigma^2),\ 0 < \sigma^2 < \infty,\ \gamma = 2$; a special case of the scaled chi), basic power half-normal ($\alpha = 1/2,\ \beta = 1/(2\sigma^2),\ 0 < \sigma^2 < \infty$), exponential ($\alpha = 1,\ \gamma = 1$), power exponential ($\alpha = 1$; for the Box-Cox power exponential, see Rigby & Stasinopoulos, 2004; Voudouris, Gilchrist, Rigby, Sedgwick & Stasinopoulos, 2012) and a parametrization of the Weibull ($\alpha = \gamma > 0$). The basic power half-normal proposed above is seen as a reparametrization of a similar case given by Gómez and Bolfarine (2015). For Box–Cox symmetric or more generally elliptical distributions including power transformations, see Ferrari and Fumes (2017), and Morán-Vásquez and Ferrari (2019). For an associated review, see Johnson, Kotz and Balakrishnan (1994, Subsection 8.7).

Bivariate and multivariate extensions of the gamma distribution have also been given in various forms (Kotz, Balakrishnan & Johnson, 2000, Chapter 48). Among the extensions satisfying marginal gammas, Cherian (1941) gave a bivariate gamma with a form of the sums of a single common gamma and a unique independent one with equal scale parameters. The common and unique independent gammas in this form correspond to the common and unique factors in the one-factor model of exploratory factor analysis (EFA; for FA, see e.g., Harman, 1976, Chapter 2; Bollen, 1989, pp. 226-232).

One of the properties of FA including latent variables is the inflation of the dimensionality of associated variables. In the Cherian bivariate gamma, the number of independent gammas is 3, which is given by the single common gamma and two unique gammas and is larger than the number of two manifest or observable gammas. In the one-factor model of EFA with *p* observable variables, we have a single common factor and *p* unique factors, yielding an inflation of the dimensionality by 1 as in the Cherian model.

Principal component analysis (PCA) is used with similar purposes as FA. However, PCA is not a latent variable model since the number of components including minor ones is equal to that of observable ones yielding no inflation of dimensionality.

The Cherian bivariate gamma has been extended to the multivariate cases by Ramabhadran (1951) when independent exponentially distributed variables are used, and by Prékopa and Szántai (1978) for various patterns for sums of independent gammas corresponding to various patterns of factor loadings in FA (see also Kotz et al., 2000, Chapter 48, Section 2.2). Mathai and Moschopoulos (1991, Definition 1) gave a multivariate gamma of the one-factor type. Recently, Furman (2008, Theorem 3.1) showed a multivariate gamma with p fully or partially common gammas and a single unique gamma for p observable variables, where the pattern of the "ladder type" corresponding to that of the factor loading/pattern matrix in FA is employed.

In order to have e.g., the pdf and moment generating function (mgf) in the multivariate gammas of the FA type, some method of dimension reduction using single or multiple integrals for partialing out extra gamma(s) is required, which is called "trivariate reduction" or "variables-in-common method" in the bivariate case (Balakrishnan & Lai, 2009, Chapter 7) and "multivariate reduction" for general cases (Furman, 2008, Section 2).

The multivariate gammas using the same number of independent gammas as that for observable variables was given by Mathai and Moschopoulos (1992, Theorem 1.1), which is seen as a PCA type model. When one of the two independent "unique" gammas is omitted in the Cherian model, the bivariate gamma of the PCA type is obtained, where the single common factor becomes equal to one of the two observable variables with no trivariate reduction required. One of the limitations of models of the PCA type is the unexchangeability of observable variables, which is easily found in the bivariate gamma of this type mentioned above. On the other hand, an advantage for the PCA type is that the vector of independent chi-squares is reconstructed from that of observable gammas with the inverse of a square matrix corresponding to the factor pattern matrix in FA.

The purpose of this paper is to introduce the multivariate power-gamma distribution with some properties using the multivariate gammas of the FA type. The remainder of this paper is organized as follows. In Section 2, moments of the univariate power-gamma are shown. Section 3 gives the multivariate power-gamma of the FA type with some properties. In Section 4, three special cases of the one-factor models are dealt with, where series or closed expressions of the pdf's are derived with numerical illustrations for bivariate cases.

2. Moments of the power gamma distribution

The pdf of the power-gamma defined in (1) gives the following properties.

Result 1. Let $Y \sim \text{Power-}\Gamma(\alpha,\beta,\gamma)$. Then, we have

$$\begin{aligned}
\mathrm{E}(Y^k) &= \int_0^\infty \frac{\beta^\alpha |\gamma| y^{\gamma\alpha-1} y^k}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma\right) \mathrm{d}y \\
&= \int_0^\infty \frac{\beta^\alpha |\gamma| y^{\gamma\{\alpha+(k/\gamma)\}-1}}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma\right) \mathrm{d}y \\
&= \frac{\Gamma\{\alpha+(k/\gamma)\}\beta^\alpha}{\Gamma(\alpha)\beta^{\alpha+(k/\gamma)}} \int_0^\infty \frac{\beta^{\alpha+(k/\gamma)} |\gamma| y^{\gamma\{\alpha+(k/\gamma)\}-1}}{\Gamma\{\alpha+(k/\gamma)\}} \exp\left(-\beta y^\gamma\right) \mathrm{d}y \\
&= \frac{\Gamma\{\alpha+(k/\gamma)\}}{\Gamma(\alpha)\beta^{k/\gamma}},
\end{aligned}$$

$$0 < \alpha + (k/\gamma),$$

where k is real-valued with $0 < \alpha + (k/\gamma)$.

$$\mathrm{E}(Y) = \frac{\Gamma\{\alpha+(1/\gamma)\}}{\Gamma(\alpha)\beta^{1/\gamma}},$$

$$\mathrm{var}(Y) = \frac{1}{\beta^{2/\gamma}}\left[\frac{\Gamma\{\alpha+(2/\gamma)\}}{\Gamma(\alpha)} - \frac{\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)}\right],$$

$$\begin{aligned}
\mathrm{sk}(Y) &= \left[\frac{\Gamma\{\alpha+(3/\gamma)\}}{\Gamma(\alpha)} - 3\frac{\Gamma\{\alpha+(2/\gamma)\}\Gamma\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)} + 2\frac{\Gamma^3\{\alpha+(1/\gamma)\}}{\Gamma^3(\alpha)}\right] \\
&\quad \times \left[\frac{\Gamma\{\alpha+(2/\gamma)\}}{\Gamma(\alpha)} - \frac{\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)}\right]^{-3/2} \\
&= \left\{\Gamma\left(\alpha+\frac{3}{\gamma}\right)\Gamma^2(\alpha) - 3\Gamma\left(\alpha+\frac{2}{\gamma}\right)\Gamma\left(\alpha+\frac{1}{\gamma}\right)\Gamma(\alpha) + 2\Gamma^3\left(\alpha+\frac{1}{\gamma}\right)\right\} \\
&\quad \times \left\{\Gamma\left(\alpha+\frac{2}{\gamma}\right)\Gamma(\alpha) - \Gamma^2\left(\alpha+\frac{1}{\gamma}\right)\right\}^{-3/2},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{kt}(Y) &= \left[\frac{\Gamma\{\alpha+(4/\gamma)\}}{\Gamma(\alpha)} - 4\frac{\Gamma\{\alpha+(3/\gamma)\}\Gamma\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)}\right. \\
&\quad \left. + 6\frac{\Gamma\{\alpha+(2/\gamma)\}\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^3(\alpha)} - 3\frac{\Gamma^4\{\alpha+(1/\gamma)\}}{\Gamma^4(\alpha)}\right] \\
&\quad \times\left[\frac{\Gamma\{\alpha+(2/\gamma)\}}{\Gamma(\alpha)} - \frac{\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)}\right]^{-2} - 3 \\
&= \left[\frac{\Gamma\{\alpha+(4/\gamma)\}}{\Gamma(\alpha)} - 4\frac{\Gamma\{\alpha+(3/\gamma)\}\Gamma\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)} - 3\frac{\Gamma^2\{\alpha+(2/\gamma)\}}{\Gamma^2(\alpha)}\right. \\
&\quad \left. + 12\frac{\Gamma\{\alpha+(2/\gamma)\}\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^3(\alpha)} - 6\frac{\Gamma^4\{\alpha+(1/\gamma)\}}{\Gamma^4(\alpha)}\right] \\
&\quad \times\left[\frac{\Gamma\{\alpha+(2/\gamma)\}}{\Gamma(\alpha)} - \frac{\Gamma^2\{\alpha+(1/\gamma)\}}{\Gamma^2(\alpha)}\right]^{-2},
\end{aligned}
$$

where $\Gamma^j(\cdot) = \{\Gamma(\cdot)\}^j$; and $\mathrm{sk}(\cdot)$ and $\mathrm{kt}(\cdot)$ are the skewness and excess kurtosis of a random quantity in parentheses, respectively. The mgf is given by a series form:

$$
\begin{aligned}
\mathrm{M}_Y(t) &= \mathrm{E}\{\exp(tY)\} = \int_0^\infty \frac{\beta^\alpha |\gamma| y^{\gamma\alpha-1}}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma + ty\right)\mathrm{d}y \\
&= \int_0^\infty \frac{\beta^\alpha |\gamma| y^{\gamma\alpha-1}}{\Gamma(\alpha)} \exp\left(-\beta y^\gamma\right)\sum_{j=0}^\infty \frac{(ty)^j}{j!}\mathrm{d}y \\
&= \sum_{j=0}^\infty \frac{t^j\Gamma\{\alpha+(j/\gamma)\}}{\beta^{j/\gamma}\Gamma(\alpha)j!}\int_0^\infty \frac{\beta^{\{\alpha+(j/\gamma)\}}|\gamma| y^{\gamma\{\alpha+(j/\gamma)\}-1}}{\Gamma\{\alpha+(j/\gamma)\}}\exp\left(-\beta y^\gamma\right)\mathrm{d}y \\
&= \sum_{j=0}^\infty \frac{t^j\Gamma\{\alpha+(j/\gamma)\}}{\Gamma(\alpha)\beta^{j/\gamma}j!} \quad (\gamma>0),
\end{aligned}
$$

which gives the second derivation of $\mathrm{E}(Y^k) = \dfrac{\Gamma\{\alpha+(k/\gamma)\}}{\Gamma(\alpha)\beta^{k/\gamma}}$ when k is a positive integer, as shown earlier. The mgf does not exist when $\gamma < 0$, which is known in the case of the inverse gamma i.e., $\gamma = -1$.

