A didactic historical review of the distributions using the Bessel function: Some extensions with unification

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Abstract: Based on a didactic historical review of the distributions using the Bessel function, some new extensions with unification are shown. Their probability density functions (pdf's) are given by several integral representations of the Bessel function, whose distributions are called the basic Bessel (BB) ones including the known Halphen type A. Power transformations of the BB (PBB) distributions are provided with their moments. The differential equations for the PBB distributions are shown and used to have their modes. A unified expression for unimodal extended distributions is also given with the differential equation.

Keywords: inverse Gaussian distribution, basic Bessel distribution, incomplete Bessel function, Halphen distribution type A, differential equation.

1. Introduction

This paper deals with distributions including the factor $\exp(-au - bu^{-1})g(u)$ $(a > 0,\ b > 0)$ in the probability density function (pdf) denoted by $f_U(u)$ for a positive random variable U, where u can be a monotonic function of u. The normalizing constant generally includes a Bessel function, which will be didactically mentioned later. The history associated with this distribution can be traced back to an observation of pollen found by Robert Brown in 1828, which is currently known as a Brownian motion. This issue has been investigated by A. Einstein in 1905 and M. C. K. Tweedie in his 1941 thesis (see Tweedie, 1945 for an essential result; and details in Tweedie, 1957a, b). The distribution associated with Brownian motion, a special case of the above pdf, was called the inverse Gaussian (IG) distribution by Tweedie in 1956 (for the history of the IG distribution, see Chhikara and Folks, 1989, Section 1.2; Seshadri, 1999, Section 1.0). The IG distribution has been used in various fields not only in the natural and physical sciences but also in the social and behavioral sciences e.g., cardiology, hydrology, demography, linguistics, employment service, labor disputes and finance typically in lifetime researches in a broad sense as noted by Chhikara and Folks (1989, Section 1.4 and Chapter 10), and additional fields as ecology, entomology, health sciences, traffic noise intensity, management sciences, actuarial sciences, histomorphometry, electrical networks, meteorology, mental health, physiology, remote sensing, market research, slug lengths in pipelines and plutonium estimation as shown by Seshadri (1999, page. v and Part II) with real life examples. Note that in the IG distribution, the Bessel function does not seem to appear though it is implicitly used as will be explained later.

Use of the (modified) Bessel function in statistics dates back to Fisher (1928, p. 663) for the multiple correlation coefficient followed by Pearson, Jeffery and Elderton (1929, pp. 165, 187; for citing Fisher's work see p. 193) and, Wishart and Bartlett (1932, Equation (12)) for the product distribution of two correlated normal variables. Note that the product distribution is a special case of the McKay Bessel distribution as noted by Wishart and Bartlett (1932, p. 459) as well as McKay (1932, p. 43) (for the product distribution, see Ogasawara, 2023b). Apart from these developments using the Bessel functions in statistics, or in Brownian motion, explicit representations of the pdf $f_U(u)$ were independently provided by Halphen (1941), a French mathematician as well as a hydrologist, which are currently summarized as the Halphen type A (HA) distribution, whose normalizing constant includes a Bessel function as noted by Halphen (1941, p. 635).

Halphen's harmonic distribution or its generalized HA distribution has been mostly unnoted in the academic community other than in the French-speaking statistical hydrologists until the Halphen family of distributions including the HA was introduced as late as in the 1980's and later (see Dvorák, Bobée, Boucher & Ashkar, 1988; Barndorff-Nielsen, Blaesild & Seshadri, 1992, Section 2; Perreault, Fortin & Bobée, 1994; Seshadri, 1997; 1999, Part II, Section F; Perreault et al.,1999a, b). Recent textbooks by statistical hydrologists on the Halphen family of distributions with extensions are e.g., El Adlouni and Bobée (2017), and Singh and Zhang (2022).

On the other hand, the IG distribution, whose pdf (see e.g., Chhikara & Folks, 1989, Equation (2.1)) denoted by $f_U^{(\mathrm{IG})}(u\,|\,\mu,\lambda)\ \ (u>0;\mu>0,\lambda>0)$ is easily found to be a special case of $f_U(u)$:

$$f_U^{(\mathrm{IG})}(u\,|\,\mu,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{1/2}u^{-3/2}\exp\left\{-\frac{\lambda(u-\mu)^2}{2\mu^2u}\right\}$$

$$=\left\{\left(\frac{\pi}{2\lambda/\mu}\right)^{1/2}\exp\left(-\frac{\lambda}{\mu}\right)\right\}^{-1}\mu^{-1}\frac{1}{2}\exp\left\{-\frac{\lambda}{2\mu}\left(\frac{u}{\mu}+\frac{\mu}{u}\right)\right\}\left(\frac{u}{\mu}\right)^{-3/2},$$

when $a=\lambda/(2\mu^2)$, $b=\lambda/2$; $g(u)=\mu^{-1}\dfrac{1}{2}\left(\dfrac{u}{\mu}\right)^{-3/2}$; and $\left\{\left(\dfrac{\pi}{2\lambda/\mu}\right)^{1/2}\exp\left(-\dfrac{\lambda}{\mu}\right)\right\}^{-1}$ is a normalizing constant. The pdf $f_U^{(\mathrm{IG})}(u\,|\,\mu,\lambda)$ looks free from the Bessel function. However, it is shown that the normalizer includes the Besel function as follows. Without loss of generality, employ the reparametrization $z=\lambda/\mu\ (z>0)$ and consider that u is given after change of variable as $u^*=u/\mu$ with μ^{-1} being the Jacobian, where μ is the scale parameter and u^* corresponds to the unit scale variable U^*. Then, it is seen that the normalizing constant is given by $K_{-1/2}(z)^{-1}$, where $K_\nu(z)\,(\nu\in\mathrm{R},\ z>0)$ is the modified Bessel function of the second kind of order ν (for $K_{-1/2}(z)=K_{1/2}(z)=\{\pi/(2z)\}^{1/2}\mathrm{e}^{-z}$, see Watson, 1944/1995, Section 3.71, Equation (13), p. 80; Abramowitz & Stegun, 1972, Formula 10.2.17, p. 444; Zwillinger, 2015, Formula 8.469-4, p. 934; DLMF, 2023, http://dlmf.nist.gov/10.39.E2), which will be didactically introduced and repeatedly used later. For clarity, the pdf's of the unit scale U^* and scaled U, using parameter *z*, are shown:

$$f_{U^*}^{(\mathrm{IG})}(u^* \mid z) = K_{-1/2}(z)^{-1} \frac{1}{2} \exp\left\{-\frac{z}{2}\left(u^* + \frac{1}{u^*}\right)\right\} u^{*-3/2} \text{ and}$$

$$f_U^{(\mathrm{IG})}(u \mid z, \mu) = K_{-1/2}(z)^{-1} \mu^{-1} \frac{1}{2} \exp\left\{-\frac{z}{2}\left(\frac{u}{\mu} + \frac{\mu}{u}\right)\right\}\left(\frac{u}{\mu}\right)^{-3/2},$$

where the Bessel function used in the pdf's is scale-free in that as shown earlier and above, even when employing a scale parameter as $u^* = u / \mu$, the Bessel function is unchanged with the additional Jacobian μ^{-1} being multiplied outside the Bessel. On the other hand, when using the most typical parametrization in the IG distribution with parameters λ and μ, the shape parameter $\lambda (= z\mu)$ is confounded with the scale parameter μ. That is, a large value of the shape parameter λ may simply be due to using a large scale parameter.

This must have been aware for researchers of the IG. Tweedie (1957a, Equations (1a) to (1d)) provided three sets of reparametrization other than $\{\mu, \lambda\}$ for the IGs, where the second one is $\{\mu, \phi(= \lambda / \mu = z)\}$, which is equal to our expression. Chhikara and Folks (1989, p. 8) stated "Both μ and λ are of the same physical dimensions as the random variable...". Seshadri (1999, p. 2) gave almost the same statement. It seems that inclusion of μ seems to have been a standard practice possibly due primarily to dealing with physical quantities whose scales are of relatively much importance. One of the advantages of using the Bessel function is its scale-freeness with its argument *z* as an un-confounded shape parameter. This is a motivation of this didactic review concerning the distributions using the Bessel function applied to various fields, where the function may be unfamiliar to beginning students/researchers in the statistical or data sciences.

The remainder of this paper is organized as follows. Section 2 introduces the modified Bessel function of the second kind and its integral representations including known ones. Based on these expressions, five distributions are defined and called basic Bessel (BB) distributions. One of the BB's is an extension of Halphen's harmonic distribution retaining harmonicity. Section 3 gives the power transformations of the BB (PBB) distributed variables, which are extensions of the BB distributions. In Section 4, moments and an associated moment generating function for the PBB's are shown, where new incomplete Bessel functions are defined and used. Section 5 provides the differential functions to derive the modes of the PBB's, where quasi-quadratic equations are also defined for iterative computation. In Section 6, some discussions including extensions with unification are given.

2. The basic Bessel distributions

The modified Bessel function of the second (or sometimes called "third") kind of order v denoted by $K_v(z)$ is given as a solution $w^*(z)$ of the following modified Bessel equation

$$z^2 \frac{\mathrm{d}^2 w^*}{\mathrm{d}z^2} + z\frac{\mathrm{d}w^*}{\mathrm{d}z} - (z^2 + v^2)w^* = 0,$$

where z and v are complex variable and parameter, respectively (Erdélyi, 1953, Section 7.2.2, Equation (11); Abramowitz & Stegun, 1972, Formula 9.6.1; DLMF, 2023, https://dlmf.nist.gov/10.25.E1). A set of integral representations of $K_v(z)$ is

"**the first integral expressions**" $K_v(z) = \frac{(z/2)^v}{2}\int_0^\infty \exp\left(-w - \frac{z^2}{4w}\right)\frac{1}{w^{v+1}}\mathrm{d}w$

$$= \frac{(z/2)^v}{2}\int_0^\infty \exp\left(-\frac{1}{w} - \frac{z^2 w}{4}\right)w^{v-1}\mathrm{d}w,\ (|\arg(z)| < \pi/4, \mathrm{Re}\,z^2 > 0)$$

(Watson, 1944/1995, Section 6.22, Equation (15); Agrest & Maksimov, 1971, Equation (4.18); Zwillinger, 2015, Formulas 3.471-12 and 8.432-6; DLMF, 2023, https://dlmf.nist.gov/10.32.E10). Among the above two representations, the first one has been exclusively used.

Alternative expressions are obtained by the following formula:

"**the second integral expressions**" $K_v(z) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(x + \frac{1}{x}\right)\right\}x^{v-1}\mathrm{d}x$

$$= \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(x + \frac{1}{x}\right)\right\}x^{-v-1}\mathrm{d}x = K_{-v}(z) \quad (v \in \mathrm{R},\ z > 0),$$

where the former expression is used in various aspects (Hurst, 1995, Equation (3.3); Perreault, Bobée & Rasmussen, 1999a, Equation (45); Finlay & Seneta, 2008, p. 170; Song, Park & Kim, 2014, Definition 1; Song, Kang & Kim, 2014, Definition 1; Gaunt, 2021, Appendix), and is easily given by the variable transformation $x = z/(2w)$ when $z \in \mathrm{R}$ in the first formula of the first integral expressions) of $K_v(z)$ whereas Song, Kang and Kim, (2014, Lemma 1) derived the earlier formula from the former expression of the second ones which they gave as the definition of $K_v(z)$. Note that the latter expression of the second ones is given by the variable transformation of the former from x to x^{-1}. Watson (1944/1995, Section 6.22, Equation (8) when $\omega = 0$) used the latter expression. The two second integral expressions give

"**the modified second integral expression**"

$$K_v(z) = \frac{1}{4}\int_0^\infty \exp\left\{-\frac{z}{2}\left(x^* + \frac{1}{x^*}\right)\right\}(x^{*v-1} + x^{*-v-1})\mathrm{d}x^* \quad (v \in \mathrm{R},\ z > 0),$$

which may look redundant but is included for later use. Note that the above integrand is unchanged by exchanging v by $-v$ and/or the variable transformation from x^* to x^{*-1}. Though this expression is easily obtained by the earlier ones, to the author's knowledge the modified expression is new.

