The product distribution of more-than-two correlated normal variables: A didactic review with some new findings (2nd version)

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Abstract: Some reviews of the distribution of the product of two correlated normal random variables and the corresponding more-than-two uncorrelated cases are given with a historical perspective. The characteristic function of the product distribution of more-than-two correlated normal variables is presented using the Cholesky decomposition and an integral representation. The probability density function (pdf) of the product of these correlated variables is shown with an improper multiple integral and discussed. The pdf of the product distribution of three normal correlated variables with zero means is obtained as a mixture of the modified Bessel function of the second kind.

Keywords: tetrad equations, Cholesky decomposition, characteristic function (cf), inversion theorem, Mellin transform, modified Bessel function, Bessel distribution.

1. Introduction

The distribution of the product of two normally distributed variables with zero means has a long history. Pearson, Jeffery and Elderton (1929, p. 187) and Wishart and Bartlett (1932, Equation (12)) gave the probability density function (pdf) in the correlated bivariate case using the Bessel function of the second kind with an imaginary argument (see McKay,1932; Watson, 1944/1995; Zwillinger, 2015, Sections 6.5 and 8.4; DLMF, 2023, https://dlmf.nist.gov/10.2.E4; for an introduction of the Bessel function see the appendix). For the pdf of the correlated case, Craig (1936, Equation (17)) derived a series expression. It is of interest to find that these earlier works were partially motivated by associated problems in the behavioral sciences e.g., "tetrad equations" for the factor analysis model in psychology (Pearson & Moul, 1927; Pearson et al., 1929, pp. 192-193). Craig (1936, p. 1) stated that he had inquiries "one from an investigator in business statistics and the other from a psychologist, concerning the probable error of the product of two quantities, each of known probable error". Currently, many applications of the product distribution are found in the natural sciences e.g., physics and engineering (see Cui, Yu, Iommelli and Kong, 2016; Gaunt, 2022; and the references therein).

The early findings seem to be unnoted by e.g., Gaunt (2019, Section 1) stating that "The exact distribution of the product $Z = XY$ has been studied since 1936", where X and Y are correlated normal variables and 1936 is Craig (1936) (see also Gaunt, 2022). This parallels Nadarajah and Pogány's (2016, Abstract) statement "We solve a problem that has remained unsolved since 1936". Later, Fischer, Gaunt and Sarantsev (2023) noted Pearson et al. (1929) as a work of the exact distribution of a sample covariance under bivariate normality in the review of the variance-gamma distribution. Seijas-Macías, Oliveira and Oliveira (2023, Section 1) referred to Wishart and Bartlett (1932) as the "First historical approach".

For general p-variate uncorrelated Gaussian cases, the pdf of the product was given by Springer and Thompson (1966, Section 6) using the Mellin transform and its inversion (for the transform see Epstein, 1948; Davies, 2002, Chapter 12; DLMF, 2023, https://dlmf.nist.gov/2.5), who stated that "it seems rather surprising that the distribution of products of more than two independent random Gaussian variables has never been derived" (p. 519) though their result was not given in closed form. Springer and Thompson (1970, Theorems 5 and 6) gave the same result using the Meijer *G*-function (DLMF, 2023, https://dlmf.nist.gov/16.17) with a recursion formula (see also, Springer, 1979, Chapter 3).

Recent researches on the correlated bivariate Gaussian case have been given by Ware and Lad (2003), Nadarajah and Pogány (2016), Cui et al. (2016), Oliveira, Oliveira and Seijas-Macías (2016), Seijas-Macías, Oliveira, Oliveira and Leiva (2020), Gaunt (2022) and Seijas-Macías et al. (2023). However, for general p-variate correlated cases, researches after Springer and Thompson (1970) do not seem to well develop as Gaunt (2022, p. 454) stated that "An exact formula for the PDF of the product of three or more correlated normal random variables is not available in the literature".

One of the purposes of this paper is to derive the characteristic function (cf) of the product distribution for correlated more-than-two normal variables with an integral expression, where the Cholesky decomposition is used for the unconstrained covariance matrix. Another purpose of this paper is to obtain a representation of the pdf of the product using improper multiple integration with discussions about the results. Then, a new result for the pdf of the correlated tri-variate case is obtained using a mixture of the Bessel function of the second kind.

2. The cf of the product of correlated normal variables

Let the random p-dimensional vector $\mathbf{Y} = (Y_1,...,Y_p)^{\mathrm{T}}$ be multivariate normally distributed with zero means, which is denoted by $\mathbf{Y} \sim \mathrm{N}_p(\mathbf{0}, \mathbf{\Sigma})$ with $\mathrm{E}(\mathbf{Y}) = \mathbf{0}$ and $\mathrm{cov}(\mathbf{Y}) = \mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is assumed to be non-singular. Consider the random quantity $Y_{i*j} \equiv \prod_{k=i}^{j} Y_k \ (1 \le i \le j \le p)$. Let $\mathrm{i} = \sqrt{-1}$.