When $\gamma = 1/m$ $(m = 1, 2, \ldots)$, the distribution of $Y \sim \text{Power-}\Gamma(\alpha,\beta,\gamma)$ reduces to that of $Y = X^m$, where X is gamma distributed with the shape and rate parameters α and β, respectively, which is denoted by $X \sim \mathrm{Gamma}(\alpha,\beta)$. Then, Result 1 or a known property of the gamma gives

$$\mathrm{E}(Y^k \mid \gamma = 1/m) = \frac{\Gamma\{\alpha + (k/\gamma)\}}{\Gamma(\alpha)\beta^{k/\gamma}} = \frac{\Gamma(\alpha + km)}{\Gamma(\alpha)\beta^{km}} = \frac{(\alpha)_{km}}{\beta^{km}}\ (k = 1,\ 2, \ldots; m = 1,\ 2, \ldots),$$

where $(\alpha)_j = \Gamma(\alpha + j)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha + j - 1)\ (j = 1,\ 2, \ldots)$ with $(\cdot)_0 = 1$ is the rising or ascending factorial using the Pochhammer notation, which yields the following results.

Result 2. Let $Y \sim \text{Power-}\Gamma(\alpha, \beta, \gamma = 1/m)\ (m = 1, 2, \ldots)$. Then, we have

$$\mathrm{E}(Y \mid \gamma = 1/m) = \frac{(\alpha)_m}{\beta^m},$$

$$\mathrm{var}(Y \mid \gamma = 1/m) = \frac{1}{\beta^{2m}}\{(\alpha)_{2m} - (\alpha)_m^2\},$$

$$\mathrm{sk}(Y \mid \gamma = 1/m) = \frac{(\alpha)_{3m} - 3(\alpha)_{2m}(\alpha)_m + 2(\alpha)_m^3}{\left\{(\alpha)_{2m} - (\alpha)_m^2\right\}^{3/2}},$$

$$\begin{aligned}\mathrm{kt}(Y \mid \gamma = 1/m) &= \frac{(\alpha)_{4m} - 4(\alpha)_{3m}(\alpha)_m + 6(\alpha)_{2m}(\alpha)_m^2 - 3(\alpha)_m^4}{\left\{(\alpha)_{2m} - (\alpha)_m^2\right\}^2} - 3 \\ &= \frac{(\alpha)_{4m} - 4(\alpha)_{3m}(\alpha)_m - 3(\alpha)_{2m}^2 + 12(\alpha)_{2m}(\alpha)_m^2 - 6(\alpha)_m^4}{\left\{(\alpha)_{2m} - (\alpha)_m^2\right\}^2},\end{aligned}$$

where $(\cdot)_m^j = \{(\cdot)_m\}^j$.

When m = 1, we see that Result 2 with $(\alpha)_1 = \alpha$ and similar simplified expressions gives the well-documented results of the usual gamma:

$$\mathrm{E}(Y \mid \gamma = 1) = \alpha/\beta,\ \mathrm{var}(Y \mid \gamma = 1) = \alpha/\beta^2,$$
$$\mathrm{sk}(Y \mid \gamma = 1) = \left\{(\alpha)_3 - 3(\alpha)_2\alpha + 2\alpha^3\right\}/\alpha^{3/2} = 2/\alpha^{1/2},$$
$$\mathrm{kt}(Y \mid \gamma = 1) = \left\{(\alpha)_4 - 4(\alpha)_3\alpha + 6(\alpha)_2\alpha^2 - 3\alpha^4\right\}\alpha^{-2} - 3 = 6/\alpha,$$

which are also obtained from the cumulant generating function (cgf) giving the j-th one:

$$\kappa_j(Y \mid \gamma = 1) = -\mathrm{d}^j \alpha \ln\left(1 - \frac{t}{\beta}\right)/\mathrm{d}t^j\,|_{t=0} = (j-1)!\,\alpha/\beta^j\,(j = 1,\ 2, \ldots).$$

Though Result 2 was presented when m is a positive integer, it is found that the expressions of Result 2 hold if an extended definition of $(\cdot)_m$ is employed as follows.

Result 3. Let $Y \sim \text{Power-}\Gamma(\alpha, \beta, \gamma = 1/m)\ (0 < m < \infty)$. Define

$(\alpha)_m = \Gamma(\alpha+m)/\Gamma(\alpha)$ with a positive real number *m*. Then, Result 2 holds when the positive integer *m* is seen as a positive real number.

Further, consider the case of a negative real number γ or equivalently a negative real *m*. Recall that Result 1 gives

$$\mathrm{E}(Y^k) = \frac{\Gamma\{\alpha+(k/\gamma)\}}{\Gamma(\alpha)\beta^{k/\gamma}} = \frac{\Gamma(\alpha+km)}{\Gamma(\alpha)\beta^{km}}, \quad 0 < \alpha+(k/\gamma) = \alpha+km\,,$$

which holds as long as the condition $0 < \alpha+(k/\gamma) = \alpha+km$ is satisfied irrespective of $-\infty < k < 0$, $-\infty < \gamma = 1/m < 0$ or both.

The case of both negative *k* and *m* is trivial in that

$$\mathrm{E}(Y^k) = \mathrm{E}\{(X^m)^k\} = \mathrm{E}(X^{km}) = \mathrm{E}(X^{|km|}) = \frac{\Gamma(\alpha+km)}{\Gamma(\alpha)\beta^{km}}\,,$$

which reduces to the case of both positive *k* and *m*, and always exists.

Result 4. Let $Y \sim \text{Power-}\Gamma(\alpha,\beta,\gamma=1/m)\ (m \neq 0)$. When $m < 0$, define $(\alpha)_m = \Gamma(\alpha+m)/\Gamma(\alpha)$ in the same expression as positive *m* in Result 3, we have

$$\mathrm{E}(Y^k) = \frac{(\alpha)_{km}}{\beta^{km}}, \quad 0 < \alpha+(k/\gamma) = \alpha+km\,.$$

The case with $m = \gamma = -1$ reduces to the inverse gamma. Then, Result 4 gives the moments when *k* is a positive integer as:

$$\begin{aligned}\mathrm{E}(Y^k \mid m=\gamma=-1) &= \frac{(\alpha)_{km}}{\beta^{km}} \\ &= (\alpha)_{-k}\beta^k = \frac{\beta^k}{(\alpha-1)(\alpha-2)\cdots(\alpha-k)}\ (k<\alpha).\end{aligned}$$

Recall that for positive integer *k*, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, while $(\alpha)_{-k}$ is the reciprocal of the descending factorial $(\alpha-1)(\alpha-2)\cdots(\alpha-k)$. When $\gamma=-1$, the role of β becomes a scale parameter, yielding β^k in the numerator of the above expression, which was located in the denominator when β was a rate parameter or the reciprocal of a scale parameter for $\gamma = 1$.

3. The multivariate power-gamma distribution of the FA type

In this section, based on the multivariate gamma distribution of the FA type, the multivariate power-gamma of the FA type is defined.

Definition 1. Let

$$\mathbf{Y}^* = (Y_1^{\gamma_1},\ldots,Y_p^{\gamma_p})^{\mathrm{T}} = \mathbf{\Lambda F} = (\mathbf{\Lambda}_0\ \mathbf{I}_p)\mathbf{F}\ (-\infty < \gamma_i < \infty,\ \gamma_i \neq 0,\ i = 1,\ldots,p)\,,$$

where $\mathbf{\Lambda}_0$ is a $p\times q$ $(1 \le q \le 2^p - n - 1)$ matrix consisting of 0 or 1; $\mathbf{F} = (F_1,\ldots,F_{p+q})^{\mathrm{T}}$ is a random vector whose $p + q$ elements are independently gamma distributed as $F_i \sim \mathrm{Gamma}(\alpha_i,\beta)\ (i = 1,\ldots,p+q)$; α_i and β are the shape and rate parameters, respectively with β common to the $p + q$ gammas; $\mathbf{I}_p$ is the $p\times p$ identity matrix; and it is assumed that each column of $\mathbf{\Lambda}_0$ has at least two 1's. The form $\mathbf{Y}^* = \mathbf{\Lambda F} = \mathbf{\Lambda}(\mathbf{F}_0^{\mathrm{T}},\mathbf{F}_1^{\mathrm{T}})^{\mathrm{T}}$ is seen as a FA model, where $\mathbf{F}_0 = (F_1,\ldots,F_q)^{\mathrm{T}}$ and $\mathbf{F}_1 = (F_{q+1},\ldots,F_{p+q})^{\mathrm{T}}$ are the vectors of common and unique factors, respectively; and $\mathbf{\Lambda}$ is the factor loading/pattern matrix. Then, $\mathbf{Y} = (Y_1,\ldots,Y_p)^{\mathrm{T}}$ is said to have the multivariate power-gamma distribution of the "FA type", which is denoted by

$$\mathbf{Y} \sim \text{Power-}\Gamma_p(\mathbf{\Lambda\alpha},\beta,\boldsymbol{\gamma}) = \text{Power-}\Gamma_p\{\mathbf{\Lambda}(\alpha_1,\ldots,\alpha_{p+q})^{\mathrm{T}},\beta,(\gamma_1,\ldots,\gamma_p)^{\mathrm{T}}\}\,.$$

In Definition 1, $2^p - n - 1$ for the upper limit of q is due to the 0/1 pattern of $\mathbf{\Lambda}_0$ excluding the single column of all zeroes, and n columns of a single 1 corresponding to unique factors.