An additional integral representation is given by

"**the third integral expressions**" $K_v(z) = \int_0^\infty \exp(-z\cosh y)\cosh(vy)\,\mathrm{d}y$

$$= \int_0^\infty \exp(-z\cosh y)\cosh(-vy)\,\mathrm{d}y \quad (|\arg(z)| < \pi/2)$$

with $\cosh(y) = (\mathrm{e}^y + \mathrm{e}^{-y})/2$ (Watson, 1944/1995, Section 6.22, Equation (5); Magnus & Oberhettinger, 1948, p. 39; Magnus, Oberhettinger & Soni, 1966, p. 85; Abramowitz & Stegun, 1972, Formula 9.6.24; a redundant expression using $K_v(z) = (1/2)\int_{-\infty}^\infty \exp(-z\cosh y)\cosh(vy)\,\mathrm{d}y$ by Barndorff-Nielsen & Halgreen, 1977, p. 310; Zwillinger, 2015, Formulas 8.432-1; DLMF, 2023, https://dlmf.nist.gov/10.32.E9). Agrest and Maksimov (1971, Equation (4.17)) and Joarder (1995, p. 118) used the third expression with slight modification:

"**the modified third integral expressions**" $K_v(z) = \frac{1}{2}\int_{-\infty}^\infty \exp(-z\cosh y^* + vy^*)\,\mathrm{d}y^*$

$$= \frac{1}{2}\int_{-\infty}^\infty \exp(-z\cosh y^* - vy^*)\,\mathrm{d}y^* = K_{-v}(z),$$

which are obtained from the first formula of the first expressions using the variable transformation $e^{y} = 2w / z$ as when Joarder (1995) first derived the closed-formula of the characteristic function (cf) of Student-*t* distribution using $K_{v}(z)$, where *v* corresponds to a half of the degrees of freedom. Note that in the third integral expression Zwillinger (2014, Formula 8.432-1) included a point for the possible area of *z* as $(|\arg z| < \pi / 2$ or $\mathrm{Re}\, z = 0$ and $v = 0)$. Since the added point does not give a finite value of the integral, the point should be deleted as described earlier. However, a limiting value of $K_{v}(z)$ when z goes to + 0 as a function of *z* exists as $\lim_{z \to +0} K_{v}(z) = \Gamma(v) 2^{v-1} z^{-v}$ $(v > 0)$ which was used by Hurst (1995, Equation (4.2)) without derivation though for this limiting value Abramowitz and Stegun (1972) was referred to. However, the latter editors do not seem to have provided their formula 9.6.9 with its derivation as noted by Song, Park and Kim (2014, Definition 1-II), who gave the proof. Using the formula, Hurst (1995) independently provided the same expression of the cf of the Student-*t* distribution.

Historically, the third expression seems to have been first derived as an integral representation of $K_{v}(z)$ by L. Schläfli in 1873 followed by the modified third, second and first ones in this order though the first one has been most extensively investigated (Watson, 1944/1995, Section 6.22, Equations (5), (7), (8) and (15)). Note that the second integral expressions are obtained from the modified third one's using the variable transformation $x = e^{y^*}$ or $x = e^{-y^*}$. An advantage of the modified third expressions over the remaining ones is $y^* \in \mathrm{R}$ i.e., two-sided while $w > 0$, $x > 0$, $x^* > 0$ and $y > 0$ for the first, second, modified second, and un-modified third expressions, respectively.

When w, x, x^*, y and y^* are seen as realized values of random variables W, X, X^*, Y and Y^* in the integral expressions, respectively, their distributions are defined as follows.

Definition 1 (the BB pdf's of the first to modified third kinds). Variables $W, X,$ X^*, Y and Y^* are defined to follow the basic Bessel (BB) distributions of the first, second, modified second, third and modified third kinds, respectively with the BB distribution for the generic expression of the five distributions with possible reparametrizations when their pdf's are "**the BB pdf of the first kind**" $f_W(w \mid v, z) = K_v(z)^{-1} \frac{(z/2)^v}{2} \exp\left(-w - \frac{z^2}{4w}\right) \frac{1}{w^{v+1}}$

$$(w > 0;\ v \in \mathrm{R},\ z > 0),$$

"**the BB pdf of the second kind**" $f_X(x \mid v, z) = K_v(z)^{-1} \frac{1}{2} \exp\left\{-\frac{z}{2}\left(x + \frac{1}{x}\right)\right\} x^{v-1}$

$$(x > 0;\ v \in \mathrm{R},\ z > 0),$$

"**the BB pdf of the modified second kind**"

$$f_{X^*}(x^* \mid v, z) = K_v(z)^{-1} \frac{1}{4} \exp\left\{-\frac{z}{2}\left(x^* + \frac{1}{x^*}\right)\right\} (x^{*v-1} + x^{*-v-1})$$

$$(x^* > 0;\ v \in \mathrm{R},\ z > 0),$$

"**the BB pdf of the third kind**" $f_Y(y \mid v, z) = K_v(z)^{-1} \exp(-z \cosh y) \cosh(vy)$

$(y > 0;\ v \in \mathrm{R},\ z > 0)$ and

"**the BB pdf of the modified third kind**" $f_{Y^*}(y^* \mid v, z) = K_v(z)^{-1} \frac{1}{2} \exp(-z \cosh y^* + vy^*)$

$$(y^* \in R;\ v \in \mathrm{R},\ z > 0),$$

where $K_v(z)^{-1} = \{K_v(z)\}^{-1}$ with $K_v(z) = K_{-v}(z)$.

The possible ranges for *v* and *z* common to the two-parameter distributions in Definition 1 are $v \in \mathrm{R}$ and $z > 0$, which will be assumed unless otherwise specified. Note that while W, X, X^* and *Y* have the common support $(0, \infty)$, that of Y^* is $(-\infty, \infty)$. Note also that, W, X and Y^* are related by variable transformations

$$\frac{z}{2w} = x = \exp(y^*).$$

Variable X^* is given by the two-point mixture of X using v and $-v$ with equal weights. Similarly, Y is obtained by the two-point mixture using the truncated Y^*'s with reduced support $(0, \infty)$ and equal weights as

$$\begin{aligned} f_Y(y \mid v, z) &= K_v(z)^{-1} \exp(-z \cosh y) \cosh(vy) \\ &= K_v(z)^{-1} \frac{1}{2} \exp(-z \cosh y + vy) + K_v(z)^{-1} \frac{1}{2} \exp(-z \cosh y - vy) \\ &= K_v(z)^{-1} \exp(-z \cosh y) \cosh(-vy) \\ &= f_Y(y \mid -v, z) \quad (0 < y < \infty). \end{aligned}$$

It is to be noted that the equalities for W, X and Y^* corresponding to $f_{X^*}(x^* \mid v, z) = f_{1/X^*}(x^* \mid v, z)$, $f_{X^*}(x^* \mid v, z) = f_{X^*}(x^* \mid -v, z)$, $f_Y(Y \mid v, z) = f_{-Y}(Y \mid v, z)$ or $f_Y(y \mid v, z) = f_Y(y \mid -v, z)$ do not generally hold unless $v = 0$ or the case of Halphen's harmonic distribution, where the variable denoted by $X_{v=0}$ (a special case of X) has the same distribution as $1/X_{v=0}$. The property $f_{X^*}(x^* \mid v, z) = f_{1/X^*}(x^* \mid v, z)$ that holds with unconstrained *v* may be called as an extended harmonicity.

The word "basic" in the BB distribution is adopted to differentiate it from the Bessel function distributions e.g., the McKay (1932, Equation (1)) Bessel function distribution (for other Bessel function distributions see Bose, 1936, Equation (6.6); Laha, 1954; McNolty, 1973). Note that a special case of the McKay Bessel function distribution can be seen as a mixture using the BB distribution for a mixing distribution. In other words, the BB distribution is implicitly used.

An explicit use of the BB distribution is found in the harmonic distribution of Halphen (1941, p. 635). Let *X* follow this distribution, then the pdf of *X* at *x* is given by

$$f_X(x \mid \Lambda) = \Lambda\, x^{-1} \exp\left(-x - \frac{1}{x}\right),$$

where Halphen stated that Λ is a normalizing constant including a Bessel function. Halphen (1941, Remarque, p. 635) also gave a generalized version of the above distribution:

$$f_X\{x \mid \Lambda(\alpha, a, b), \alpha, a, b\} = \Lambda\, x^{\alpha-1} \exp\left(-ax - \frac{b}{x}\right).$$

We derive the actual form of the normalizer $\Lambda(\alpha, a, b)$ in the above generalized distribution. Employing the scale parameter $\beta > 0$ in the BB distribution of the second kind, we have its pdf using the same notations *X* and *x* for simplicity

$$f_X(x\,|\,v,z,\beta)=K_v(z)^{-1}\frac{1}{2\beta^v}\exp\left\{-\frac{z}{2}\left(\frac{x}{\beta}+\frac{\beta}{x}\right)\right\}x^{v-1}\ (x>0;\ v\in\mathrm{R},\ z>0,\beta>0),$$

which is the pdf of a 3-parameter or scaled BB distribution of the second kind. Halphen's pdf $f_X\{x\,|\,\Lambda(\alpha,a,b),\alpha,a,b\}$ is given from $f_X(x\,|\,v,z,\beta)$ when $v=\alpha$, $z=2(ab)^{1/2}$ and x in the second scaled BB pdf is re-expressed by $2a\beta x/z$ with $\mathrm{d}(2a\beta x/z)/\mathrm{d}x=2a\beta/z$ as

$$\begin{aligned}f_X(x\,|\,\alpha,a,b)&=K_\alpha(z)^{-1}\frac{1}{2\beta^\alpha}\exp\left(-ax-\frac{b}{x}\right)(2a\beta x/z)^{\alpha-1}(2a\beta/z)\\&=K_\alpha(z)^{-1}\frac{(2a/z)^\alpha}{2}x^{\alpha-1}\left(-ax-\frac{b}{x}\right)\\&=K_\alpha\{2(ab)^{1/2}\}^{-1}\frac{(a/b)^{\alpha/2}}{2}x^{\alpha-1}\exp\left(-ax-\frac{b}{x}\right).\end{aligned}$$

It is found that the normalizer in Halphen's pdf $f_X(x\,|\,\Lambda,a,b)$ is given by

$$\Lambda=K_\alpha\{2(ab)^{1/2}\}^{-1}(a/b)^{\alpha/2}/2\,.$$

After Halphen's death in 1954, Morlat (1956, Section 2.3) investigated the 3-parameter scaled BB distribution of the second kind with the pdf $f_X(x\,|\,v,z,\beta)$ shown earlier, and called the distribution as the Halphen type A (HA) distribution. Guillot (1964, pp. 64-66) gave the pdf of the logarithmic transformation of Halphen's harmonic-distributed variable, which is exactly the same as one of the BB pdf's of the modified third kind with the unit scale parameter i.e.,

$$f_U(u\,|\,v,z)=K_v(z)^{-1}(1/2)\exp(-z\,\mathrm{ch}\,u+vu),$$

where $\mathrm{ch}\,x\equiv\cosh u$ was used by Guillot. Perreault et al. (1999a, p. 190) introduced the work of Guillot as a power transformation of Halphen's variable though the actual one is logarithmic as addressed earlier.

Good (1953) seems to be the first rediscovery of Halphen's (1941) harmonic distribution as noted by Jørgensen (1982, p. 1). Perreault et al. (1999a, p. 189) also stated that "The generalized inverse Gaussian distribution is commonly accredited to Good (1953)", where the cited generalized distribution is the same as the reparametrized Halphen distribution as will be addressed later. Good (1953, Equation (50)) gave an expression of the pdf $f(p)=sAp^\alpha\exp(-\beta p-\varepsilon p^{-1})$ $(\beta>0,\ \varepsilon>0)$ with $\int_0^1 f(p)\,\mathrm{d}p=s$ under possible truncation, where p is an argument of the pdf and sA is seen as a normalizing factor i.e., a function of α, β and ε. Note that Good's pdf is the same as Halphen's $f_X\{x \mid \Lambda(\alpha,a,b),\alpha,a,b\}$ with slight reparametrization and the removal of truncation, which is also a reparametrized scaled BB distribution of the second kind. Rukhin (1974/1978, Equation (0.2)) rediscovered Halphen's scaled and shifted harmonic distribution with the logarithmic change of variable or Guillot's distribution using the pdf $f_X(x) = \dfrac{\beta}{2K_0(\alpha)}\exp\{-\alpha \mathrm{ch}(x-\theta_0)\}$ with $\mathrm{ch}(\cdot)$ defined earlier. Rukhin's distribution is a special case of the BB of the modified third kind when $\nu = 0$ in $K_\nu(\cdot)$. The distributions using $K_0(\cdot)$ including Halphen's harmonic one are also called the hyperbola distributions (Barndorff-Nielsen, 1978, Section 5; Jørgensen, 1982, pp. 1-2; Perreault et al.,1999a, Equation (1)). Sichel (1975, Equation (2.1)) gave a reparametrized distribution of the Halphen distribution. Also, Barndorff-Nielsen (1977, Equation (7.1)), and Barndorff-Nielsen and Halgreen (1977, Equation (1)) rediscovered the Halphen distribution with the pdf $f_X(x \mid \alpha,a,b)$ when $\alpha = \lambda$, $a = \psi/2$ and $b = \chi/2$ as a member of "the family of the generalized inverse Gaussian (GIG) distributions":

$$
\begin{aligned}
&f_X(x \mid \lambda,\psi,\chi) \\
&= K_\lambda\{(\psi\chi)^{1/2}\}^{-1}\frac{(\psi/\chi)^{\lambda/2}}{2}x^{\lambda-1}\exp\left(-\psi x - \frac{\chi}{x}\right)\ (x>0;\ \lambda \in \mathrm{R}, \psi>0, \chi>0)
\end{aligned}
$$

(for usages of this distribution as a mixing distribution for normal variance-mean mixtures or the class of generalized hyperbolic distributions, see also Barndorff-Nielsen, 1978, Equation (4.1); Barndorff-Nielsen, Kent & Sørensen,1982, Equation (2.4)).

3. Power basic-Bessel (PBB) distributions

Power transformations of random variables of well-known distributions such as the normal distribution have been extensively investigated and used as the Box-Cox (1964) or power-normal (Goto & Inoue, 1980) transformations. Recently, Ferrari and Fumes (2017) and Morán-Vásquez and Ferrari (2019) gave the Box-Cox transformations for symmetric and elliptical distributions, respectively. On the other hand, Takei and Matsunawa (2001, Equation (1.6)) showed the power-gamma distribution as a prior one, where the variable is given when a power of the variable follows the gamma distribution. Independently, Ogasawara (2022, Section 9.2) obtained the same distribution with a multivariate version (Ogasawara, 2023a). Note that the power-gamma distribution is a reparametrization of Stacy's (1962, Equation (1)) generalized gamma with an extension of the real-valued power over the positive one used by the generalized gamma. Stacy's generalized gamma has been rediscovered or restated as e.g., the three-parameter generalized gamma or the Krisky-Menkel distributions (TPGG; Singh & Zhang, 2022, Section 6.1). The power-gamma distribution is also seen as a special case of the Amoroso (1925) distribution (for this distribution see also Crooks, 2015). For the Halphen type A distribution, whose special case is the IG distribution (Folks & Chhikara, 1978; Chhikara & Folks, 1989) with some reparametrization and variable transformation, Iwase and Hirano (1990, Equation (10)) obtained the distribution of the power-transformation of the IG called the power-IG (PIG) distribution. Further, Takei and Matsunawa (2001, Equation (1.7)) presented the power-GIG (PGIG) distribution, where the GIG was mentioned earlier as a reparametrized Halphen type A distribution. For clarity, the pdf of Takei and Matsunawa's PGIG is repeated without changing notations:

$$q^*(x) = \frac{|\gamma|\, x^{\gamma\beta-1}}{2(ab)^b K_\beta\left(\sqrt{b/a}\right)} \exp\left\{-\frac{1}{2}\left(\frac{x^\gamma}{a} + \frac{b}{x^\gamma}\right)\right\}, \left(\begin{matrix} x>0,\ a>0,\ b>0 \\ -\infty<\beta<\infty,\ \gamma\in R-\{0\} \end{matrix}\right).$$

We consider the power transformations of the BB distributed variables, which are found to be given by the PGIG distributions using variable transformations, reparametrizations or mixtures.

Definition 2 (the power-BB (PBB) pdf's of the first to modified third kinds). The power-BB distributions are defined when their pdf's are given as follows:

"t**he PBB pdf of the first kind**"

$$f_W(w\,|\,v,z,\gamma) = K_v(z)^{-1}\frac{(z/2)^v}{2}\exp\left(-w^\gamma - \frac{z^2}{4w^\gamma}\right)\frac{|\gamma|}{w^{\gamma v+1}},$$

"**the PBB pdf of the second kind**"

$$f_X(x\,|\,v,z,\gamma) = K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\, x^{\gamma v-1},$$

"**the PBB pdf of the modified second kind**"

$$f_{X^*}(x^* \mid v,z,\gamma) = K_v(z)^{-1}\frac{1}{4}\exp\left\{-\frac{z}{2}\left(x^{*\gamma}+\frac{1}{x^{*\gamma}}\right)\right\}|\gamma|(x^{*\gamma v-1}+x^{*-\gamma v-1}),$$

"**the PBB pdf of the third kind**"

$$f_Y(y \mid v,z,\gamma) = K_v(z)^{-1}\exp\{-z\cosh(\gamma y)\}\cosh(v\gamma y)|\gamma|$$

and "**the PBB pdf of the modified third kind**"

$$f_{Y^*}(y^* \mid v,z,\gamma) = K_v(z)^{-1}\frac{1}{2}\exp\{-z\cosh(\gamma y^*)+v\gamma y^*\}|\gamma|$$

$$(w>0,\ x>0,\ x^*>0, y>0,\ y^*\in \mathrm{R};\ v\in \mathrm{R},\ z>0, -\infty<\gamma<\infty,\ \gamma\neq 0).$$

The pdf's in Definition 2 are given by variable transformations of $w,\ x,\ x^*, y$ and y^* by $w^\gamma, x^\gamma, x^{*\gamma}, \gamma y$ and γy^* with the Jacobians $|\gamma|w^{\gamma-1}, |\gamma|x^{\gamma-1}, |\gamma|x^{*\gamma-1}$, $|\gamma|$ and $|\gamma|$, respectively. Note that for the second last pdf, the form γy yielding the power-transformation of e^y i.e., $e^{y\gamma}$ is used rather than y^γ considering practical use with similar results for γy^* having the extended support.

4. Moments of the PBB distributions

In this section, the moments or the moment generating functions (mgf's) are given except the mgf of Y and the moments of e^Y, which will be shown later.

Lemma 1. *Some moments of the PBB distributions of the first, second and modified third kinds of real-valued order k with* $\mathrm{M}_{Y^*}(k \mid v,z,\gamma)$ *being the mgf of* Y^* *are*

$$\mathrm{E}(W^k \mid v,z,\gamma) = \frac{K_{v-(k/\gamma)}(z)}{K_v(z)(z/2)^{-k/\gamma}},$$

$$\mathrm{E}(X^k \mid v,z,\gamma) = \mathrm{E}\{\exp(kY^*) \mid v,z,\gamma\} = \mathrm{M}_{Y^*}(k \mid v,z,\gamma) = \frac{K_{v+(k/\gamma)}(z)}{K_v(z)},$$

$$\mathrm{E}(X^{*k} \mid v, z, \gamma) = \frac{1}{2}\frac{K_{v+(k/\gamma)}(z) + K_{v-(k/\gamma)}(z)}{K_v(z)}$$

$$= \frac{1}{2}\{\mathrm{E}(X^k \mid v, z, \gamma) + \mathrm{E}(X^k \mid -v, z, \gamma)\}$$

$$= \frac{1}{2}\left[\mathrm{E}\{\exp(kY^*) \mid v, z, \gamma\} + \mathrm{E}\{\exp(kY^*) \mid -v, z, \gamma\}\right]$$

$$= \frac{1}{2}\left\{\mathrm{M}_{Y^*}(k \mid v, z, \gamma) + \mathrm{M}_{Y^*}(k \mid -v, z, \gamma)\right\}$$

$(-\infty < k < \infty)$ and

$$\mathrm{E}(Y^{*j} \mid v, z, \gamma) = \frac{\mathrm{d}^j K_{v+(k/\gamma)}(z) / \mathrm{d}k^j \,|_{k=0}}{K_v(z)} \quad (j = 1, 2, \ldots).$$

respectively.

The proofs of Lemma 1 and some other results will be provided in the appendix. In Lemma 1, $\mathrm{M}_{Y^*}(k \mid v, z, \gamma) = \mathrm{E}\{\exp(kY^*) \mid v, z, \gamma\}$ may also be called the *k*-th order exponential moment of Y^*. Note that generally $\mathrm{E}\{\exp(kY) \mid v, z, \gamma\} \neq \mathrm{E}\{\exp(kY^*) \mid v, z, \gamma\}$. The two negative signs in the expression of $\mathrm{E}(W^k \mid v, z, \gamma)$ become positive when we use $W^\# \equiv W^{-1}$ before power-transformation as

$$\mathrm{E}(W^{\#k} \mid v, z, \gamma) = \frac{K_{v+(k/\gamma)}(z)}{K_v(z)(z/2)^{k/\gamma}},$$

which is obtained by using the pdf of $W^\#$:

$$f_{W^\#}(w^\# \mid v, z) = K_v(z)^{-1} \frac{(z/2)^v}{2} \exp\left(-\frac{1}{w^\#} - \frac{z^2 w^\#}{4}\right) w^{\#v-1}.$$

Though the variable $W^\#$ seems to be unused in literatures, its additional advantage is the factor $w^{\#v-1} (= (w^\#)^{v-1})$ of the same form as x^{v-1} in the BB pdf of the second kind. An indirect derivation of the expression of $\mathrm{E}(W^{\#k} \mid v, z, \gamma)$ with positive signs is given by considering the moments of order $-k$ for W, which are seen as those of order *k* for $W^\#$ by definition.

Conversely, if we consider the variable $X^\# \equiv X^{-1}$, we have

$$\mathrm{E}(X^{\#k} \mid v, z, \gamma) = \frac{K_{v-(k/\gamma)}(z)}{K_v(z)} \quad (-\infty < k < \infty)$$

including a negative sign, which is found to hold by the indirect method mentioned earlier. The corresponding pdf for $X^{\#}$ becomes

$$f_{X^{\#}}(x^{\#} \mid v, z) = K_v(z)^{-1} \frac{1}{2} \exp\left\{ -\frac{z}{2}\left(x^{\#} + \frac{1}{x^{\#}} \right) \right\} x^{\#-v-1},$$

which seems to be comparable to the pdf of X whose factor corresponding to x^{*-v-1} is x^{v-1} as addressed earlier (note the Jacobian $|\mathrm{d}x / \mathrm{d}x^{\#}| = x^{\#-2}$ or $|\mathrm{d}x^{\#} / \mathrm{d}x| = x^{-2}$). The pdf of $X^{\#}$ was used by Song, Park and Kim (2014, Definition 1-I) to derive $K_v(z) = K_{-v}(z)$ as addressed in Definition 1.

Now, we derive the mgf of Y, which is relatively involved, using the pdf's of Y^{*} and X. Note that the pdf of Y^{*} is not an even function or an odd one and that the support of Y is $(0, \infty)$ corresponding to the truncated one $(1, \infty)$ of X while the support of Y^{*} is $(-\infty, \infty)$ corresponding to the untruncated one $(0, \infty)$ for X. We also use upper incomplete functions of the modified Bessel of the second kind defined by Cicchetti and Faraone (2004, Equation (2)), and Jones (2007, Equation (2.4)):

$$K_v(z, \omega) = \int_{\omega}^{\infty} \exp(-z\cosh y)\cosh(vy)\,\mathrm{d}y\,.$$

Definition 3 (the types 1 to m3 upper incomplete functions of the modified Bessell of the second kind). Recall the BB pdf of the third kind denoted by $f_Y(y \mid v, z)$ and that $f_Y(y \mid v, z)$ is the sum of two distributions of Y^{*} with the reduced or truncated support $(0, \infty)$.

"**The types 3 and m3 upper incomplete Bessel functions**": The above equality for the definition of $K_v(z, \omega)$ can be re-expressed as

$$\begin{aligned} K_v(z,\omega) &\equiv K_{(3),v}(z,\omega) = \int_{\omega}^{\infty} \exp(-z\cosh y)\cosh(vy)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\omega}^{\infty} \exp(-z\cosh y + vy)\,\mathrm{d}y + \frac{1}{2}\int_{\omega}^{\infty} \exp(-z\cosh y - vy)\,\mathrm{d}y\,. \\ &\equiv K_{(\mathrm{m}3),v}(z,\omega) + K_{(\mathrm{m}3),-v}(z,\omega). \end{aligned}$$

The integrals $K_{(3),v}(z,\omega)(= K_v(z,\omega))$ and $K_{(\mathrm{m}3),v}(z,\omega)$ defined above are called the types 3 and m3 upper incomplete functions of the modified Bessell of the second kind, respectively, where "3" and "m3" indicate the corresponding BB pdf's of the un-modified and modified third kinds, respectively. Though in the above equation $0 \le \omega$ is considered to have the relationship between $K_{(3),\nu}(z,\omega)$ and $K_{(\mathrm{m}3),\nu}(z,\omega)$, the range of ω in $K_{(\mathrm{m}3),\nu}(z,\omega)$ is $\omega \in \mathrm{R}$. Further, for later use and convenience, define

$$
\begin{aligned}
&K_{(3),\nu}(z,\omega) = \int_{\omega}^{\infty} \exp(-z\cosh y)\cosh(\nu y)\,\mathrm{d}y \\
&= \int_{\omega/\lambda}^{\infty} \exp(-z\cosh \lambda y)\cosh(\nu\lambda y)\,\lambda \mathrm{d}y \\
&\equiv K_{(3),\nu,\lambda}(z,\omega/\lambda)\ (0 \le \omega, 0 < \lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
&K_{(\mathrm{m}3),\nu}(z,\omega) = \frac{1}{2}\int_{\omega}^{\infty} \exp(-z\cosh y^{*} + \nu y^{*})\,\mathrm{d}y^{*} \\
&= \frac{1}{2}\int_{\omega/\lambda}^{\infty} \exp\{-z\cosh(\lambda y^{*}) + \nu\lambda y^{*}\}\,|\lambda|\,\mathrm{d}y^{*} \\
&\equiv K_{(\mathrm{m}3),\nu,\lambda}(z,\omega/\lambda)\ (\omega \in \mathrm{R},\ \lambda \in \mathrm{R},\ \lambda \ne 0).
\end{aligned}
$$