Lemma 1. *Define* $\boldsymbol{\lambda}_{\cdot j} = (\lambda_{1j},\ldots,\lambda_{pj})^{\mathrm{T}}$ *as the j-th column of* $\mathbf{\Lambda}_0$ $(j = 1,\ldots,q)$. *Then, the moment generating function (mgf) of* $\mathbf{Y}^*$ *is*

$$\mathrm{M}_{\mathbf{Y}^*}(\mathbf{t}) = \left[\prod_{j=1}^{q}\frac{1}{\{1-(\mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot j}/\beta)\}^{\alpha_j}}\right]\prod_{i=1}^{p}\frac{1}{\{1-(t_i/\beta)\}^{\alpha_{q+i}}}$$

$$(0 \le \mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot j}/\beta < 1,\ j = 1,\ldots,q;\ 0 \le t_i/\beta < 1,\ i = 1,\ldots,p).$$

Proof. The mgf of $\mathbf{Y}^*$ is given from that of $\mathbf{F}$:

$$\mathrm{M}_{\mathbf{F}}(\mathbf{t}^*) = \left[\prod_{j=1}^{q}\frac{1}{\{1-(t_j^*/\beta)\}^{\alpha_j}}\right]\prod_{i=1}^{p}\frac{1}{\{1-(t_{q+i}^*/\beta)\}^{\alpha_{q+i}}}\,.$$

Since $\mathbf{Y}^* = \mathbf{\Lambda F}$, the mgf of $\mathbf{Y}^*$ is

$$\begin{aligned}
&\mathrm{M}_{\mathbf{Y}^*}(\mathbf{t}) = \mathrm{E}\{\exp(\mathbf{t}^{\mathrm{T}}\mathbf{Y}^*)\} = \mathrm{E}\{\exp(\mathbf{t}^{\mathrm{T}}\mathbf{\Lambda}_0\mathbf{F}_0 + \mathbf{t}^{\mathrm{T}}\mathbf{F}_1)\} \\
&= \left\{\prod_{j=1}^{q}\mathrm{E}(\mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot j}F_j)\right\}\prod_{i=1}^{p}\mathrm{E}(t_i F_{q+i}) \\
&= \left[\prod_{j=1}^{q}\frac{1}{\{1-(\mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot j}/\beta)\}^{\alpha_j}}\right]\prod_{i=1}^{p}\frac{1}{\{1-(t_i/\beta)\}^{\alpha_{q+i}}} \\
&(0 \le \mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot j}/\beta < 1,\ j = 1,\dots,q;\ 0 \le t_i/\beta < 1,\ i = 1,\dots,p),
\end{aligned}$$

which gives the required result. Q.E.D.

A special case of Lemma 1 in the case of the one-factor model including location parameters was given by Mathai and Moschopoulos (1991, Equation (2)).

Result 5. Lemma 1 gives the following.

(i) Let $\boldsymbol{\lambda}_{i\cdot}^{\mathrm{T}} = (\lambda_{i1},\dots,\lambda_{i,p+q})$ be the *i*-th row of $\mathbf{\Lambda}$ $(i = 1,\dots,p)$. The mgf of the marginal distribution of Y_i^* is given when $\mathbf{t} = (0,\dots,0,t_i,0,\dots,0)^{\mathbf{T}}$ $(i = 1,\dots,p)$ as

$$\begin{aligned}
\mathrm{M}_{Y_i^*}(t_i) &= \left[\prod_{j=1}^{q}\frac{1}{\{1-(t_i 1\{\lambda_{ij}=1\}/\beta)\}^{\alpha_j}}\right]\frac{1}{\{1-(t_i/\beta)\}^{\alpha_{q+i}}} \\
&= \frac{1}{\{1-(t_i/\beta)\}^{\alpha_{q+i}+\sum_{j=1}^{q}1\{\lambda_{ij}=1\}\alpha_j}} = \frac{1}{\{1-(t_i/\beta)\}^{\boldsymbol{\lambda}_i^{\mathrm{T}}\boldsymbol{\alpha}}},
\end{aligned}$$

where $1\{\cdot\}$ is the indicator function. Then, it is found that the marginal distributions are gamma:

$$Y_i^{\gamma_i} = Y_i^* \sim \mathrm{Gamma}(\boldsymbol{\lambda}_i^{\mathrm{T}}\boldsymbol{\alpha},\beta) \equiv \mathrm{Gamma}(\alpha_{(i)},\beta)\ (i = 1,\dots,p).$$

The above result is expected since Y_i^* is the sum of independent gammas with the same rate parameters, whose number of variables and the sum of their shape parameters are $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{1}_{p+q}$ and $\boldsymbol{\lambda}_i^{\mathrm{T}}\boldsymbol{\alpha}$, respectively.

(ii) Let $\boldsymbol{\alpha}_0 = (\alpha_1,\dots,\alpha_q)^{\mathrm{T}}$ and $\boldsymbol{\alpha}_1 = (\alpha_{q+1},\dots,\alpha_{p+q})^{\mathrm{T}}$. Then, the mean and variance of Y_i^*, and the covariance matrix of $\mathbf{Y}^*$ are

$$\mathrm{E}(Y_i^*) = \alpha_{(i)} / \beta,\ \mathrm{var}(Y_i^*) = \alpha_{(i)} / \beta^2\ \ (i = 1, ..., p),$$
$$\mathrm{cov}(\mathbf{Y}^*) = \boldsymbol{\Lambda}\mathrm{diag}(\boldsymbol{\alpha})\boldsymbol{\Lambda}^{\mathrm{T}} / \beta^2 = \{\boldsymbol{\Lambda}_0\mathrm{diag}(\boldsymbol{\alpha}_0)\boldsymbol{\Lambda}_0^{\mathrm{T}} + \mathrm{diag}(\boldsymbol{\alpha}_1)\} / \beta^2.$$

The last result of $\mathrm{cov}(\mathbf{Y}^*)$ shows a typical covariance structure of the orthogonal FA model, where $\mathrm{diag}(\boldsymbol{\alpha}_0) / \beta^2$ and $\mathrm{diag}(\boldsymbol{\alpha}_1) / \beta^2$ are the covariance matrices of q orthogonal common factors and p unique factors, respectively.

(iii) The k-th order moments (k = 1, 2,…) of Y_i^* is given by $\mathrm{M}_{\mathbf{Y}^*}(\mathbf{t})$ obtained in Lemma 1 or more easily by $\mathrm{M}_{Y_i^*}(t_i)$ given in (i) as a mgf of a gamma:

$$\mathrm{E}(Y_i^{*k}) = \frac{\mathrm{d}^k}{\mathrm{d}t_i^k}\mathrm{M}_{Y_i^*}(t_i)\,|_{t_i=0} = \frac{\mathrm{d}^k}{\mathrm{d}t_i^k}\frac{1}{\{1-(t_i/\beta)\}^{\alpha_{(i)}}}\,|_{t_i=0}$$
$$= \alpha_{(i)}(\alpha_{(i)}+1)\cdots(\alpha_{(i)}+k-1)/\beta^k = \frac{(\alpha_{(i)})_k}{\beta^k} = \frac{\Gamma(\alpha_{(i)}+k)}{\beta^k\Gamma(\alpha_{(i)})}\ \ (k = 1, 2, ...).$$

The real-valued k-th moment $(k \neq 0,\ \alpha_{(i)} + k > 0)$ should be given from the definition:

$$\mathrm{E}(Y_i^{*k}) = \int_0^\infty \frac{\beta^{\alpha_{(i)}} y_i^{*k} y_i^{*(\alpha_{(i)}-1)}}{\Gamma(\alpha_{(i)})}\exp(-\beta y_i^*)\mathrm{d}y_i^*$$
$$= \frac{\beta^{\alpha_{(i)}}\Gamma(\alpha_{(i)}+k)}{\beta^{\alpha_{(i)}+k}\Gamma(\alpha_{(i)})}\int_0^\infty \frac{\beta^{\alpha_{(i)}+k} y_i^{*(\alpha_{(i)}+k-1)}}{\Gamma(\alpha_{(i)}+k)}\exp(-\beta y_i^*)\mathrm{d}y_i^*$$
$$= \frac{\Gamma(\alpha_{(i)}+k)}{\beta^k\Gamma(\alpha_{(i)})} = \frac{(\alpha_{(i)})_k}{\beta^k},$$

where the extended notation defined by $(\alpha_{(i)})_k = \Gamma(\alpha_{(i)}+k)/\Gamma(\alpha_{(i)})$ with possibly negative real value k introduced in Result 4 is used.

The cumulant generating function for Y_i^* is given by

$$\mathrm{K}_{Y_i^*}(t_i) = \ln \mathrm{M}_{Y_i^*}(t_i) = -\alpha_{(i)}\ln\{1-(t_i/\beta)\},$$

which gives the k-th cumulant:

$$\frac{\mathrm{d}^k}{\mathrm{d}t_i^k}\mathrm{K}_{Y_i^*}(t_i)\,|_{t_i=0} = -\frac{\mathrm{d}^k}{\mathrm{d}t_i^k}\alpha_{(i)}\ln\{1-(t_i/\beta)\}\,|_{t_i=0} = \frac{\alpha_{(i)}(k-1)!}{\beta^k}\ (k = 1, 2, ...).$$

The $(k_1, ..., k_r)$ th product cumulant of $Y_i^*(i = 1, ..., r;\ r = 2, ..., p)$ is given by

$$\frac{\mathrm{d}^{k_1+\ldots+k_r}}{\mathrm{d}t_1^{k_1}\cdots\mathrm{d}t_r^{k_r}}\mathrm{K}_{\mathbf{Y}^*}(\mathbf{t})\,|_{\mathbf{t}=\mathbf{0}}$$
$$=-\frac{\mathrm{d}^{k_1+\ldots+k_r}}{\mathrm{d}t_1^{k_1}\cdots\mathrm{d}t_r^{k_r}}\left[\sum_{m=1}^{q}\alpha_m\ln\{1-(\mathbf{t}^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot m}/\beta)\}+\sum_{m=1}^{p}\alpha_{q+m}\ln\{1-(t_m/\beta)\}\right]|_{\mathbf{t}=\mathbf{0}}$$
$$=\sum_{m=1}^{q}\alpha_m\frac{1\{\lambda_{1m}\times\cdots\times\lambda_{rm}=1\}(k_1+\ldots+k_r-1)!}{\beta^{k_1+\ldots+k_r}}$$
$$(k_i=1,2,\ldots),$$

where the results for the first r variables are given without loss of generality excluding the case of $r = 1$. As expected, the last result shows that the cross product cumulants depend only on those of the common factors (compare the above results with the corresponding ones of Mathai and Moschopoulos, 1992, Section 3 and Furman, 2008, Section 3).