"**The types 2 and m2 upper incomplete Bessel functions**": Using the variable transformation $\mathrm{e}^{y^{*}} = x$, and then replacing x by x^{γ} $(\gamma \in \mathrm{R}, \gamma \ne 0)$, we have

$$
\begin{aligned}
&K_{(\mathrm{m}3),\nu}(z,\omega) = K_{(2),\nu}\{z,\exp\,(\omega)\} = \int_{\exp\,(\omega)}^{\infty} \frac{1}{2}\exp\left\{-\frac{z}{2}\left(x+\frac{1}{x}\right)\right\} x^{\nu-1}\mathrm{d}x \\
&= \int_{\exp\,(\omega/\gamma)}^{\infty} \frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^{\gamma}+\frac{1}{x^{\gamma}}\right)\right\} x^{\gamma\nu-1}|\gamma|\mathrm{d}x \equiv K_{(2),\nu,\gamma}\{z,\exp\,(\omega/\gamma)\},
\end{aligned}
$$

where $K_{(2),\nu}\{z,\exp\,(\omega)\}$ is called the type 2 upper incomplete function of the modified Bessell of the second kind. The type m2 counterpart is given by $K_{(2),\nu}\{z,\exp\,(\omega)\}$ as

$$
\begin{aligned}
&K_{(\mathrm{m}2),\nu}\{z,\exp\,(\omega)\} = \int_{\exp\,(\omega)}^{\infty} \frac{1}{4}\exp\left\{-\frac{z}{2}\left(x^{*}+\frac{1}{x^{*}}\right)\right\}(x^{*\nu-1}+x^{*-\nu-1})\mathrm{d}x \\
&= \frac{1}{2}\left[K_{(2),\nu}\{z,\exp\,(\omega)\} + K_{(2),-\nu}\{z,\exp\,(\omega)\}\right] \\
&= \int_{\exp\,(\omega/\gamma)}^{\infty} \frac{1}{4}\exp\left\{-\frac{z}{2}\left(x^{*\gamma}+\frac{1}{x^{*\gamma}}\right)\right\}(x^{\gamma\nu-1}+x^{-\gamma\nu-1})|\gamma|\mathrm{d}x \\
&= \frac{1}{2}\left[K_{(2),\nu,\gamma}\{z,\exp\,(\omega/\gamma)\} + K_{(2),-\nu,\gamma}\{z,\exp\,(\omega/\gamma)\}\right] \\
&\equiv K_{(\mathrm{m}2),\nu,\gamma}\{z,\exp\,(\omega/\gamma)\}.
\end{aligned}
$$

"**The type 1 upper incomplete Bessel function**": The remaining type 1 counterpart

$K_{(1),v}\{z,(z/2)\exp(-\omega)\}$ is defined by using $\frac{z}{2w}=x$

$$\begin{aligned}
&K_{(\mathrm{m}3),v}(z,\omega)=K_{(2),v}\{z,\exp(\omega)\}=K_{(1),v}\{z,(z/2)\exp(-\omega)\} \\
&=\frac{(z/2)^{v}}{2}\int_{(z/2)\exp(-\omega)}^{\infty}\exp\left(-w-\frac{z^{2}}{4w}\right)\frac{1}{w^{v+1}}\mathrm{d}w \\
&=\frac{(z/2)^{v}}{2}\int_{(z/2)^{1/\gamma}\exp(-\omega/\gamma)}^{\infty}\exp\left(-w^{\gamma}-\frac{z^{2}}{4w^{\gamma}}\right)\frac{|\gamma|}{w^{\gamma v+1}}\mathrm{d}w \\
&\equiv K_{(1),v,\gamma}\{z,(z/2)^{1/\gamma}\exp(-\omega/\gamma)\}.
\end{aligned}$$

It is known that the integral formula corresponding to an incomplete Bessel function of the second kind dates back to Binet (1841) a century before Halphen (1941) (for Binet's contribution, see Watson, 1944/1999, Section 6.22, p. 183; Agrest & Maksimov, 1971, p. 21), where the integral of the form $\int_a^b y^{2r}\exp\left(-\frac{p}{y^2}-qy^2\right)\mathrm{d}y$ with a and b being unconstrained was focused on. It is found that this formula is obtained by a special case of the type 2 incomplete function $K_{(2),v,\gamma}\{z,\exp(\gamma\omega)\}$ in Definition 3 using the square transformation i.e., $\gamma=2$. Currently, as mentioned earlier, $K_{(3),v}(z,\omega)(=K_v(z,\omega))$ seems to be exclusively used in literatures.

Lemma 2. *The mgf of Y with the PBB pdf of the third kind is*

$$\begin{aligned}
&\mathrm{E}\{\exp(kY)\,|\,v,z,\gamma\}=\mathrm{M}_Y(k\,|\,v,z,\gamma) \\
&=\frac{K_{(2),v+(k/\gamma),\gamma}(z,1)}{K_v(z)}+\frac{K_{(2),-v+(k/\gamma),\gamma}(z,1)}{K_v(z)}\equiv\frac{K_{(2),\{\pm v\}+(k/\gamma),\gamma}(z,1)}{K_v(z)}\quad(k\in\mathrm{R}).
\end{aligned}$$

In Lemma 2, $\Pr\{Y^*>0\,|\,\cdot\}\ (=\Pr\{X>1\,|\,\cdot\})$ is a value of a survival function. In the following lemma, we partially employ the scale parameter β for generality and didactic purposes. Define the partial (not conditional) expectation of $(\gamma^{-1}\ln X)^j$ over the range $X>\omega^*$ as

$$\begin{aligned}
&\mathrm{E}\{(\gamma^{-1}\ln X)^j\,|\,v+(k/\gamma),z;X>\omega^*\} \\
&=\int_{\omega^*}^{\infty}K_{v+(k/\gamma)}(z)^{-1}(\gamma^{-1}\ln x)^j\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x+\frac{1}{x}\right)\right\}x^{v+(k/\gamma)-1}\mathrm{d}x
\end{aligned}$$

$$\left(= \int_{\omega^{*(1/\gamma)}}^{\infty} K_{v+(k/\gamma)}(z)^{-1} (\ln x)^j \frac{1}{2} \exp\left\{ -\frac{z}{2}\left(x^{\gamma} + \frac{1}{x^{\gamma}} \right) \right\} x^{\gamma\{v+(k/\gamma)\}-1} |\gamma| \mathrm{d}x \right)$$

$(-\infty < k < \infty; j = 1,2,...).$

Then, we have the following results.

Lemma 3.

$$\sum_{m=0}^{k} \binom{j}{m} (\gamma^{-1} \ln \beta)^m \frac{1}{K_{v+(k/\gamma)}(z)} \frac{\mathrm{d}^{j-m} K_{v+(k/\gamma)}(z)}{\mathrm{d}k^{j-m}} = \mathrm{E}\{(\gamma^{-1} \ln X)^j \mid v + (k/\gamma), z, \beta\}$$

and

$$\sum_{m=0}^{k} \binom{j}{m} (\gamma^{-1} \ln \beta)^m \frac{1}{K_{v+(k/\gamma)}(z)} \frac{\mathrm{d}^{j-m} K_{(2),v+(k/\gamma),\gamma}(z, \omega^*)}{\mathrm{d}k^{j-m}}$$

$= \mathrm{E}\{(\gamma^{-1} \ln X)^j \mid v + (k/\gamma), z, \beta; X > \omega^*\}$ $(-\infty < k < \infty; j = 1,2,...).$

Remark 1. Penneault et al. (1999a, Equation (65)) gave the derivation of the special case of Lemma 2 when $j = 1$, $v + (k/\gamma) = k$ and $\gamma = 1$. That is, using our notation, they gave

$$\ln \beta + \frac{1}{K_k(z)} \frac{\mathrm{d}K_k(z)}{\mathrm{d}t} = \mathrm{E}\{(\ln X) \mid k, z, \beta\}.$$

Note that in the case of unit scale i.e., $\beta = 1$, Lemma 3 becomes considerably simple:

$$\frac{1}{K_{v+(k/\gamma)}(z)} \frac{\mathrm{d}^j K_{v+(k/\gamma)}(z)}{\mathrm{d}k^j} = \mathrm{E}\{(\gamma^{-1} \ln X)^j \mid v + (k/\gamma), z\} \text{ and}$$

$$\frac{1}{K_{v+(k/\gamma)}(z)} \frac{\mathrm{d}^j K_{v+(k/\gamma)}(z, \omega^*)}{\mathrm{d}k^j} = \mathrm{E}\{(\gamma^{-1} \ln X)^j \mid v + (k/\gamma), z; X > \omega^*\}.$$

The above formulas are integral representations. The closed-form expression seems to be available only when $j = 1$ i.e., $\mathrm{d}K_k(z)/\mathrm{d}k$ using series expressions depending on whether k is an integer or a non-integer with the modified Bessel functions of the first kind in the latter case (Magnus et al., 1966, Sections 3.1.3 to 3.3.3, pp. 69-75; Zwillinger, 2015, Formulas 8.486 (1)-4, 5, 9, 11, pp. 938-939; DLMF, 2023, Section 10.38).

Define $\mathrm{E}(Y^{*j} \mid v, z, \gamma) = \dfrac{\mathrm{d}^j K_{v+(k/\gamma)}(z) / \mathrm{d}k^j \mid_{k=0}}{K_v(z)} \equiv \dfrac{K_{v,\gamma}^{(j)}(z)}{K_v(z)}$,

$$\mathrm{E}\{(\gamma^{-1}\ln X)^j \mid \nu+(k/\gamma), z; X>\omega^*\} = \frac{\mathrm{d}^j K_{(2),\nu+(k/\gamma),\gamma}(z,\omega^*)/\mathrm{d}k^j \,|_{k=0}}{K_\nu(z)}$$

$$\equiv \frac{K^{(j)}_{(2),\nu,\gamma}(z,\omega^*)}{K_\nu(z)} \; (-\infty<k<\infty),$$

$K^{(j)}_{(2),\{\pm\nu\},\gamma}(z,1) \equiv K^{(j)}_{(2),\nu,\gamma}(z,1) + K^{(j)}_{(2),-\nu,\gamma}(z,1)$, $K^{(j)m}_{(2),\{\pm\nu\},\gamma}(z,1) = \{K^{(j)}_{(2),\{\pm\nu\},\gamma}(z,1)\}^m \, (m=2,3,...)$

and $K^j_\nu(z) = \{K_\nu(z)\}^j \;\; (j=1,2,...)$ for simplicity of notation. Then, we have the results for some moments and their functions of the PBB distributions, which will be given in Theorem 1 of the appendix. Relationships between two sets of variables "$W, W^{\#}(=W^{-1})$" and "$X, X^{\#}(=X^{-1})$" in the PBB variables are important, which are summarized as

Corollary 1. *Some relationships between the raw moments, sk's and kt's of the PBB-distributed variables of the first and second kinds are*

$$\begin{aligned}
&\mathrm{E}(W^k \mid \nu,z,\gamma) = (z/2)^{k/\gamma}\,\mathrm{E}(X^{\#k} \mid \nu,z,\gamma),\\
&\mathrm{E}(X^k \mid \nu,z,\gamma) = (z/2)^{k/\gamma}\,\mathrm{E}(W^{\#k} \mid \nu,z,\gamma) \; (k\in\mathrm{R}),\\
&\mathrm{sk}(W) = \mathrm{sk}(X^{\#}),\ \mathrm{sk}(W^{\#}) = \mathrm{sk}(X),\\
&\mathrm{kt}(W) = \mathrm{kt}(X^{\#}),\ \mathrm{kt}(W^{\#}) = \mathrm{kt}(X).
\end{aligned}$$

Proof. The results are due to the proportionalities $W=(z/2)^{1/\gamma}X^{\#}$ and $W^{\#}=(z/2)^{-1/\gamma}X \;\; (z>0, -\infty<\gamma<\infty, \gamma\neq 0)$. Q.E.D.

5. Modes and quantiles of the PBB distributions

In this section, the modes and quantiles of the PBB distributions are considered. Dvorák, Bobée, Boucher and Ashkar (1988, Equation (4)) showed a unified expression of the differential equations or log-density derivatives for the Halphen type A and other two Halphen distributions as

$$\frac{1}{f(u)}\frac{\mathrm{d}f(u)}{\mathrm{d}u} = \frac{a_0+a_1u+a_2u^2}{u^q},$$

where $f(u)$ is the generic expression of the Halphen distributions (see also Perreault et al.,1999a, Equation (8) and Table 1; El Adlouni & Bobée, 2017, Equation (2.1); Singh & Zhang, 2022, Equation (3.16)). In the case of the unit scale Halphen type A or the BB distribution of the second kind, we have

$$\frac{\mathrm{d}\ln f_X(x\,|\,v,z)}{\mathrm{d}x} = -\frac{z}{2}\left(1-\frac{1}{x^2}\right)+\frac{v-1}{x} = \left\{-\frac{z}{2}x^2+(v-1)x+\frac{z}{2}\right\}/\,x^2\,,$$

yielding $a_0 = z/2,\ a_1 = v-1$ and $a_2 = -z/2$ with $q=2$. Recalling that by definition $x>0$ as well as $v \in \mathrm{R}$ and $z>0$, and using $\mathrm{d}\ln f_X(x\,|\,v,z)/\mathrm{d}x = 0$, it is found that a single mode denoted by $\mathrm{Mo}(X\,|\,v,z)$ exists:

$$\mathrm{Mo}(X\,|\,v,z) = \frac{1}{z}\left[v-1+\{(v-1)^2+z^2\}^{1/2}\right].$$

Note that among the PBB-distributed variables, those of the first, second and modified third kinds are given by power transformations of *x* as $x^{\lambda}\,(\lambda \in \mathrm{R}, \lambda \neq 0)$ for the Halphen type A or BB distribution of the second kind, followed by changes of variable: $x^{\lambda} = \dfrac{z}{2w^{\lambda}} = \mathrm{e}^{\lambda y^*}$.