The distribution of $\mathbf{Y}^* = \boldsymbol{\Lambda}\mathbf{F}$ is a special case of $\text{Power-}\Gamma_p(\boldsymbol{\Lambda}\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})$ when $\boldsymbol{\gamma}=\mathbf{1}_p$, where $\mathbf{1}_p$ is the $p\times 1$ vector of 1's. Then, $\mathbf{Y}^*$ is said to have the multivariate gamma-distribution of the FA type:

$$\mathbf{Y}^* \sim \text{Power-}\Gamma_p(\boldsymbol{\Lambda}\boldsymbol{\alpha},\beta,\mathbf{1}_p)\equiv\Gamma_p(\boldsymbol{\Lambda}\boldsymbol{\alpha},\beta).$$

The multivariate power chi-square distribution is defined as a special case of the multivariate power-gamma as

$$\text{Power-}\Gamma_p(\boldsymbol{\Lambda}\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})=\text{Power-}\Gamma_p(\boldsymbol{\Lambda}\mathbf{n}/2,\,1/2,\boldsymbol{\gamma})\equiv\text{Power-}\chi_p^2(\boldsymbol{\Lambda}\mathbf{n},\boldsymbol{\gamma})$$

with $\mathbf{n}=(n_1,\ldots,n_{p+q})^{\mathrm{T}}$, which is obtained when

$$F_i \sim \text{gamma}(\alpha_i,\beta)=\text{gamma}(n_i/2,1/2)\ (i=1,\ldots,p+q)$$

i.e., F_i is chi-square distributed with n_i degrees of freedom (df), denoted by $F_i \sim \chi^2(n_i)$. Note that $Y_i^* = \boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{F}\sim\chi^2(\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{n})$, which gives the definition of the multivariate chi-square of the FA type:

$$\mathbf{Y}^* \sim \text{Power-}\chi_p^2(\boldsymbol{\Lambda}\mathbf{n},\mathbf{1}_p)=\chi_p^2(\boldsymbol{\Lambda}\mathbf{n}).$$

Typically, n_i's are positive integers though positive real values can also be used.

Recall the definitions $\boldsymbol{\alpha}_0=(\alpha_1,\ldots,\alpha_q)^{\mathrm{T}}$ and $\boldsymbol{\alpha}_1=(\alpha_{q+1},\ldots,\alpha_{p+q})^{\mathrm{T}}$ with the corresponding notations $\mathbf{F}_0$ and $\mathbf{F}_1$. Recall also that $\boldsymbol{\lambda}_{\cdot j}$ is the j-th column of

$\mathbf{\Lambda}_0$ $(j = 1,...,q)$. Let $\alpha_0 = \mathbf{1}_q^{\mathrm{T}} \boldsymbol{\alpha}_0$, $\alpha_+ = \mathbf{1}_p^{\mathrm{T}} \boldsymbol{\alpha}_1$, $\boldsymbol{\lambda}_{0i\cdot} = (\lambda_{i1},...,\lambda_{iq})^{\mathrm{T}}$ $(i = 1,..., p)$, and $y_+^* = \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*$.

Lemma 2. *The pdf of* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{\Lambda}\boldsymbol{\alpha}, \beta)$ *of the FA type at* $\mathbf{Y}^* = \mathbf{y}^*$ *is*

$$
\begin{aligned}
&g_{\Gamma,p}(\mathbf{y}^* \mid \mathbf{\Lambda}\boldsymbol{\alpha}, \beta) \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*)}{\prod_{i=1}^{p+q} \Gamma(\alpha_i)} \\
&\times \sum_{j=0}^{\infty} \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} (y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0)^{\alpha_{q+i}-1} \right\} \prod_{i=1}^{q} \frac{\{\beta(\mathbf{1}_p^{\mathrm{T}} \boldsymbol{\lambda}_{\cdot i} - 1)\}^j f_i^{\alpha_i + j - 1}}{j!} \mathrm{d}\mathbf{f}_0.
\end{aligned}
$$

Proof. The FA model gives

$$Y_i^* = \boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{F} = \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{F}_0 + F_{q+i} \ (i = 1,..., p).$$

Consider the joint distribution of $\mathbf{F}$ and use the variable transformation from $\mathbf{F}_1$ to $\mathbf{Y}^* = \mathbf{\Lambda}\mathbf{F} = \mathbf{\Lambda}_0 \mathbf{F}_0 + \mathbf{F}_1$ with unchanged $\mathbf{F}_0$ and the unit Jacobian. Let $g_{\Gamma}(f_i \mid \alpha_i, \beta)$ be the pdf of $F_i \sim \mathrm{Gamma}(\alpha_i, \beta)$ at $F_i = f_i$ $(i = 1,..., p + q)$. Noting that $\mathbf{F}_0$ and the elements of $\mathbf{Y}^* - \mathbf{\Lambda}_0 \mathbf{F}_0 = \mathbf{F}_1$ are independently distributed, the pdf of $\mathbf{Y}^*$ is obtained by integrating out $\mathbf{F}_0$:

$$
\begin{aligned}
&g_{\Gamma,p}(\mathbf{y}^* \mid \mathbf{\Lambda}\boldsymbol{\alpha}, \beta) \\
&= \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} g_{\Gamma}(y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0 \mid \alpha_{q+i}, \beta) \right\} \prod_{i=1}^{q} g_{\Gamma}(f_i \mid \alpha_i, \beta) \, \mathrm{d}\mathbf{f}_0 \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*)}{\prod_{i=1}^{p+q} \Gamma(\alpha_i)} \\
&\times \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} (y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0)^{\alpha_{q+i}-1} \right\} \left(\prod_{i=1}^{q} f_i^{\alpha_i - 1} \right) \exp\{\beta(\mathbf{1}_p^{\mathrm{T}} \mathbf{\Lambda}_0 \mathbf{f}_0 - \mathbf{1}_q^{\mathrm{T}} \mathbf{f}_0)\} \mathrm{d}\mathbf{f}_0,
\end{aligned}
$$

where $\int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} (\cdot) \mathrm{d}\mathbf{f}_0$ indicates the multiple integral with respect to the elements of $\mathbf{f}_0 = (f_1,..., f_q)^{\mathrm{T}}$ over the region $\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0$ i.e., $\bigcap_{i=1}^{q} \{y_i^* > \boldsymbol{\lambda}_{0i\cdot}^{\mathrm{T}} \mathbf{f}_0\}$. Using

$$\exp\{\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\Lambda}_0\mathbf{f}_0 - \mathbf{1}_q^{\mathrm{T}}\mathbf{f}_0)\} = \exp\left\{\sum_{i=1}^{q}\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot i} - 1)f_i\right\}$$
$$= \prod_{i=1}^{q}\sum_{j=0}^{\infty}\left\{\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot i} - 1)f_i\right\}^j / j!,$$

the previous result becomes

$$\int_{\mathbf{y}^*>\boldsymbol{\Lambda}_0\mathbf{f}_0}\left\{\prod_{i=1}^{p} g_{\Gamma}(y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}}\mathbf{f}_0 \mid \alpha_{q+i},\beta)\right\}\prod_{i=1}^{q} g_{\Gamma}(f_i \mid \alpha_i,\beta)\,\mathrm{d}\mathbf{f}_0$$
$$= \frac{\beta^{\alpha_+ + \alpha_0}\exp(-\beta\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^*)}{\prod_{i=1}^{p+q}\Gamma(\alpha_i)}$$
$$\times\int_{\mathbf{y}^*>\boldsymbol{\Lambda}_0\mathbf{f}_0}\left\{\prod_{i=1}^{p}(y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}}\mathbf{f}_0)^{\alpha_{q+i}-1}\right\}\prod_{i=1}^{q} f_i^{\alpha_i-1}\sum_{j=0}^{\infty}\frac{\left\{\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot i} - 1)f_i\right\}^j}{j!}\mathrm{d}\mathbf{f}_0$$
$$= \frac{\beta^{\alpha_+ + \alpha_0}\exp(-\beta\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^*)}{\prod_{i=1}^{p+q}\Gamma(\alpha_i)}$$
$$\times\sum_{j=0}^{\infty}\int_{\mathbf{y}^*>\boldsymbol{\Lambda}_0\mathbf{f}_0}\left\{\prod_{i=1}^{p}(y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}}\mathbf{f}_0)^{\alpha_{q+i}-1}\right\}\prod_{i=1}^{q}\frac{\left\{\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot i} - 1)\right\}^j f_i^{\alpha_i+j-1}}{j!}\mathrm{d}\mathbf{f}_0.$$

Q.E.D.

Among the FA type models, consider the one-factor model with $q = 1$, $\boldsymbol{\Lambda}_0 = \mathbf{1}_p$,

$$\boldsymbol{\Lambda}\mathbf{F} = (\boldsymbol{\Lambda}_0\ \ \mathbf{I}_p)\mathbf{F} = \mathbf{1}_p F_0 + \mathbf{F}_1 \text{ and } \boldsymbol{\Lambda}\boldsymbol{\alpha} = \mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1,$$

where $\mathbf{F} = (F_0, \mathbf{F}_1^{\mathrm{T}})^{\mathrm{T}} = (F_0, F_1, ..., F_p)^{\mathrm{T}}$ and $\boldsymbol{\alpha} = (\alpha_0, \boldsymbol{\alpha}_1^{\mathrm{T}})^{\mathrm{T}} = (\alpha_0, \alpha_1, ..., \alpha_p)^{\mathrm{T}}$ with $\alpha_0, \alpha_1, ..., \alpha_p$ redefined.

Lemma 3. *The pdf of* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor type at* $\mathbf{Y}^* = \mathbf{y}^*$ *is*

$$g_{\Gamma,p}(\mathbf{y}^* \mid \mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta)$$
$$= \frac{\beta^{\alpha_+ + \alpha_0}\exp(-\beta y_+^*)}{\prod_{i=0}^{p}\Gamma(\alpha_i)}\sum_{j=0}^{\infty}\int_0^{\min\{\mathbf{y}^*\}}\left\{\prod_{i=1}^{p}(y_i^* - f_0)^{\alpha_i-1}\right\}\frac{\{\beta(p-1)\}^j f_0^{\alpha_0+j-1}}{j!}\mathrm{d}f_0.$$

Proof. The one-factor model gives

$$Y_i^* = \boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{F} = F_0 + F_i \ (i = 1, ..., p).$$

Consider the joint distribution of **F** and use the variable transformation from $\mathbf{F}_1$ to

$\mathbf{Y}^* = \mathbf{\Lambda F} = \mathbf{1}_p F_0 + \mathbf{F}_1$ with unchanged F_0 and the unit Jacobian. Let $g_{\Gamma}(f_0 \,|\, \alpha_0, \beta)$ be the pdf of $F_0 \sim \mathrm{gamma}(\alpha_0, \beta)$ at $F_0 = f_0$. Noting that F_0 and the elements of $\mathbf{Y}^* - \mathbf{1}_p F_0 = \mathbf{F}_1$ are independently distributed, the pdf of $\mathbf{Y}^*$ is obtained by integrating out F_0:

$$
\begin{aligned}
& g_{\Gamma,p}(\mathbf{y}^* \,|\, \mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta) \\
&= \int_0^{\min\{y_1^*,\ldots,y_1^*\}} \left\{ \prod_{i=1}^{p} g_{\Gamma}(y_i^* - f_0 \,|\, \alpha_i, \beta) \right\} g_{\Gamma}(f_0 \,|\, \alpha_0, \beta)\, \mathrm{d}f_0 \\
&= \frac{\beta^{\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1 + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \int_0^{\min\{\mathbf{y}^*\}} \left\{ \prod_{i=1}^{p} (y_i^* - f_0)^{\alpha_i - 1} \right\} f_0^{\alpha_0 - 1} \exp\{\beta(p-1) f_0\}\, \mathrm{d}f_0 \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \int_0^{\min\{\mathbf{y}^*\}} \left\{ \prod_{i=1}^{p} (y_i^* - f_0)^{\alpha_i - 1} \right\} f_0^{\alpha_0 - 1} \sum_{j=0}^{\infty} \frac{\{\beta(p-1) f_0\}^j}{j!}\, \mathrm{d}f_0 \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \sum_{j=0}^{\infty} \int_0^{\min\{\mathbf{y}^*\}} \left\{ \prod_{i=1}^{p} (y_i^* - f_0)^{\alpha_i - 1} \right\} \frac{\{\beta(p-1)\}^j f_0^{\alpha_0 + j - 1}}{j!}\, \mathrm{d}f_0.
\end{aligned}
$$

Q.E.D.

Mathai and Moschopoulos (1991, Section 5) gave an expression for a model essentially equivalent to that in Lemma 3 with no integral though they use (p + 1)-fold infinite series.

Result 6. *The pdf of* $Y_+^* \equiv \mathbf{1}_p^{\mathrm{T}} \mathbf{Y}^*$ *with* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor type at* $Y_+^* = y_+^* = \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*$ when $\mathbf{Y}^* = \mathbf{y}^*$ *is*

$$
\begin{aligned}
& g_{Y_+^*}(y_+^* \,|\, \mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta) \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\Gamma(\alpha_+)\Gamma(\alpha_0)} \int_0^{\min\{\mathbf{y}^*\}} (y_+^* - p f_0)^{\alpha_+ - 1} f_0^{\alpha_0 - 1} \exp\{\beta(p-1) f_0\}\, \mathrm{d}f_0 \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*) y_+^{*(\alpha_+ + \alpha_0 - 1)}}{\Gamma(\alpha_+)\Gamma(\alpha_0) p^{\alpha_0}} \sum_{j=0}^{\infty} \frac{\{\beta(p-1) y_+^*\}^j}{p^j j!} \mathrm{B}(p \min\{\mathbf{y}^*\} / y_+^* \,|\, \alpha_0 + j, \alpha_+),
\end{aligned}
$$

where $\min\{\mathbf{y}^*\} = \min\{y_1^*, \ldots, y_p^*\}$ *and* $\mathrm{B}(c \,|\, a, b) = \int_0^c t^{a-1}(1-t)^{b-1} \mathrm{d}t \quad (0 \le c \le 1)$ *is the incomplete beta function*.

Proof. Noting that $\mathbf{1}_p^{\mathrm{T}}\mathbf{Y}^* - pf_0 \sim \mathrm{Gamma}(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1, \beta)$, as in Lemma 4, we have:

$$\int_0^{\min\{\mathbf{y}^*\}} g_{\Gamma}(\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^* - pf_0 \,|\, \mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1, \beta) g_{\Gamma}(f_0 \,|\, \alpha_0, \beta)\,\mathrm{d}f_0$$

$$= \frac{\beta^{\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1+\alpha_0}\exp(-\beta\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^*)}{\Gamma(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1)\Gamma(\alpha_0)}\int_0^{\min\{\mathbf{y}^*\}} (\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^* - pf_0)^{\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\alpha}_1-1} f_0^{\alpha_0-1}\exp\{\beta(p-1)f_0\}\,\mathrm{d}f_0$$

$$= \frac{\beta^{\alpha_+ +\alpha_0}\exp(-\beta y_+^*)}{\Gamma(\alpha_+)\Gamma(\alpha_0)}\int_0^{\min\{\mathbf{y}^*\}} (y_+^* - pf_0)^{\alpha_+-1} f_0^{\alpha_0-1}\exp\{\beta(p-1)f_0\}\,\mathrm{d}f_0.$$

Define $e_0 = pf_0 / y_+^*$ with $\mathrm{d}f_0 / \mathrm{d}e_0 = y_+^* / p$. Then, the above integral becomes

$$\int_0^{\min\{\mathbf{y}^*\}} (y_+^* - pf_0)^{\alpha_+-1} f_0^{\alpha_0-1}\exp\{\beta(p-1)f_0\}\,\mathrm{d}f_0$$

$$= \int_0^{p\min\{\mathbf{y}^*\}/y_+^*} (y_+^* - y_+^* e_0)^{\alpha_+-1}(y_+^* e_0 / p)^{\alpha_0-1}(y_+^* / p)\exp\{\beta(p-1)y_+^* e_0 / p\}\,\mathrm{d}e_0$$

$$= \frac{y_+^{*(\alpha_+ +\alpha_0-1)}}{p^{\alpha_0}}\int_0^{p\min\{\mathbf{y}^*\}/y_+^*} (1-e_0)^{\alpha_+-1} e_0^{\alpha_0-1}\sum_{j=0}^{\infty}\frac{\{\beta(p-1)y_+^* e_0\}^j}{p^j j!}\,\mathrm{d}e_0$$

$$= \frac{y_+^{*(\alpha_+ +\alpha_0-1)}}{p^{\alpha_0}}\sum_{j=0}^{\infty}\frac{\{\beta(p-1)y_+^*\}^j}{p^j j!}\int_0^{p\min\{\mathbf{y}^*\}/y_+^*} e_0^{\alpha_0+j-1}(1-e_0)^{\alpha_+-1}\,\mathrm{d}e_0$$

$$= \frac{y_+^{*(\alpha_+ +\alpha_0-1)}}{p^{\alpha_0}}\sum_{j=0}^{\infty}\frac{\{\beta(p-1)y_+^*\}^j}{p^j j!}\mathrm{B}(p\min\{\mathbf{y}^*\} / y_+^* \,|\, \alpha_0 + j, \alpha_+),$$

which gives the required result. Q.E.D.

In Result 6, note that $\mathbf{1}_p^{\mathrm{T}}\mathbf{Y}^* - pf_0 = f_1 + \ldots + f_p$ is gamma distributed, whereas $\mathbf{1}_p^{\mathrm{T}}\mathbf{Y}^* = f_1 + \ldots + f_p + pf_0$ is not gamma since the latter is a linear combination of independent gammas with equal scales rather than their sum unless $p = 1$.

Theorem 1. *The pdf of multivariate power-gamma distributed* $\mathbf{Y} = (Y_1, \ldots, Y_p)^{\mathrm{T}}$ *at* $\mathbf{Y} = \mathbf{y} = (y_1, \ldots, y_p)^{\mathrm{T}}$ *with* $\mathbf{y}^* = (y_1^{\gamma_1}, \ldots, y_p^{\gamma_p})^{\mathrm{T}}$ *when* $\mathbf{Y}^* = (Y_1^{\gamma_1}, \ldots, Y_p^{\gamma_p})^{\mathrm{T}}$ $\sim \Gamma_p(\boldsymbol{\Lambda\alpha}, \beta)$ *of the FA type is*

$$g_{\text{Power-}\Gamma,p}(\mathbf{y}\,|\,\boldsymbol{\Lambda\alpha},\beta,\boldsymbol{\gamma})$$
$$=\frac{\beta^{\alpha_+ +\alpha_0}\exp(-\beta\mathbf{1}_p^{\mathrm{T}}\mathbf{y}^*)\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}}{\prod_{i=1}^{p+q}\Gamma(\alpha_i)}$$
$$\times\sum_{j=0}^{\infty}\int_{\mathbf{y}^*>\boldsymbol{\Lambda}_0\mathbf{f}_0}\left\{\prod_{i=1}^{p}(y_i^*-\boldsymbol{\lambda}_{0i}^{\mathrm{T}}\mathbf{f}_0)^{\alpha_{q+i}-1}\right\}\prod_{i=1}^{q}\frac{\{\beta(\mathbf{1}_p^{\mathrm{T}}\boldsymbol{\lambda}_{\cdot i}-1)\}^j f_i^{\alpha_i+j-1}}{j!}\mathrm{d}\mathbf{f}_0.$$

Proof. Noting that the Jacobian of the variable transformation from $\mathbf{Y}^*$ to $\mathbf{Y}$ is $\prod_{i=1}^{p}\mathrm{d}y_i^*/\mathrm{d}y_i=\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}$, Lemma 2 gives the required pdf. Q.E.D.