Lemma 4. *The differential equations for the PBB distributed variables W, X and Y^* with the generic expression of their pdf's denoted by $f(t)$ are given by*

$$\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t} = \left\{a_0\lambda+(a_1\lambda+\lambda-1)t^{\lambda}+a_2\lambda t^{2\lambda}\right\}t^{-\lambda-1} \text{ for } W \text{ and } \underline{X},$$

$$\text{and } \frac{\mathrm{d}\ln f(t)}{\mathrm{d}t} = \{a_0+(a_1+1)\exp(t\lambda)+a_2\exp(2t\lambda)\}\lambda\exp(-t\lambda) \text{ for } Y^*,$$

where the coefficients a_0, a_1 and a_2 are shown in Table 1.

Table 1. The coefficients in the differential equations for the PBB distributions for W, X and Y^*

PBB distribution of	a_0	a_1	a_2
the 1st kind, *W*	$z^2/4$	$-v-1$	-1
the 2nd kind, *X*	$z/2$	$v-1$	$-z/2$
the modified 3rd kind, Y^*	$z/2$	$v-1$	$-z/2$

Theorem 2. The PBB distributions of W, X and Y^* are unimodal with the modes

$$\mathrm{Mo}(W\,|\,v,z,\lambda) = \frac{1}{2^{1/\lambda}}\left[-(v\lambda+1)+\{(v\lambda+1)^2+z^2\}^{1/2}\right]^{1/\lambda},$$

$$\mathrm{Mo}(X \mid v, z, \lambda) = \frac{1}{z^{1/\lambda}} \left[v\lambda - 1 + \{(v\lambda - 1)^2 + z^2\}^{1/2} \right]^{1/\lambda} \text{ and}$$

$$\mathrm{Mo}(Y^* \mid v, z, \lambda) = \frac{\ln\left[\left\{ v + (v^2 + z^2)^{1/2} \right\} / z \right]}{\lambda}.$$

Proof. The unimodalities of the distributions are given indirectly due to the monotonic transformation of the Halphen type A or the BB distributions of the second kind having unimodality. The modes are obtained by the differential equations set to zero, which reduce to the quadratic equations for monotonic functions of *t*. A direct derivation of the unimodalities is found by the negative values of the product i.e., a_0 / a_2 of the solutions of the quadratic equations (see Table 1), where only the single positive solutions give the modes after inverse transformation of associated monotonic functions of *t*. Q.E.D.

The limiting values of $\mathrm{Mo}(W \mid v, z, \lambda), \mathrm{Mo}(X \mid v, z, \lambda)$ and $\mathrm{Mo}(Y^* \mid v, z, \lambda)$ when the values of *v* and *z* become large or small will be shown in Property 1 of the appendix. For the modes of the remaining PBB distributed variables X^* and Y of the modified second and un-modified third kinds, respectively, the following results are provided.

Lemma 5. *The differential equations for the PBB distributed variables* X^* *and* Y *with the generic expression of their pdf's denoted by* $f(t)$ *are given by*

$$\frac{\mathrm{d} \ln f(t)}{\mathrm{d}t} = \left[a_0 + \{a_1(t^\lambda) + 1 - \lambda^{-1}\} t^\lambda + a_2 t^{2\lambda} \right] \lambda t^{-\lambda - 1} \quad \textit{for } X^*,$$

and $$\frac{\mathrm{d} \ln f(t)}{\mathrm{d}t} = \left[a_0 + a_1\{\exp(t\lambda)\} \exp(t\lambda) + a_2 \exp(2t\lambda) \right] \lambda \exp(-t\lambda) \quad \textit{for } Y,$$

where the coefficients/function a_0, $a_1(\cdot)$ (*or* $a_1\{\cdot\}$) *and* a_2 *are shown in Table 2.*

Table 2. The coefficients or functions in the differential equations for the PBB distributions for X^* *and* Y

PBB distribution of	a_0	$a_1(u)$ or $a_1\{u\}$	a_2
the modified 2nd kind, X^*	$z/2$	$\frac{(v-1)u^{v-1} - (v+1)u^{-v-1}}{u^{v-1} + u^{-v-1}}$	$-z/2$
the 3rd kind, Y	$z/2$	$\frac{v(u^v - u^{-v})}{u^v + u^{-v}}$	$-z/2$

Theorem 3. *The PBB distributions of* X^* *and* Y *are unimodal.* $\mathrm{Mo}(X^* \mid v, z, \lambda)$ *is given by the converged value of t in the following iterative computation*:

$$t\,|_{(j+1)-\mathrm{th}} = \left(\frac{a_1(t^\lambda) + 1 - \lambda^{-1} + \left[\{a_1(t^\lambda) + 1 - \lambda^{-1}\}^2 + z^2 \right]^{1/2}}{z} \right)^{1/\lambda} |_{j-\mathrm{th}} \;\; (j = 0, 1, \ldots),$$

where $t\,|_{(j+1)-\mathrm{th}}$ *indicates the revised value of t in the* (*j*+1)*-th* $(j = 0, 1, \ldots)$ *iteration. Similarly,* $\mathrm{Mo}(Y \mid v, z, \lambda)$ *is obtained by the following iterative computation*:

$$t\,|_{(j+1)-\mathrm{th}} = \frac{1}{\lambda} \ln \left(\frac{a_1\{\exp(t\lambda)\} + \left(\left[a_1\{\exp(t\lambda)\} \right]^2 + z^2 \right)^{1/2}}{z} \right) |_{j-\mathrm{th}} \;\; (j = 0, 1, \ldots),$$

where $a_1\{\cdot\}$ *is the function of the argument in braces.*

Remark 2. The α- th ($0 < \alpha < 1$) quantiles denoted by $Q_U(\alpha)$ of the generic variable U for the PBB distributions are given by the upper incomplete BB functions satisfying

$$\alpha = \int_0^{Q_U(\alpha)} f_U(u \mid v, z, \gamma)\mathrm{d}u = 1 - \frac{K_{(j),v,\gamma}\{z, Q_U(\alpha)\}}{K_v(z)}$$
$$\{(U, u, j) \in \{(W, w, 1), (X, x, 2), (X^*, x^*, \mathrm{m}2), (Y, y, 3), (Y^*, y^*, \mathrm{m}3)\}\}.$$

Since $K_{(j),v,\gamma}\{z, Q_U(\alpha)\}$'s are decreasing functions of $Q_U(\alpha)$, the solutions of $Q_U(\alpha)$ given α are numerically obtained by e.g., the bisection method. Note that as mentioned earlier, variables W, X and Y^* have the relationships $X^\lambda = \dfrac{z}{2W^\lambda} = \mathrm{e}^{\lambda Y^*}$, which gives

$$Q_X(\alpha)^\lambda = \frac{z}{2Q_W(1-\alpha)^\lambda} = \mathrm{e}^{\lambda Q_{Y^*}(\alpha)}.$$

6. Discussion

(a) The symmetric reciprocal Bessel distributions

As mentioned earlier, Takei and Matsunawa (2001, p. 175) defined a family of symmetric reciprocal (SR) distributions that includes the factor $\dfrac{x^\gamma}{a} + \dfrac{b}{x^\gamma}$ $(x > 0,\ a > 0,\ b > 0,\ -\infty < \gamma < \infty)$ in the pdf with some associated regularity conditions, where x is the argument of the pdf of a random variable. It is found that the PBB distributions of the first and second kinds belong to the SR family while the those of the third and modified third kinds do not, though they belong to the family by the logarithmic transformation. Due to this property, we have the following extension of the SR family based on a BB distribution.

Definition 4 (the symmetric reciprocal Bessel (SRB) distributions). Suppose that $h(U\,|\,\boldsymbol{\gamma})$ is a positive and differentiable function of random U over the support $U \in \mathrm{S}$, and that $h(U\,|\,\boldsymbol{\gamma})$ has the BB distribution of the second kind, where $\boldsymbol{\gamma}$ is the vector of parameters of an appropriate dimension. Then, U is defined to follow the symmetric reciprocal Bessel (SRB) distribution, whose pdf is given by

$$f_U(u\,|\,v,z,\boldsymbol{\gamma}) = K_v(z)^{-1}\frac{1}{2}\exp\left[-\frac{z}{2}\left\{h(u\,|\,\boldsymbol{\gamma}) + \frac{1}{h(u\,|\,\boldsymbol{\gamma})}\right\}\right]h(u\,|\,\boldsymbol{\gamma})^{v-1}\,|\,\mathrm{d}h(u\,|\,\boldsymbol{\gamma})/\,\mathrm{d}u\,|\,.$$

It is found that the family of the SRB distributions includes the BB, PBB and some of the SR distributions.

(b) Unimodal symmetric reciprocal (USR) distributions and quasi-quadratic equations

As found in Theorem 3, the differential equations set equal to zeroes take a form of the quasi-quadratic one in that the coefficient a_1 is a function of the associated variable. This suggests a pdf of a generalized (Bessel) distribution of the form similar to that of the SRB:

$$f_U(u\,|\,z,\boldsymbol{\gamma}^*) = C(z,\boldsymbol{\gamma}^*)\exp\left[-\frac{z}{2}\left\{g(u\,|\,\boldsymbol{\gamma}^*) + \frac{1}{g(u\,|\,\boldsymbol{\gamma}^*)}\right\}\right]h(u\,|\,\boldsymbol{\gamma}^*),$$

where $z > 0$ as before; $\boldsymbol{\gamma}^*$ is a vector of other parameters; $C(z,\boldsymbol{\gamma}^*)$ is a normalizing constant possibly including a Bessel function; $g(u\,|\,\boldsymbol{\gamma}^*) > 0$ with $g'(u\,|\,\boldsymbol{\gamma}^*)$ $\equiv \mathrm{d}g(u\,|\,\boldsymbol{\gamma}^*)/\,\mathrm{d}u > 0$ is a strictly increasing function of u without loss of generality in that a decreasing function $g(u\,|\,\boldsymbol{\gamma}^*)$, if desired, can be replaced by its reciprocal; $h(u\,|\,\boldsymbol{\gamma}^*) > 0$; and the support, which can be negative, depends on cases. This distribution is called unimodal symmetric reciprocal (USR) distributions. Then, the corresponding differential equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\ln f_U(u\,|\,z,\boldsymbol{\gamma}^*) = -\frac{z}{2}\left\{g'(u\,|\,\boldsymbol{\gamma}^*) - \frac{g'(u\,|\,\boldsymbol{\gamma}^*)}{g(u\,|\,\boldsymbol{\gamma}^*)^2}\right\} + \frac{h'(u\,|\,\boldsymbol{\gamma}^*)}{h(u\,|\,\boldsymbol{\gamma}^*)}$$

$$= \frac{\dfrac{z}{2}g'(u\,|\,\boldsymbol{\gamma}^*) + \dfrac{h'(u\,|\,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*)}{h(u\,|\,\boldsymbol{\gamma}^*)}g(u\,|\,\boldsymbol{\gamma}^*) - \dfrac{z}{2}g'(u\,|\,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*)^2}{g(u\,|\,\boldsymbol{\gamma}^*)^2}$$

$$= \frac{c_0(u\,|\,z,\boldsymbol{\gamma}^*) + c_1(u\,|\,z,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*) + c_2(u\,|\,z,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*)^2}{g(u\,|\,\boldsymbol{\gamma}^*)^2}.$$

The equation to have the mode is given by the numerator set equal to zero:

$$c_0(u\,|\,z,\boldsymbol{\gamma}^*) + c_1(u\,|\,z,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*) + c_2(u\,|\,z,\boldsymbol{\gamma}^*)g(u\,|\,\boldsymbol{\gamma}^*)^2 = 0,$$

which is seen as a quasi-quadratic equation in terms of $g(u\,|\,\boldsymbol{\gamma}^*)$ with all the coefficients $c_j(u\,|\,z,\boldsymbol{\gamma}^*)$ $(j = 0,1,2)$ being functions of *u*. The two solutions of the equation always consist of a positive one and a negative one since the product of the two solutions is $c_0(u\,|\,z,\boldsymbol{\gamma}^*)\,/\,c_2(u\,|\,z,\boldsymbol{\gamma}^*) = -1$ as before. The solution to be employed becomes

$$g(u\,|\,\boldsymbol{\gamma}^*) = \frac{-c_1(u\,|\,z,\boldsymbol{\gamma}^*) + \left\{c_1(u\,|\,z,\boldsymbol{\gamma}^*)^2 + z^2 g'(u\,|\,\boldsymbol{\gamma}^*)^2\right\}^{-1}}{z\,g'(u\,|\,\boldsymbol{\gamma}^*)},$$

which is found to be positive due to the conditions $z > 0$ and $g'(u\,|\,\boldsymbol{\gamma}^*) > 0$. The left-hand side of the above equation is also seen as an undated value of $g(u\,|\,\boldsymbol{\gamma}^*)$. Then, the updated *u* is given by the inverse function $g^{-1}(u\,|\,\boldsymbol{\gamma}^*)$ with the assumption of the existence of its unique value addressed earlier. Then, using this *u*, the right-hand side of the above equation is updated. The iteration is repeated until convergence. It is easily found that the family of the SRB distributions in the previous Subsection is a sub-family of the USR distributions.