Corollary 1. *The pdf of multivariate power-gamma distributed* $\mathbf{Y}=(Y_1,...,Y_p)^{\mathrm{T}}$ *at* $\mathbf{Y}=\mathbf{y}=(y_1,...,y_p)^{\mathrm{T}}$ *with* $\mathbf{y}^*=(y_1^{\gamma_1},...,y_p^{\gamma_p})^{\mathrm{T}}$ *when* $\mathbf{Y}^*=(Y_1^{\gamma_1},...,Y_p^{\gamma_p})^{\mathrm{T}}$ $\sim\Gamma_p(\mathbf{1}_p\alpha_0+\boldsymbol{\alpha}_1,\beta)$ *of the one-factor type is*

$$g_{\text{Power-}\Gamma,p}(\mathbf{y}\,|\,\mathbf{1}_p\alpha_0+\boldsymbol{\alpha}_1,\beta,\boldsymbol{\gamma})$$
$$=\frac{\beta^{\alpha_+ +\alpha_0}\exp(-\beta y_+^*)\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}}{\prod_{i=0}^{p}\Gamma(\alpha_i)}\sum_{j=0}^{\infty}\int_0^{\min\{\mathbf{y}^*\}}\left\{\prod_{i=1}^{p}(y_i^*-f_0)^{\alpha_i-1}\right\}\frac{\{\beta(p-1)\}^j f_0^{\alpha_0+j-1}}{j!}\mathrm{d}f_0.$$

Proof. Using the same Jacobian as in Theorem 1, Lemma 3 gives the required pdf. Q.E.D.

Recall that $\boldsymbol{\lambda}_i^{\mathrm{T}}$ is the *i*-th row of $\boldsymbol{\Lambda}$ in the FA type model with the definition of $\alpha_{(i)}=\boldsymbol{\lambda}_i^{\mathrm{T}}\boldsymbol{\alpha}\quad(i=1,..,p)$.

Theorem 2. *The k-th order moment with k being possibly non-integer and/or negative value for the marginal distribution of power-gamma distributed* Y_i *when* $Y_i^*=Y_i^{\gamma_i}\sim\text{Gamma}(\alpha_{(i)},\beta)\quad(i=1,...,p)$ *in the FA type model is*

$$\mathrm{E}(Y_i^k)=(\alpha)_{k/\gamma_i}/\beta^{k/\gamma_i},\ \ 0<\alpha+(k/\gamma_i)\ (\gamma_i\neq 0,\ i=1,...,p).$$

Proof. Note that $Y_i\sim\text{Power-}\Gamma(\alpha_{(i)},\beta,\gamma_i)\ (\gamma_i\neq 0)$. Using the extended notation $(\alpha_{(i)})_{1/\gamma_i}=\Gamma\{\alpha_{(i)}+(1/\gamma_i)\}/\Gamma(\alpha_{(i)})\ \ (0<\alpha_{(i)}+(1/\gamma_i))$ for possibly non-integer and/or negative γ_i *and k*, Result 4 gives the required result. Q.E.D.

Lemma 4. *The product moment of multivariate power-gamma distributed*

$\mathbf{Y} = (Y_1, ..., Y_p)^{\mathrm{T}}$ *at* $\mathbf{Y} = \mathbf{y} = (y_1, ..., y_p)^{\mathrm{T}}$ *with* $\mathbf{y}^* = (y_1^{\gamma_1}, ..., y_p^{\gamma_p})^{\mathrm{T}}$ *when* $\mathbf{Y}^* = (Y_1^{\gamma_1}, ..., Y_p^{\gamma_p})^{\mathrm{T}} \sim \Gamma_p(\mathbf{\Lambda\alpha}, \beta)$ *of the FA type is*

$$\mathrm{E}\left(\prod_{i=1}^{p} Y_i^{k_i}\right) = \mathrm{E}\left(\prod_{i=1}^{p} Y_i^{*k_i/\gamma_i}\right).$$

Proof. By definition.

$$\mathrm{E}\left(\prod_{i=1}^{p} Y_i^{k_i}\right) = \int_{\mathbf{0}}^{\infty} \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*) \prod_{i=1}^{p} |\gamma_i| \, y_i^{\gamma_i + k_i - 1}}{\prod_{i=1}^{p+q} \Gamma(\alpha_i)}$$

$$\times \sum_{j=0}^{\infty} \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} (y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0)^{\alpha_{q+i} - 1} \right\} \prod_{i=1}^{q} \frac{\{\beta(\mathbf{1}_p^{\mathrm{T}} \boldsymbol{\lambda}_{\cdot i} - 1)\}^j f_i^{\alpha_i + j - 1}}{j!} \mathrm{d}\mathbf{f}_0 \mathrm{d}\mathbf{y}.$$

Employ the variable transformation from $\mathbf{y}$ to $\mathbf{y}^*$ with the Jacobian $\prod_{i=1}^{p} \mathrm{d}y_i / \mathrm{d}y_i^* = \prod_{i=1}^{p} |1/\gamma_i| \, y_i^{*\{(1/\gamma_i) - 1\}}$. Then, the above result becomes

$$\mathrm{E}\left(\prod_{i=1}^{p} Y_i^{k_i}\right) = \int_{\mathbf{0}}^{\infty} \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*) \prod_{i=1}^{p} y_i^{*\{\gamma_i + k_i - 1\}/\gamma_i} y_i^{*\{(1/\gamma_i) - 1\}}}{\prod_{i=1}^{p+q} \Gamma(\alpha_i)}$$

$$\times \sum_{j=0}^{\infty} \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} (y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0)^{\alpha_{q+i} - 1} \right\} \prod_{i=1}^{q} \frac{\{\beta(\mathbf{1}_p^{\mathrm{T}} \boldsymbol{\lambda}_{\cdot i} - 1)\}^j f_i^{\alpha_i + j - 1}}{j!} \mathrm{d}\mathbf{f}_0 \mathrm{d}\mathbf{y}^*$$

$$= \int_{\mathbf{0}}^{\infty} \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta \mathbf{1}_p^{\mathrm{T}} \mathbf{y}^*) \prod_{i=1}^{p} y_i^{*k_i/\gamma_i}}{\prod_{i=1}^{p+q} \Gamma(\alpha_i)}$$

$$\times \sum_{j=0}^{\infty} \int_{\mathbf{y}^* > \mathbf{\Lambda}_0 \mathbf{f}_0} \left\{ \prod_{i=1}^{p} (y_i^* - \boldsymbol{\lambda}_{0i}^{\mathrm{T}} \mathbf{f}_0)^{\alpha_{q+i} - 1} \right\} \prod_{i=1}^{q} \frac{\{\beta(\mathbf{1}_p^{\mathrm{T}} \boldsymbol{\lambda}_{\cdot i} - 1)\}^j f_i^{\alpha_i + j - 1}}{j!} \mathrm{d}\mathbf{f}_0 \mathrm{d}\mathbf{y}^*$$

$$= \mathrm{E}\left(\prod_{i=1}^{p} Y_i^{*k_i/\gamma_i}\right).$$

Q.E.D.

The result of Lemma 4 is expected since the variable transformation used in the proof restores the original gammas.

Theorem 3. *The* $(k_1, ..., k_p)$*-th product moment and cumulant of* $Y_1, ..., Y_p$ *are equal to the corresponding* $(k_1/\gamma_1, ..., k_p/\gamma_p)$*-th ones of* $Y_1^*, ..., Y_p^*$. *When*

$k_i / \gamma_i \equiv k_i^* (i=1,...,p)$ *are positive integers, the cumulants are given by Result 5 (iii) when* k_i *is replaced by* $k_i^* (i=1,...,p)$.

Theorem 3 is not trivial in that k_i and γ_i can be real-valued and/or negative as long as the condition is satisfied.

4. Special cases of the one-factor model and numerical illustrations

In this section, three special cases of the one-factor model are shown with their simplified properties. The first special case has a restriction of integer-valued shape parameters for unique factors i.e., $F_i (i=1,...,p)$ with an unconstrained single common factor F_0.

Lemma 5. *The pdf of* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{1}_p \alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor model at* $\mathbf{Y}^* = \mathbf{y}^*$, *when the elements of* $\boldsymbol{\alpha}_1 = (\alpha_1,...,\alpha_p)^{\mathrm{T}}$ *are all positive integers, is*

$$\begin{aligned}
& g_{\Gamma,p}(\mathbf{y}^* | \mathbf{1}_p \alpha_0 + \boldsymbol{\alpha}_1, \beta) \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\Gamma(\alpha_0) \prod_{i=1}^{p} (\alpha_i - 1)!} \sum_{h_1=0}^{\alpha_1 - 1} \cdots \sum_{h_p=0}^{\alpha_p - 1} \left\{ \prod_{i=1}^{p} \binom{\alpha_i - 1}{h_i} y_i^{*(\alpha_i - 1 - h_i)} (-1)^{h_i} \right\} \\
&\times \sum_{j=0}^{\infty} \frac{\{\beta(p-1)\}^j (\min\{\mathbf{y}^*\})^{\alpha_0 + j + h_1 + ... + h_p}}{j!(\alpha_0 + j + h_1 + ... + h_p)}.
\end{aligned}$$

Proof. When $\alpha_1,...,\alpha_p$ are positive integers, the pdf derived in Lemma 3 for the one-factor model can be expanded as

$$\begin{aligned}
& g_{\Gamma,p}(\mathbf{y}^* | \mathbf{1}_p \alpha_0 + \boldsymbol{\alpha}_1, \beta) \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \sum_{j=0}^{\infty} \int_0^{\min\{\mathbf{y}^*\}} \left\{ \prod_{i=1}^{p} (y_i^* - f_0)^{\alpha_i - 1} \right\} \frac{\{\beta(p-1)\}^j f_0^{\alpha_0 + j - 1}}{j!} \mathrm{d}f_0 \\
&= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \\
&\times \sum_{j=0}^{\infty} \int_0^{\min\{\mathbf{y}^*\}} \left\{ \prod_{i=1}^{p} \sum_{h_i=0}^{\alpha_i - 1} \binom{\alpha_i - 1}{h_i} y_i^{*(\alpha_i - 1 - h_i)} (-f_0)^{h_i} \right\} \frac{\{\beta(p-1)\}^j f_0^{\alpha_0 + j - 1}}{j!} \mathrm{d}f_0
\end{aligned}$$

$$= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \sum_{h_1=0}^{\alpha_1 - 1} \cdots \sum_{h_p=0}^{\alpha_p - 1} \left\{ \prod_{i=1}^{p} \binom{\alpha_i - 1}{h_i} y_i^{*(\alpha_i - 1 - h_i)} (-1)^{h_i} \right\}$$

$$\times \sum_{j=0}^{\infty} \int_0^{\min\{\mathbf{y}^*\}} \frac{\{\beta(p-1)\}^j f_0^{\alpha_0 + j + h_1 + \ldots + h_p - 1}}{j!} \mathrm{d}f_0$$

$$= \frac{\beta^{\alpha_+ + \alpha_0} \exp(-\beta y_+^*)}{\prod_{i=0}^{p} \Gamma(\alpha_i)} \sum_{h_1=0}^{\alpha_1 - 1} \cdots \sum_{h_p=0}^{\alpha_p - 1} \left\{ \prod_{i=1}^{p} \binom{\alpha_i - 1}{h_i} y_i^{*(\alpha_i - 1 - h_i)} (-1)^{h_i} \right\}$$

$$\times \sum_{j=0}^{\infty} \frac{\{\beta(p-1)\}^j (\min\{\mathbf{y}^*\})^{\alpha_0 + j + h_1 + \ldots + h_p}}{j!(\alpha_0 + j + h_1 + \ldots + h_p)}.$$

Q.E.D.