(c) Multivariate extensions

While the Halphen type A is defined for a family of univariate distributions, multivariate extensions of the IG and GIG are available (see Barndorff-Nielsen, Blaesild, Jensen & Jørgensen, 1982, Example 4; Chhikara & Folks, 1989, Sections 11.1 and 11.2; Barndorff-Nielsen et al., 1992, Sections 3 and 4). It is expected that the multivariate extensions of the PBB distributions corresponding to the known multivariate GIG are similarly obtained using reparametrizations with change of variables.

(d) Other remaining issues

The problems of parameter estimation, hypothesis testing for parameters and goodness-of-fit have not been dealt with due to the emphasis on the historical review and extensions (for estimation in the Halphen type A, see Perreault et al., 1999b; El Adlouni & Bobée, 2017, Chapter 4; Singh & Zhang, 2022, Section 3.5). The parameters in the PBB except the scale one are v, z and $\lambda\,(v \in \mathrm{R},\ z > 0,\ \lambda \in \mathrm{R},\ \lambda \neq 0)$. Among them, several candidate values may be chosen in practice for v and λ rather than estimating them by e.g., maximum likelihood. In Section 1, the IG was found to be a special case of the Halphen type A or the PBB of the second kind with fixed $v = -1/2$ and $\lambda = 1$. It is known that when v is a half integer, $K_{-v}(z)(= K_v(z))$ is given by the following finite series:

$$K_{k+(1/2)}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \sum_{j=0}^{k} \frac{(k+j)!}{(k-j)!\,j!}(2z)^{-j}\mathrm{e}^{-z} \quad (k = 0,1,\ldots)$$

(Watson, 1944/1995, Section 3.71, Equation (12), p. 80; Abramowitz & Stegun, 1972, Formula 10.2.15-17, p. 444; Barndorff-Nielsen,1978, Equation (3.1); Zwillinger, 2015, Formula 8.468, p. 934; DLMF, 2023, http://dlmf.nist.gov/10.39.E2, https://dlmf.nist.gov/10.47#ii, https://dlmf.nist.gov/10.49#ii; Gaunt & Li, 2023, Equation (24)). When $k = 0, 1, 2$ and 3, the above formula gives

$$K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}, K_{3/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}(1 + z^{-1}),$$

$$K_{5/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}(1 + 3z^{-1} + 3z^{-2}) \text{ and } K_{7/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \mathrm{e}^{-z}(1 + 6z^{-1} + 15z^{-2} + 15z^{-3}).$$

As addressed earlier, in the IG, only the simplest one $K_{1/2}(z)$ is used. This limitation has been relaxed using $K_v(z)$ $(v \in \mathrm{R})$ in the GIG and PBB distributions, where the subfamilies with $K_{k+(1/2)}(z)$ $(k = 0,1,\ldots)$ may be of practical use.

Appendix

Proofs of Lemma 1.

Proof 1. Using the definitions of the moments, we obtain

$$\mathrm{E}(W^k \mid v,z,\gamma) = \int_0^\infty K_v(z)^{-1}\frac{(z/2)^v}{2}\exp\left(-w^\gamma - \frac{z^2}{4w^\gamma}\right)\frac{|\gamma|\,w^k}{w^{\gamma v+1}}\mathrm{d}w$$

$$= \frac{K_{v-(k/\gamma)}(z)(z/2)^v}{K_v(z)(z/2)^{v-(k/\gamma)}}\int_0^\infty K_{v-(k/\gamma)}(z)^{-1}\frac{(z/2)^{v-(k/\gamma)}}{2}\exp\left(-w^\gamma - \frac{z^2}{4w^\gamma}\right)\frac{|\gamma|}{w^{\gamma\{v-(k/\gamma)\}+1}}\mathrm{d}w$$

$$= \frac{K_{v-(k/\gamma)}(z)}{K_v(z)(z/2)^{-k/\gamma}},$$

$$\mathrm{E}(X^k \mid v,z,\gamma) = \int_0^\infty K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\,x^{\gamma v+k-1}\mathrm{d}x$$

$$= \frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\int_0^\infty K_{v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\,x^{\gamma\{v+(k/\gamma)\}-1}\mathrm{d}x$$

$$= \frac{K_{v+(k/\gamma)}(z)}{K_v(z)},$$

$$\mathrm{E}(X^{*k} \mid v,z,\gamma) = \int_0^\infty K_v(z)^{-1}\frac{1}{4}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\,(x^{\gamma v+k-1} + x^{-\gamma v+k-1})\mathrm{d}x$$

$$= \frac{1}{2}\frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\int_0^\infty K_{v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\,x^{\gamma\{v+(k/\gamma)\}-1}\mathrm{d}x$$

$$+ \frac{1}{2}\frac{K_{-v+(k/\gamma)}(z)}{K_v(z)}\int_0^\infty K_{-v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma + \frac{1}{x^\gamma}\right)\right\}|\gamma|\,x^{\gamma\{-v+(k/\gamma)\}-1}\mathrm{d}x$$

$$= \frac{1}{2}\frac{K_{v+(k/\gamma)}(z) + K_{-v+(k/\gamma)}(z)}{K_v(z)} \equiv \frac{\bar{K}_{\{\pm v\}+(k/\gamma)}(z)}{K_v(z)}$$

$$= \frac{1}{2}\{\mathrm{E}(X^k \mid v,z,\gamma) + \mathrm{E}(X^k \mid -v,z,\gamma)\}$$

with the remaining expressions for $\mathrm{E}(X^{*k} \mid v,z,\gamma)$ being obtained similarly, and

$$\mathrm{E}\{\exp(kY^*) \mid v,z,\gamma\} = \mathrm{M}_{Y^*}(k \mid v,z,\gamma)$$

$$= \int_{-\infty}^\infty K_v(z)^{-1}\frac{1}{2}\exp\{-z\cosh(\gamma y^*) + v\gamma y^*\}|\gamma|\,\mathrm{e}^{ky^*}\mathrm{d}y^*$$

$$= \frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\int_{-\infty}^\infty K_{v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\{-z\cosh(\gamma y^*) + \{v+(k/\gamma)\}\gamma y^*\}|\gamma|\,\mathrm{d}y^*$$

$$= \frac{K_{v+(k/\gamma)}(z)}{K_v(z)}.$$

The above expressions with the property of the mgf of Y^* yielding its raw moments give the required results. Q.E.D.

Proof 2. An alternative indirect proof of $\mathrm{E}(X^k \mid v,z,\gamma)=\mathrm{M}_{Y^*}(k \mid v,z,\gamma)$ is shown. By definition we have as before

$$\mathrm{E}(X^k \mid v,z,\gamma)=\int_0^\infty K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(x^\gamma+\frac{1}{x^\gamma}\right)\right\}|\gamma|\,x^{\gamma v+k-1}\mathrm{d}x\,.$$

Using the variable transformation $x=\mathrm{e}^{y^*}$, the above equation becomes

$$\begin{aligned}
\mathrm{E}(X^k \mid v,z,\gamma)&=\int_{-\infty}^\infty K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(\mathrm{e}^{y^*\gamma}+\frac{1}{\mathrm{e}^{y^*\gamma}}\right)\right\}|\gamma|\,\mathrm{e}^{y^*(\gamma v+k-1)}\mathrm{e}^{y^*}\mathrm{d}y^*\\
&=\int_{-\infty}^\infty K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(\mathrm{e}^{y^*\gamma}+\frac{1}{\mathrm{e}^{y^*\gamma}}\right)\right\}|\gamma|\,\mathrm{e}^{ky^*}\mathrm{e}^{y^*\gamma v}\mathrm{d}y^*\\
&=\int_{-\infty}^\infty K_v(z)^{-1}\frac{1}{2}\exp\{-z\cosh(y^*\gamma)+y^*\gamma v\}|\gamma|\,\mathrm{e}^{ky^*}\mathrm{d}y^*\\
&=\mathrm{E}\{\exp(kY^*)\mid v,z,\gamma\},
\end{aligned}$$

which gives the required result. Q.E.D.

Proof of Lemma 2.

Using the definition of the mgf, we obtain

$$\begin{aligned}
&\mathrm{E}\{\exp(kY)\mid v,z,\gamma\}=\mathrm{M}_Y(k\mid v,z,\gamma)\\
&=\int_0^\infty K_v(z)^{-1}\exp\{-z\cosh(\gamma y)\}\cosh(v\gamma y)|\gamma|\,\mathrm{e}^{ky}\mathrm{d}y\\
&=\int_0^\infty K_v(z)^{-1}\frac{1}{2}\exp\left[-z\cosh(\gamma y)+\{v+(k/\gamma)\}\gamma y\right]|\gamma|\,\mathrm{d}y\\
&+\int_0^\infty K_v(z)^{-1}\frac{1}{2}\exp\left[-z\cosh(\gamma y)-\{v-(k/\gamma)\}\gamma y\right]|\gamma|\,\mathrm{d}y\\
&=\frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\int_0^\infty K_{v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\{-z\cosh(\gamma y^*)+\{v+(k/\gamma)\}\gamma y^*\}|\gamma|\,\mathrm{d}y^*\\
&+\frac{K_{v-(k/\gamma)}(z)}{K_v(z)}\int_0^\infty K_{-v+(k/\gamma)}(z)^{-1}\frac{1}{2}\exp\{-z\cosh(\gamma y^*)-\{v-(k/\gamma)\}\gamma y^*\}|\gamma|\,\mathrm{d}y^*\\
&=\frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\Pr\{Y^*>0\mid v+(k/\gamma),z,\gamma\}+\frac{K_{v-(k/\gamma)}(z)}{K_v(z)}\Pr\{Y^*>0\mid -v+(k/\gamma),z,\gamma\}\\
&=\frac{K_{v+(k/\gamma)}(z)}{K_v(z)}\Pr\{X>1\mid v+(k/\gamma),z,\gamma\}+\frac{K_{v-(k/\gamma)}(z)}{K_v(z)}\Pr\{X>1\mid -v+(k/\gamma),z,\gamma\}\\
&=\frac{K_{(2),v+(k/\gamma),\gamma}(z,1)}{K_v(z)}+\frac{K_{(2),-v+(k/\gamma),\gamma}(z,1)}{K_v(z)}=\frac{K_{(2),\{\pm v\}+(k/\gamma),\gamma}(z,1)}{K_v(z)},
\end{aligned}$$

where $K_{v-(k/\gamma)}(z)=K_{-v+(k/\gamma)}(z)$ is used. The last expression gives the required result.

Proof of Lemma 3.

Since $1=\int_0^\infty K_v(z)^{-1}\frac{1}{2}\exp\left\{-\frac{z}{2}\left(\frac{x}{\beta}+\frac{\beta}{x}\right)\right\}\frac{x^{v-1}}{\beta^v}\mathrm{d}x$, we have

$$\frac{\mathrm{d}^j\beta^v K_v(z)}{\mathrm{d}v^j}=\frac{\mathrm{d}^j}{\mathrm{d}v^j}\int_0^\infty\frac{1}{2}\exp\left\{-\frac{z}{2}\left(\frac{x}{\beta}+\frac{\beta}{x}\right)\right\}x^{v-1}\mathrm{d}x\ (j=1,2,...).$$

Using $\frac{\mathrm{d}^j g(k)h(k)}{\mathrm{d}k^j}=\sum_{m=0}^{j}\binom{j}{m}\frac{\mathrm{d}^m g(k)}{\mathrm{d}k^m}\frac{\mathrm{d}^{j-m}h(k)}{\mathrm{d}k^{j-m}}$, where $g(k)$ and $h(k)$ are differentiable function of k, we obtain

$$\frac{\mathrm{d}^j\beta^{v+(k/\gamma)}K_{v+(k/\gamma)}(z)}{\mathrm{d}k^j}=\beta^{v+(k/\gamma)}\sum_{m=0}^{j}\binom{j}{m}(\gamma^{-1}\ln\beta)^m\frac{\mathrm{d}^{j-m}K_{v+(k/\gamma)}(z)}{\mathrm{d}v^{j-m}}$$
$$=\int_0^\infty(\gamma^{-1}\ln x)^j\frac{1}{2}\exp\left[-\frac{z}{2}\left(\frac{x}{\beta}+\frac{\beta}{x}\right)\right]x^{v+(k/\gamma)-1}\mathrm{d}x.$$

Dividing both sides of the last equation by $\beta^{v+(k/\gamma)}K_{v+(k/\gamma)}(z)$, the first result follows as

$$\sum_{m=0}^{j}\binom{j}{m}(\gamma^{-1}\ln\beta)^m\frac{1}{K_{v+(k/\gamma)}(z)}\frac{\mathrm{d}^{j-m}K_{v+(k/\gamma)}(z)}{\mathrm{d}t^{j-m}}$$
$$=\int_0^\infty(\gamma^{-1}\ln x)^j K_{v+(k/\gamma)}(z)^{-1}\frac{1}{2\beta^{v+(k/\gamma)}}\exp\left[-\frac{z}{2}\left(\frac{x}{\beta}+\frac{\beta}{x}\right)\right]x^{v+(k/\gamma)-1}\mathrm{d}x$$
$$=\mathrm{E}\{(\gamma^{-1}\ln X)^j\,|\,v+(k/\gamma),z,\beta\}.$$

The second result is similarly obtained using the definition of the partial expectation instead of the full one, which gives the required result.

Some moments and their functions of the PBB distributions

Theorem 1. *The expectations, variances, skewnesses (sk's) and excess kurtoses (kt's) of the power BB-distributed variables of the first, second and modified third kinds are*

$$\mathrm{E}(W\,|\,v,z,\gamma)=\frac{K_{v-(1/\gamma)}(z)}{K_v(z)(z/2)^{-\gamma}},$$

$$\operatorname{var}(W \mid v, z, \gamma)=\left(\frac{K_{v-(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v-(1/\gamma)}^2(z)}{K_v^2(z)}\right)\frac{1}{(z/2)^{-2\gamma}},$$

$$\operatorname{sk}(W \mid v, z, \gamma)$$
$$=\left(\frac{K_{v-(3/\gamma)}(z)}{K_v(z)}-3\frac{K_{v-(2/\gamma)}(z)K_{v-(1/\gamma)}(z)}{K_v^2(z)}+2\frac{K_{v-(1/\gamma)}^3(z)}{K_v^3(z)}\right)\left(\frac{K_{v-(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v-(1/\gamma)}^2(z)}{K_v^2(z)}\right)^{-3/2},$$

$$\operatorname{kt}(W \mid v, z, \gamma)$$
$$=\left(\frac{K_{v-(4/\gamma)}(z)}{K_v(z)}-4\frac{K_{v-(3/\gamma)}(z)K_{v-(1/\gamma)}(z)}{K_v^2(z)}+6\frac{K_{v-(2/\gamma)}K_{v-(1/\gamma)}^2(z)}{K_v^3(z)}-3\frac{K_{v-(1/\gamma)}^4(z)}{K_v^4(z)}\right)$$
$$\times\left(\frac{K_{v-(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v-(1/\gamma)}^2(z)}{K_v^2(z)}\right)^{-2}-3,$$

$$\mathrm{E}(X \mid v, z, \gamma)=\frac{K_{v+(1/\gamma)}(z)}{K_v(z)},$$

$$\operatorname{var}(X \mid v, z, \gamma)=\left(\frac{K_{v+(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v+(1/\gamma)}^2(z)}{K_v^2(z)}\right),$$

$$\operatorname{sk}(X \mid v, z, \gamma)$$
$$=\left(\frac{K_{v+(3/\gamma)}(z)}{K_v(z)}-3\frac{K_{v+(2/\gamma)}(z)K_{v+(1/\gamma)}(z)}{K_v^2(z)}+2\frac{K_{v+(1/\gamma)}^3(z)}{K_v^3(z)}\right)$$
$$\times\left(\frac{K_{v+(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v+(1/\gamma)}^2(z)}{K_v^2(z)}\right)^{-3/2},$$

$$\operatorname{kt}(X \mid v, z, \gamma)$$
$$=\left(\frac{K_{v+(4/\gamma)}(z)}{K_v(z)}-4\frac{K_{v+(3/\gamma)}(z)K_{v+(1/\gamma)}(z)}{K_v^2(z)}+6\frac{K_{v+(2/\gamma)}K_{v+(1/\gamma)}^2(z)}{K_v^3(z)}-3\frac{K_{v+(1/\gamma)}^4(z)}{K_v^4(z)}\right)$$
$$\times\left(\frac{K_{v+(2/\gamma)}(z)}{K_v(z)}-\frac{K_{v+(1/\gamma)}^2(z)}{K_v^2(z)}\right)^{-2}-3,$$

$$\mathrm{E}(X^* \mid v, z, \gamma)=\frac{\overline{K}_{\{\pm v\}+(1/\gamma)}(z)}{K_v(z)},$$

$$\operatorname{var}(X^* \mid v, z, \gamma)=\left(\frac{\overline{K}_{\{\pm v\}+(2/\gamma)}(z)}{K_v(z)}-\frac{\overline{K}_{\{\pm v\}+(1/\gamma)}^2(z)}{K_v^2(z)}\right),$$

$$\mathrm{sk}(X^* \mid v, z, \gamma)$$

$$= \left(\frac{\bar{K}_{\{\pm v\}+(3/\gamma)}(z)}{K_v(z)} - 3\frac{\bar{K}_{\{\pm v\}+(2/\gamma)}(z)\bar{K}_{\{\pm v\}+(1/\gamma)}(z)}{K_v^2(z)} + 2\frac{\bar{K}^3_{\{\pm v\}+(1/\gamma)}(z)}{K_v^3(z)} \right),$$

$$\times \left(\frac{\bar{K}_{\{\pm v\}+(2/\gamma)}(z)}{K_v(z)} - \frac{\bar{K}^2_{\{\pm v\}+(1/\gamma)}(z)}{K_v^2(z)} \right)^{-3/2},$$

$$\mathrm{kt}(X^* \mid v, z, \gamma)$$

$$= \left(\frac{\bar{K}_{\{\pm v\}+(4/\gamma)}(z)}{K_v(z)} - 4\frac{\bar{K}_{\{\pm v\}+(3/\gamma)}(z)\bar{K}_{\{\pm v\}+(1/\gamma)}(z)}{K_v^2(z)} + 6\frac{\bar{K}_{\{\pm v\}+(2/\gamma)}(z)\bar{K}^2_{\{\pm v\}+(1/\gamma)}(z)}{K_v^3(z)} \right.$$

$$\left. -3\frac{\bar{K}^4_{\{\pm v\}+(1/\gamma)}(z)}{K_v^4(z)} \right) \left(\frac{\bar{K}_{\{\pm v\}+(2/\gamma)}(z)}{K_v(z)} - \frac{\bar{K}^2_{\{\pm v\}+(1/\gamma)}(z)}{K_v^2(z)} \right)^{-2} - 3,$$

$$\mathrm{E}(Y \mid v, z, \gamma) = \frac{K^{(1)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)},$$

$$\mathrm{var}(Y \mid v, z, \gamma) = \left(\frac{K^{(2)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)} - \frac{K^{(1)2}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^2(z)} \right),$$

$$\mathrm{sk}(Y \mid v, z, \gamma)$$

$$= \left(\frac{K^{(3)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)} - 3\frac{K^{(2)}_{(2),\{\pm v\},\gamma}(z,1)K^{(1)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^2(z)} + 2\frac{K^{(1)3}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^3(z)} \right)$$

$$\times \left(\frac{K^{(2)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)} - \frac{K^{(1)2}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^2(z)} \right)^{-3/2},$$

$$\mathrm{kt}(Y \mid v, z, \gamma) = \left(\frac{K^{(4)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)} - 4\frac{K^{(3)}_{(2),\{\pm v\},\gamma}(z,1)K^{(1)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^2(z)} \right.$$

$$\left. +6\frac{K^{(2)}_{(2),\{\pm v\},\gamma}(z,1)K^{(1)2}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^3(z)} - 3\frac{K^{(1)4}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^4(z)} \right) \left(\frac{K^{(2)}_{(2),\{\pm v\},\gamma}(z,1)}{K_v(z)} - \frac{K^{(1)2}_{(2),\{\pm v\},\gamma}(z,1)}{K_v^2(z)} \right)^{-2}$$

$$-3.$$

$$\mathrm{E}(Y^* \mid v, z, \gamma) = \frac{K^{(1)}_{v,\gamma}(z)}{K_v(z)},$$

$$\mathrm{var}(Y^* \mid v, z, \gamma) = \left(\frac{K^{(2)}_{v,\gamma}(z)}{K_v(z)} - \frac{K^{(1)2}_{v,\gamma}(z)}{K_v^2(z)} \right),$$

$$\mathrm{sk}(Y^* \mid v, z, \gamma)$$
$$= \left(\frac{K_{v,\gamma}^{(3)}(z)}{K_v(z)} - 3\frac{K_{v,\gamma}^{(2)}(z)K_{v,\gamma}^{(1)}(z)}{K_v^2(z)} + 2\frac{K_{v,\gamma}^{(1)3}(z)}{K_v^3(z)} \right)$$
$$\times \left(\frac{K_{v,\gamma}^{(2)}(z)}{K_v(z)} - \frac{K_{v,\gamma}^{(1)2}(z)}{K_v^2(z)} \right)^{-3/2},$$

$$\mathrm{kt}(Y^* \mid v, z, \gamma)$$
$$= \left(\frac{K_{v,\gamma}^{(4)}(z)}{K_v(z)} - 4\frac{K_{v,\gamma}^{(3)}(z)K_{v,\gamma}^{(1)}(z)}{K_v^2(z)} + 6\frac{K_{v,\gamma}^{(2)}(z)K_{v,\gamma}^{(1)2}(z)}{K_v^3(z)} - 3\frac{K_{v,\gamma}^{(1)4}(z)}{K_v^4(z)} \right)$$
$$\times \left(\frac{K_{v,\gamma}^{(2)}(z)}{K_v(z)} - \frac{K_{v,\gamma}^{(1)2}(z)}{K_v^2(z)} \right)^{-2} - 3.$$

Proof. The results are given by Lemmas 1to 3. Q.E.D.

Proof of Lemma 4.

For *W* and *X*, the coefficients a_0, a_1 and a_2 in Table 1 are given by definition. Then, we obtain

$$\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{(-a_0u^{-1} + a_1 \ln u + a_2u)\,|_{u=t^\lambda} + \ln|\mathrm{d}t^\lambda / \mathrm{d}t|\right\}$$
$$= \frac{a_0 + a_1u + a_2u^2}{u^2}\,|_{u=t^\lambda}\,\frac{\mathrm{d}t^\lambda}{\mathrm{d}t} + (\lambda - 1)t^{-1}$$
$$= (a_0t^{-2\lambda} + a_1t^{-\lambda} + a_2)\lambda t^{\lambda-1} + (\lambda-1)t^{-1}$$
$$= a_0\lambda t^{-\lambda-1} + (a_1\lambda + \lambda - 1)t^{-1} + a_2\lambda t^{\lambda-1}$$
$$= \left\{a_0\lambda + (a_1\lambda + \lambda - 1)t^\lambda + a_2\lambda t^{2\lambda}\right\} t^{-\lambda-1}.$$

For Y^*, we have in a similar manner

$$\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{(-a_0u^{-1} + a_1 \ln u + a_2u)\,|_{u=\exp(t\lambda)} + \ln\frac{\mathrm{d}\exp(t\lambda)}{\mathrm{d}t}\right\}$$
$$= \frac{a_0 + a_1u + a_2u^2}{u^2}\,|_{u=\exp(t\lambda)}\,\frac{\mathrm{d}\exp(t\lambda)}{\mathrm{d}t} + \lambda$$
$$= \{a_0\exp(-2t\lambda) + a_1\exp(-t\lambda) + a_2\}\lambda\exp(t\lambda) + \lambda$$
$$= \{a_0 + (a_1+1)\exp(t\lambda) + a_2\exp(2t\lambda)\}\lambda\exp(-t\lambda).$$

The above expressions give the required results.

Some limiting values of $\mathrm{Mo}(W \mid v,z,\lambda)$, $\mathrm{Mo}(X \mid v,z,\lambda)$ **and** $\mathrm{Mo}(Y^* \mid v,z,\lambda)$**.**

Property 1. Consider the limiting values of $\mathrm{Mo}(W \mid v,z,\lambda)$, $\mathrm{Mo}(X \mid v,z,\lambda)$ and $\mathrm{Mo}(Y^* \mid v,z,\lambda)$ when the values of *v* and *z* become large or small.

(i) When an arbitrary positive *z* and fixed $\lambda \in \mathrm{R}\,(\lambda \neq 0)$ are given,

$$\lim_{v\to-\infty} \mathrm{Mo}(W \mid v,z,\lambda>0) = \infty,$$

$$\lim_{v\to-\infty} \mathrm{Mo}(W \mid v,z,\lambda<0) = \lim_{v\to-\infty}\left(\frac{2}{-(v\lambda+1)+\{(v\lambda+1)^2+z^2\}^{1/2}}\right)^{1/|\lambda|}$$

$$= \lim_{v\to-\infty}\left[2\frac{v\lambda+1+\{(v\lambda+1)^2+z^2\}^{1/2}}{z^2}\right]^{1/|\lambda|} = \infty,$$

$$\lim_{v\to-\infty} \mathrm{Mo}(X \mid v,z,\lambda\neq 0) = 0,$$

$$\lim_{v\to-\infty} \mathrm{Mo}(Y^* \mid v,z,\lambda>0) = \lim_{v\to-\infty}\frac{\ln\left[\left\{v+(v^2+z^2)^{1/2}\right\}/z\right]}{\lambda} = -\infty,$$

$$\lim_{v\to-\infty} \mathrm{Mo}(Y^* \mid v,z,\lambda<0) = \infty,$$

$$\lim_{v\to\infty} \mathrm{Mo}(W \mid v,z,\lambda\neq 0) = 0,\ \lim_{v\to\infty} \mathrm{Mo}(X \mid v,z,\lambda\neq 0) = \infty,$$

$$\lim_{v\to\infty} \mathrm{Mo}(Y^* \mid v,z,\lambda>0) = \infty,\ \lim_{v\to\infty} \mathrm{Mo}(Y^* \mid v,z,\lambda<0) = -\infty$$

with

$$\frac{\partial \mathrm{Mo}(W \mid v,z,\lambda\neq 0)}{\partial v} = \frac{\partial \mathrm{Mo}(W \mid v,z,\lambda\neq 0)}{\partial (v\lambda)}\lambda$$

$$= \frac{1}{2^{1/\lambda}}\left[-(v\lambda+1)+\{(v\lambda+1)^2+z^2\}^{1/2}\right]^{(1/\lambda)-1}\left[-1+(v\lambda+1)\{(v\lambda+1)^2+z^2\}^{-1/2}\right] < 0,$$

$$\frac{\partial \mathrm{Mo}(X \mid v,z,\lambda\neq 0)}{\partial v}$$

$$= \frac{1}{z^{1/\lambda}}\left[v\lambda-1+\{(v\lambda-1)^2+z^2\}^{1/2}\right]^{(1/\lambda)-1}\left[1+(v\lambda-1)\{(v\lambda-1)^2+z^2\}^{-1/2}\right] > 0$$

and

$$\frac{\partial \mathrm{Mo}(Y^* \mid v,z,\lambda\neq 0)}{\partial v} = \frac{\partial}{\partial v}\frac{\ln\left[\left\{v+(v^2+z^2)^{1/2}\right\}/z\right]}{\lambda} = \frac{1+v(v^2+z^2)^{-1/2}}{v+(v^2+z^2)^{1/2}} > 0$$

indicating that $\mathrm{Mo}(W \mid v, z, \lambda \neq 0)$, $\mathrm{Mo}(X \mid v, z, \lambda \neq 0)$ and $\mathrm{Mo}(Y^* \mid v, z, \lambda \neq 0)$ are strictly decreasing, increasing and increasing functions of *v,* respectively.