When $\alpha_i = 2$ in the case of Lemma 5, F_i is $1/\sqrt{2\beta}$ times chi-distributed. The second special case is given when $\alpha_i = 1\ (i = 1, \ldots, p)$. That is, all the unique factors are $1/\beta$ times exponentially distributed.

Lemma 6. *The pdf of* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{1}_p \alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor model at* $\mathbf{Y}^* = \mathbf{y}^*$, *when the elements of* $\boldsymbol{\alpha}_1 = (\alpha_1, \ldots, \alpha_p)^{\mathrm{T}}$ *are all one i.e.,* $\boldsymbol{\alpha}_1 = \mathbf{1}_p$, *is*

$$g_{\Gamma,p}\{\mathbf{y}^* \mid \mathbf{1}_p(\alpha_0 + 1), \beta\}$$

$$= \frac{\beta^{p+\alpha_0} \exp(-\beta y_+^*)}{\Gamma(\alpha_0)} \int_0^{\min\{\mathbf{y}^*\}} f_0^{\alpha_0 - 1} \exp\{\beta(p-1) f_0\} \mathrm{d}f_0$$

$$= \frac{\beta^{p+\alpha_0} \exp(-\beta y_+^*)}{\Gamma(\alpha_0)} \sum_{j=0}^{\infty} \frac{\{\beta(p-1)\}^j (\min\{\mathbf{y}^*\})^{\alpha_0 + j}}{j!(\alpha_0 + j)}.$$

Proof. When the additional condition of $\boldsymbol{\alpha}_1 = \mathbf{1}_p$ in Lemma 5 is imposed, the requited result follows. Alternatively, the result is directly obtained from the above integral expression. Q.E.D.

As mentioned earlier, Ramabhadran (1951) dealt with the case of $\alpha_0 = 1$ with $\beta = 1$ as well as $\boldsymbol{\alpha}_1 = \mathbf{1}_p$, which indicates that all the *p* + 1 common/unique factors are exponentially distributed. Then, consider a scaled Ramabhadran's case with $0 < \beta < \infty$.

Lemma 7. *The pdf of* $\mathbf{Y}^* \sim \Gamma_p(\mathbf{1}_p \alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor model at* $\mathbf{Y}^* = \mathbf{y}^*$, *when the elements of* $\boldsymbol{\alpha} = (\alpha_0, \boldsymbol{\alpha}_1^{\mathrm{T}})^{\mathrm{T}} = (\alpha_0, \alpha_1, ..., \alpha_p)^{\mathrm{T}}$ *are all one, is*

$$g_{\Gamma,p}(\mathbf{y}^* \mid 2\mathbf{1}_p, \beta) = \beta^p \frac{\exp\{\beta(p-1)\min\{\mathbf{y}^*\}\} - 1}{\exp(\beta y_+^*)(p-1)}.$$

Proof. When the condition of α_0 is added in Lemma 6, we have

$$\begin{aligned}
&g_{\Gamma,p}(\mathbf{y}^* \mid \mathbf{1}_p(\alpha_0+1), \beta) = g_{\Gamma,p}(\mathbf{y}^* \mid 2\mathbf{1}_p, \beta) \\
&= \beta^{p+1}\exp(-\beta y_+^*)\sum_{j=0}^{\infty}\frac{\{\beta(p-1)\}^j(\min\{\mathbf{y}^*\})^{j+1}}{(j+1)!} \\
&= \beta^{p+1}\exp(-\beta y_+^*)\frac{1}{\beta(p-1)}\sum_{j=0}^{\infty}\frac{\{\beta(p-1)\}^{j+1}(\min\{\mathbf{y}^*\})^{j+1}}{(j+1)!} \\
&= \beta^p \frac{\exp\{\beta(p-1)\min\{\mathbf{y}^*\}\} - 1}{\exp(\beta y_+^*)(p-1)},
\end{aligned}$$

which is also given from the expression

$$\beta^{p+1}\exp(-\beta y_+^*)\int_0^{\min\{\mathbf{y}^*\}}\exp\{\beta(p-1)f_0\}\,\mathrm{d}f_0.$$

Q.E.D.

Lemmas 5, 6 and 7 yield the following results.

Theorem 4. *(i) The pdf of multivariate power-gamma distributed* $\mathbf{Y} = (Y_1, ..., Y_p)^{\mathrm{T}}$ *at* $\mathbf{Y} = \mathbf{y} = (y_1, ..., y_p)^{\mathrm{T}}$ *with* $\mathbf{y}^* = (y_1^{\gamma_1}, ..., y_p^{\gamma_p})^{\mathrm{T}}$ *when* $\mathbf{Y}^* = (Y_1^{\gamma_1}, ..., Y_p^{\gamma_p})^{\mathrm{T}}$ $\sim \Gamma_p(\mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta)$ *of the one-factor model with the elements of* $\boldsymbol{\alpha}_1 = (\alpha_1, ..., \alpha_p)^{\mathrm{T}}$ *being all positive integers, is*

$$\begin{aligned}
&g_{\text{Power-}\Gamma,p}(\mathbf{y} \mid \mathbf{1}_p\alpha_0 + \boldsymbol{\alpha}_1, \beta, \boldsymbol{\gamma}) \\
&= \frac{\beta^{\alpha_+ + \alpha_0}\exp(-\beta y_+^*)\prod_{i=1}^p |\gamma_i|\, y_i^{\gamma_i - 1}}{\Gamma(\alpha_0)\prod_{i=1}^p(\alpha_i - 1)!}\sum_{h_1=0}^{\alpha_1-1}\cdots\sum_{h_p=0}^{\alpha_p-1}\left\{\prod_{i=1}^p\binom{\alpha_i - 1}{h_i}y_i^{*(\alpha_i-1-h_i)}(-1)^{h_i}\right\} \\
&\quad\times\sum_{j=0}^{\infty}\frac{\{\beta(p-1)\}^j(\min\{\mathbf{y}^*\})^{\alpha_0+j+h_1+...+h_p}}{j!(\alpha_0+j+h_1+...+h_p)}.
\end{aligned}$$

(ii) In the above case, if the additional condition of $\boldsymbol{\alpha}_1 = \mathbf{1}_p$ *is employed, we have*

$$g_{\text{Power-}\Gamma,p}(\mathbf{y}\,|\,\mathbf{1}_p(\alpha_0+1),\beta,\boldsymbol{\gamma})$$
$$=\frac{\beta^{p+\alpha_0}\exp(-\beta y_+^*)\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}}{\Gamma(\alpha_0)}\int_0^{\min\{\mathbf{y}^*\}} f_0^{\alpha_0-1}\exp\{\beta(p-1)f_0\}\mathrm{d}f_0$$
$$=\frac{\beta^{p+\alpha_0}\exp(-\beta y_+^*)\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}}{\Gamma(\alpha_0)}\sum_{j=0}^{\infty}\frac{\{\beta(p-1)\}^j(\min\{\mathbf{y}^*\})^{\alpha_0+j}}{j!(\alpha_0+j)}.$$

(iii) When all the elements of $\boldsymbol{\alpha}=(\alpha_0,\alpha_1,...,\alpha_p)^{\mathrm{T}}$ *are one, we obtain*

$$g_{\text{Power-}\Gamma,p}(\mathbf{y}\,|\,2\mathbf{1}_p,\beta,\boldsymbol{\gamma})=\beta^p\frac{\left[\exp\{\beta(p-1)\min\{\mathbf{y}^*\}\}-1\right]\prod_{i=1}^{p}|\gamma_i\,|\,y_i^{\gamma_i-1}}{\exp(\beta y_+^*)(p-1)}$$

Before presenting numerical illustrations, we provide the following properties for clarity.

Result 7. Consider $\mathbf{Y}^*=(Y_1^*,...,Y_p^*)^{\mathrm{T}}\sim\Gamma_p(\mathbf{1}_p\alpha_0+\boldsymbol{\alpha}_1,\beta)$ i.e., variables before power transformation for the one-factor model.

(i) $0<\alpha_0/\beta^2=\mathrm{cov}(Y_i^*,Y_j^*)$
$<\min\{\mathrm{var}(Y_i^*),\mathrm{var}(Y_i^*)\}=(\alpha_0+\min\{\alpha_i,\alpha_j\})/\beta^2\ \ (i\neq j).$

(ii) When α_0 goes to +0, F_0 approaches the one-point distribution of zero in probability with $\mathbf{Y}^*$ approaching p independent gammas. When $\boldsymbol{\alpha}_1\to\mathbf{0}$, $Y_i^*(i=1,...,p)$ approach the identical gamma $\Gamma(\alpha_0,\beta)$.

(iii) $\mathrm{cov}(1/Y_i^*,Y_i^*)<0$ when the covariance exists. This property is confirmed as

$$\mathrm{cov}(1/Y_i^*,Y_i^*)=1-\mathrm{E}(1/Y_i^*)\mathrm{E}(Y_i^*)$$
$$=1-\frac{\Gamma(\alpha_0+\alpha_i-1)\beta}{\Gamma(\alpha_0+\alpha_i)}\frac{\Gamma(\alpha_0+\alpha_i+1)}{\Gamma(\alpha_0+\alpha_i)\beta}=1-\frac{\alpha_0+\alpha_i}{\alpha_0+\alpha_i-1}=-\frac{1}{\alpha_0+\alpha_i-1}<0$$
$(\alpha_0+\alpha_i>1).$

Result 7 (i) of positive covariances smaller than the associated variances is a limitation of the model in applications as noted by Prékopa and Szántai (1978), which can be relaxed by considering the sum of gammas with different scale parameters as in the usual FA model or linear combinations of equal-scale gammas. However, these relaxations generally yield non-gamma distributed marginals with complicated pdf's (see Mathai, 1982; Moschopoulos, 1985). The obvious properties of Result 7 (ii) show that the one-factor model is situated between those of a single gamma and independent p gammas. The negative covariance of Result 7 (iii) is expected since $1/Y_i^*$ is a decreasing function in the support.