(ii) When *v* and λ are given, the results are shown by cases.

(ii-a) $v\lambda < -1$, $\lim_{z \to +0} \mathrm{Mo}(W \mid v, z, \lambda \neq 0) = \{-(v\lambda + 1)\}^{1/\lambda}$;

$$v\lambda < 1,\ \lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda > 0) = \left(\lim_{z \to +0} \frac{\partial \left[v\lambda - 1 + \{(v\lambda - 1)^2 + z^2\}^{1/2} \right] / \partial z}{\partial z / \partial z} \right)^{1/\lambda} = 0 \text{ and}$$

$\lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda < 0) = \infty$ using L'Hôpital's rule;

$v < 0$, $\lim_{z \to +0} \mathrm{Mo}(Y^* \mid v, z, \lambda > 0) = -\infty$ and $\lim_{z \to +0} \mathrm{Mo}(Y^* \mid v, z, \lambda < 0) = \infty$.

(ii-b) $v\lambda = -1$, $\lim_{z \to +0} \mathrm{Mo}(W \mid v, z, \lambda > 0) = 0$ and $\lim_{z \to +0} \mathrm{Mo}(W \mid v, z, \lambda < 0) = \infty$;

$v\lambda = 1$, $\lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda \neq 0) = 1$;

$v = 0$, $\lim_{z \to +0} \mathrm{Mo}(Y^* \mid v, z, \lambda \neq 0) = 0$.

(ii-c) $v\lambda > -1$, $\lim_{z \to +0} \mathrm{Mo}(W \mid v, z, \lambda > 0) = 0$ and $\lim_{z \to +0} \mathrm{Mo}(W \mid v, z, \lambda < 0) = \infty$;

$v\lambda > 1$, $\lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda > 0) = \infty$ and $\lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda < 0) = 0$;

$v > 0$, $\lim_{z \to +0} \mathrm{Mo}(Y^* \mid v, z, \lambda > 0) = \infty$ and $\lim_{z \to +0} \mathrm{Mo}(X \mid v, z, \lambda < 0) = -\infty$.

(iii) When an arbitrary *v* is given, $\lim_{z \to \infty} \mathrm{Mo}(W \mid v, z, \lambda > 0) = \infty$, $\lim_{z \to \infty} \mathrm{Mo}(W \mid v, z, \lambda < 0) = 0$, $\lim_{z \to \infty} \mathrm{Mo}(X \mid v, z, \lambda \neq 0) = 1$ and $\lim_{z \to \infty} \mathrm{Mo}(Y^* \mid v, z, \lambda \neq 0) = 0$.

In (ii) and (iii), note that

$$\frac{\partial \mathrm{Mo}(W \mid v, z, \lambda > 0)}{\partial z} = \frac{1/\lambda}{2^{1/\lambda}} \left[-(v\lambda + 1) + \{(v\lambda + 1)^2 + z^2\}^{1/2} \right]^{(1/\lambda)-1} z\{(v\lambda + 1)^2 + z^2\}^{-1/2} > 0,$$

$$\frac{\partial \mathrm{Mo}(W \mid v, z, \lambda < 0)}{\partial z} < 0; \text{ and}$$

$$\frac{\partial \mathrm{Mo}\{X \mid v, z, (v\lambda - 1)\lambda < 0\}}{\partial z} = \frac{\partial}{\partial z} \frac{1}{z^{1/\lambda}} \left[v\lambda - 1 + \{(v\lambda - 1)^2 + z^2\}^{1/2} \right]^{1/\lambda}$$

$$= \frac{1}{\lambda}\left(\frac{1}{z}\left[v\lambda - 1 + \{(v\lambda-1)^2 + z^2\}^{1/2}\right]\right)^{(1/\lambda)-1}$$

$$\times\left(\frac{1}{z}z\{(v\lambda-1)^2+z^2\}^{-1/2} - \frac{1}{z^2}\left[v\lambda - 1 + \{(v\lambda-1)^2 + z^2\}^{1/2}\right]\right)$$

$$= \frac{1}{\lambda}\left(\frac{1}{z}\left[v\lambda - 1 + \{(v\lambda-1)^2 + z^2\}^{1/2}\right]\right)^{(1/\lambda)-1}$$

$$\times\left(\{(v\lambda-1)^2+z^2\}^{-1/2} - \frac{z^2}{z^2\left[-(v\lambda-1) + \{(v\lambda-1)^2 + z^2\}^{1/2}\right]}\right)$$

$$= \frac{1}{\lambda}\left(\frac{1}{z}\left[v\lambda - 1 + \{(v\lambda-1)^2 + z^2\}^{1/2}\right]\right)^{(1/\lambda)-1} \frac{-(v\lambda-1)}{-(v\lambda-1) + \{(v\lambda-1)^2 + z^2\}^{1/2}} > 0;$$

$\frac{\partial \mathrm{Mo}(X \mid v, z, v\lambda = 1)}{\partial z} = 0$; and $\frac{\partial \mathrm{Mo}\{X \mid v, z, (v\lambda-1)\lambda > 0\}}{\partial z} < 0$;

and

$$\frac{\partial \mathrm{Mo}(Y^* \mid v, z, v\lambda < 0)}{\partial z} = \frac{\partial}{\partial z}\frac{\ln\left[\left\{v + (v^2+z^2)^{1/2}\right\}/z\right]}{\lambda}$$

$$= \frac{-z^{-1} + \left\{v + (v^2+z^2)^{1/2}\right\}^{-1} z(v^2+z^2)^{-1/2}}{\lambda}$$

$$= \frac{z^2 - \left\{v + (v^2+z^2)^{1/2}\right\}(v^2+z^2)^{1/2}}{z\lambda\left\{v + (v^2+z^2)^{1/2}\right\}(v^2+z^2)^{1/2}}$$

$$= \frac{-v\left\{v + (v^2+z^2)^{1/2}\right\}}{z\lambda\left\{v + (v^2+z^2)^{1/2}\right\}(v^2+z^2)^{1/2}} > 0;$$

$\frac{\partial \mathrm{Mo}(Y^* \mid v, z, v = 0)}{\partial z} = 0$ and $\frac{\partial \mathrm{Mo}(Y^* \mid v, z, v\lambda > 0)}{\partial z} < 0$.

Then, it is found that $\mathrm{Mo}(W \mid v, z, \lambda > 0)$ and $\mathrm{Mo}(W \mid v, z, \lambda < 0)$ are strictly increasing and decreasing functions of z;

when $(v\lambda - 1)\lambda < 0$, $v\lambda = 1$ and $(v\lambda-1)\lambda > 0$, $\mathrm{Mo}(X \mid v, z, \lambda)$ is a strictly increasing, constant and strictly decreasing functions of z, respectively; and

when $v\lambda < 0$, $v = 0$ and $v\lambda > 0$, $\mathrm{Mo}(Y^* \mid v, z, \lambda)$ is a strictly increasing, constant and strictly decreasing functions of z, respectively.

Proof of Lemma 5.

For X^*, the coefficients/function a_0, $a_1(\cdot)$ and a_2 in Table 2 are obtained by the corresponding pdf

$$f_{X^*}(t\,|\,v,z,\gamma)=K_v(z)^{-1}\frac{1}{4}\exp\left\{-\frac{z}{2}\left(u+\frac{1}{u}\right)\right\}(u^{v-1}+u^{-v-1})\,|_{u=t^\lambda}\,|\,\mathrm{d}t^\lambda/\mathrm{d}t\,|\quad(0<t<\infty)$$

as follows

$$\begin{aligned}
\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\left\{(-a_0u^{-1}+a_2u)\,|_{u=t^\lambda}+\ln(u^v+u^{-v})+\ln|\,\mathrm{d}t^\lambda/\mathrm{d}t\,|\right\}\\
&=\frac{a_0+a_1(u)u+a_2u^2}{u^2}\,|_{u=t^\lambda}\,\frac{\mathrm{d}t^\lambda}{\mathrm{d}t}+(\lambda-1)t^{-1}\\
&=\{a_0t^{-2\lambda}+a_1(t^\lambda)t^{-\lambda}+a_2\}\lambda t^{\lambda-1}+(\lambda-1)t^{-1}\\
&=a_0\lambda t^{-\lambda-1}+\{a_1(t^\lambda)\lambda+\lambda-1\}t^{-1}+a_2\lambda t^{\lambda-1}\\
&=\left[a_0+\{a_1(t^\lambda)+1-\lambda^{-1}\}t^\lambda+a_2t^{2\lambda}\right]\lambda t^{-\lambda-1},
\end{aligned}$$

where the following is used

$$\begin{aligned}
\frac{\mathrm{d}\ln(u^{v-1}+u^{-v-1})}{\mathrm{d}t}\,|_{u=t^\lambda}&=\frac{(v-1)u^{v-2}-(v+1)u^{-v-2}}{u^{v-1}+u^{-v-1}}\frac{\mathrm{d}t^\lambda}{\mathrm{d}t}\\
&=\frac{(v-1)u^{v-1}-(v+1)u^{-v-1}}{u^{v-1}+u^{-v-1}}\frac{1}{u}\frac{\mathrm{d}t^\lambda}{\mathrm{d}t}\equiv a_1(u)\frac{1}{u}\frac{\mathrm{d}t^\lambda}{\mathrm{d}t}
\end{aligned}$$

For Y, noting that $f_Y(t\,|\,v,z,\gamma)=K_v(z)^{-1}\exp\left\{-\frac{z}{2}\left(u+\frac{1}{u}\right)\right\}\frac{u^v+u^{-v}}{2}\,|_{u=\exp(\lambda t)}\,|\,\mathrm{d}\lambda t/\mathrm{d}t\,|$ $(0<t<\infty)$, we have in a similar manner

$$\begin{aligned}
\frac{\mathrm{d}\ln f(t)}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\{-a_0u^{-1}+a_1(u)\ln u+a_2u)/\}\,|_{u=\exp(t\lambda)}+\ln|\,\mathrm{d}t\lambda/\mathrm{d}t\,|\right]\\
&=\frac{a_0+a_1(u)u+a_2u^2}{u^2}u\lambda\,|_{u=\exp(t\lambda)}\\
&=\left[a_0\exp(-2t\lambda)+a_1\{\exp(t\lambda)\}\exp(-t\lambda)+a_2\right]\lambda\exp(t\lambda)\\
&=\left[a_0+a_1\{\exp(t\lambda)\}\exp(t\lambda)+a_2\exp(2t\lambda)\right]\lambda\exp(-t\lambda),
\end{aligned}$$

where

$$\frac{\mathrm{d}\ln(u^v+u^{-v})}{\mathrm{d}t}\,|_{u=\exp(\lambda t)}=\frac{v(u^v-u^{-v})}{u^v+u^{-v}}\frac{1}{u}u\frac{\mathrm{d}t\lambda}{\mathrm{d}t}=\frac{v(u^v-u^{-v})}{u^v+u^{-v}}\lambda\equiv a_1\{u\}\lambda$$

is used. The above expressions give the required results.

Proof of Theorem 3.

Lemma 5 provides the conditions for the modes of X^* and Y to satisfy:

$$a_0 + \{a_1(t^\lambda) + 1 - \lambda^{-1}\}t^\lambda + a_2 t^{2\lambda} = 0 \text{ and}$$

$$a_0 + a_1\{\exp(t\lambda)\}\exp(t\lambda) + a_2 \exp(2t\lambda) = 0, \text{ respectively.}$$

When $a_1(t^\lambda)$ and $a_1\{\exp(t\lambda)\}$ are seen as fixed constants using the current values in iterative computation, the equations become usual quadratic ones with respect to $t^\lambda > 0$ and $\exp(t\lambda) > 0$. Since the products $a_0 / a_2 = -1$ of the two solutions are both negative, the positive solution is employed. In iterations, when converged, the solutions satisfy the conditions of the single positive modes.

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