Numerical illustrations of the pdf's of the bivariate power-gammas when the one-factor models of Lemmas 6 and 7 hold are shown in Figures 1 and 2, respectively. In Figure 1 with a single common gamma, $\alpha_0 = 0.2$ is used with two unique independent exponentials, which gives $\mathrm{cor}(Y_1^*, Y_2^*) = \alpha_0 / (\alpha_0 + 1) = 1/6$, while in Figure 2 with three independent common/unique exponentials, we always have $\mathrm{cor}(Y_1^*, Y_2^*) = 1/2$. Three pairs of powers $\boldsymbol{\gamma} = (-2, -1)^{\mathrm{T}}, (-2, 2)^{\mathrm{T}}, (2, 2)^{\mathrm{T}}$ are used for transformation in the figures. The first case of $\boldsymbol{\gamma} = (-2, -1)^{\mathrm{T}}$ gives two decreasing transformations of the original gammas while the second case of $\boldsymbol{\gamma} = (-2, 2)^{\mathrm{T}}$ yields a decreasing and an increasing transforms as in Result 7 (ii). The rate parameter $\beta = 0.8$ is used for all the cases. The pdf's are shown by a mesh with 30^2 points of $Y_i = 0\ (0.1)\, 3\ (i = 1, 2)$ using finite approximations of the series expression in Lemma 6. As suggested by Result 7 (iii), in Figure 1, the relatively less correlated Y_i^*'s by construction are found to be conveyed to the corresponding transformed ones. In Figure 2, increased correlations are shown in the transformed variables.

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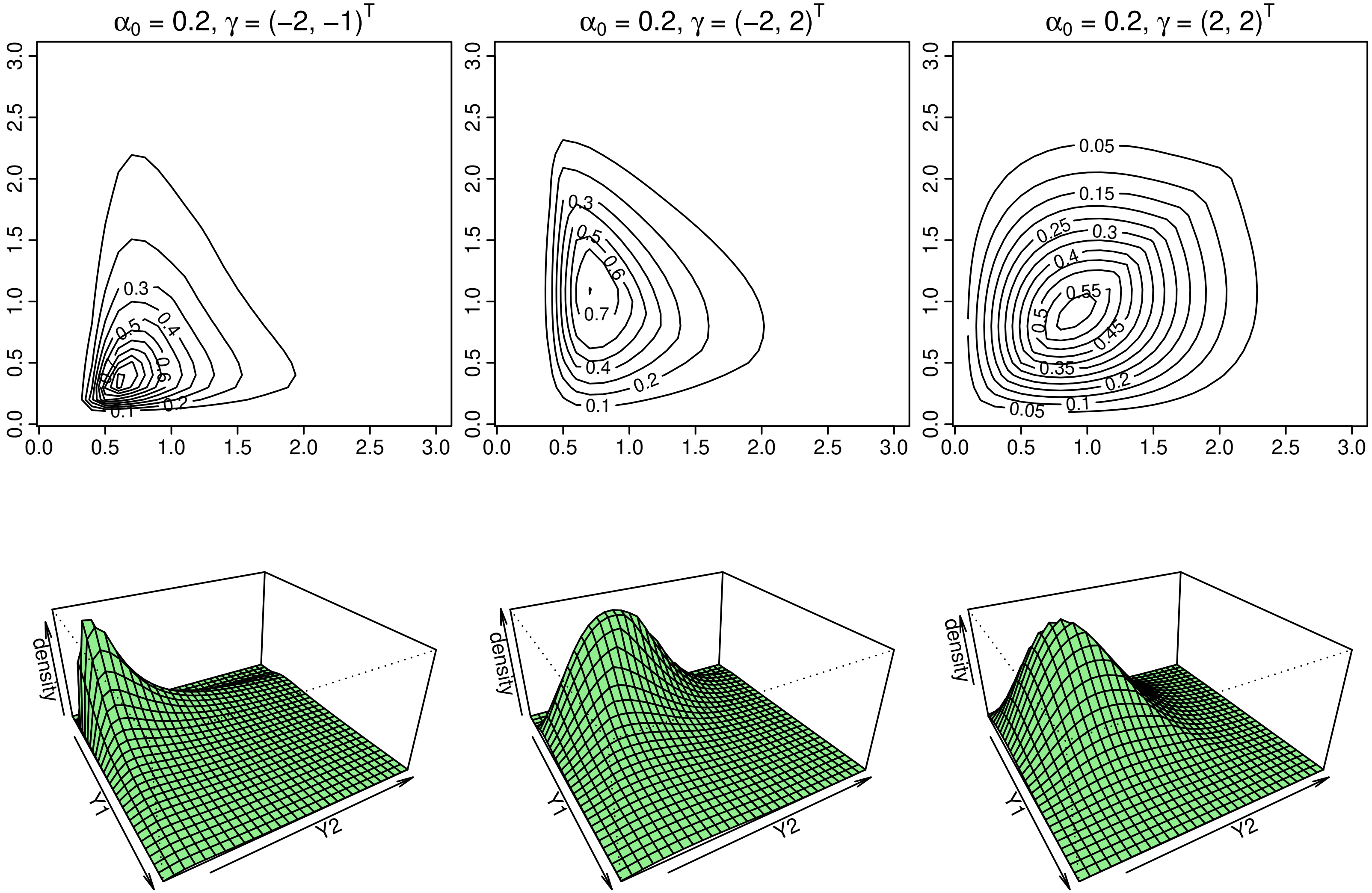

Figure 1. Density plots of the bivariate power−gamma distributions
with a common gamma and unique exponentials
(Y_1 = the horizontal axis, Y_2 = the vertical axis for upper plots, $\beta = 0.8$)

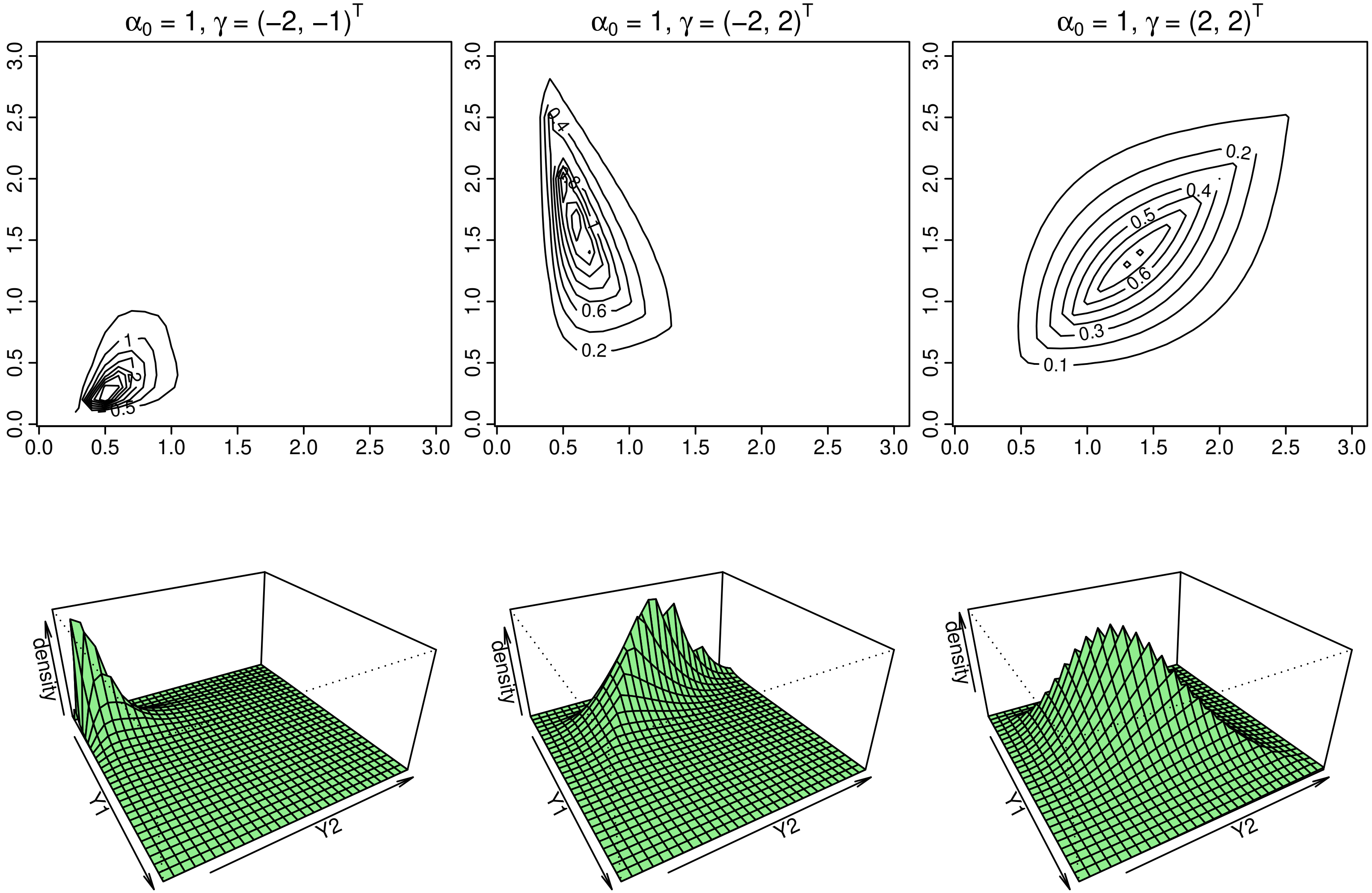

Figure 2. Density plots of the bivariate power−gamma distributions
with common and unique exponentials
(Y_1 = the horizontal axis, Y_2 = the vertical axis for upper plots, $\beta = 0.8$)