Result 1. We obtain an integral expression for the cf of $Z \equiv Y_{1*p}$. By definition, the cf is given by

$$\varphi_Z(t) = \mathrm{E}\{\exp(\mathrm{i}tY_{1*p})\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(\mathrm{i}ty_{1*p})\, \phi_p(\mathbf{y} \mid \mathbf{0}, \mathbf{\Sigma}) \mathrm{d}y_1 \cdots \mathrm{d}y_p$$
$$\equiv \int_{-\infty}^{\infty} \exp(\mathrm{i}ty_{1*p})\, \phi_p(\mathbf{y} \mid \mathbf{0}, \mathbf{\Sigma}) \mathrm{d}\mathbf{y},$$

where y_{1*p} is defined similarly to Y_{1*p} and $\phi_p(\mathbf{y} \mid \mathbf{0}, \mathbf{\Sigma})$ $= (2\pi)^{-p/2} \mid \mathbf{\Sigma} \mid^{-1/2} \exp(-\mathbf{y}^{\mathrm{T}} \mathbf{\Sigma}^{-1} \mathbf{y} / 2)$ is the pdf of $\mathbf{Y} = \mathbf{y}$. Let $\mathbf{Y} = \mathbf{\Sigma}^{1/2} \mathbf{X}$, where

$\mathbf{\Sigma} = \mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{\mathrm{T}/2}$ is the Cholesky decomposition with $\mathbf{\Sigma}^{\mathrm{T}/2} \equiv (\mathbf{\Sigma}^{1/2})^{\mathrm{T}}$; and $\mathbf{\Sigma}^{1/2} = \{\sigma^*_{ij}\}\ (i, j = 1, ..., p)$ is a lower triangular matrix whose diagonal elements are defined to be positive for identification of the decomposition. Then, it can be shown that $\mathbf{X} = (\mathbf{\Sigma}^{1/2})^{-1}\mathbf{Y} \equiv \mathbf{\Sigma}^{-1/2}\mathbf{Y} \sim \mathrm{N}_p(0, \mathbf{I}_p)$ with $\mathbf{I}_p$ being the $p \times p$ identity matrix. Using $\mathbf{X}$, we have

$$Y_{1*p} = \prod\nolimits_{i=1}^{p} (\sigma^*_{i1}, ..., \sigma^*_{ii})(X_1, ..., X_i)^{\mathrm{T}} \equiv \prod\nolimits_{i=1}^{p} \boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{X}_{(i)}$$

with $(\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}, 0, ..., 0)$ being the *i*-th row of $\mathbf{\Sigma}^{1/2}$. Define $\boldsymbol{\sigma}^{1/2}_{(i)j}(i = 1, ..., p; j = 1, ..., i)$ as the subvector of $\boldsymbol{\sigma}^{1/2}_{(i)}$ consisting of the first *j* elements of $\boldsymbol{\sigma}^{1/2}_{(i)}$. Then, we have

$$\begin{aligned}
&\varphi_Z(t) = \varphi_{Y_{1*p}}(t) \\
&= \mathrm{E}\left\{\exp\left(\mathrm{i}t\prod\nolimits_{i=1}^{p}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{X}_{(i)}\right)\right\} = \int_{-\infty}^{\infty}\exp\left(\mathrm{i}t\prod\nolimits_{i=1}^{p}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\phi_p(\mathbf{x}\,|\,\mathbf{0}, \mathbf{I}_p)\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-p/2}\exp\left\{\mathrm{i}t\left(\prod\nolimits_{i=1}^{p}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right) - \frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}}{2}\right\}\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-(p-1)/2}\exp\left\{\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-1}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p)p-1}\mathbf{x}_{(p-1)} - \frac{\mathbf{x}^{\mathrm{T}}_{(p-1)}\mathbf{x}_{(p-1)}}{2}\right\} \\
&\quad\times\int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-1}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\sigma^*_{pp}x_p - \frac{x_p^2}{2}\right\}\mathrm{d}x_p\mathrm{d}\mathbf{x}_{(p-1)} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-(p-1)/2}\exp\left[\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-1}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p)p-1}\mathbf{x}_{(p-1)} - \frac{\mathbf{x}^{\mathrm{T}}_{(p-1)}\mathbf{x}_{(p-1)}}{2}\right. \\
&\qquad\qquad\left. + \left\{\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-1}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\sigma^*_{pp}\right\}^2(1/2)\right]\mathrm{d}\mathbf{x}_{(p-1)} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-(p-1)/2}\exp\left[\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-2}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p-1)p-2}\mathbf{x}_{(p-2)}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p)p-2}\mathbf{x}_{(p-2)} - \frac{\mathbf{x}^{\mathrm{T}}_{(p-2)}\mathbf{x}_{(p-2)}}{2}\right. \\
&\quad + \mathrm{i}t\left(\prod\nolimits_{i=1}^{p-2}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\sigma^*_{p-1,p-1}x_{p-1}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p)p-2}\mathbf{x}_{(p-2)} + \mathrm{i}t\left(\prod\nolimits_{i=1}^{p-2}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p-1)p-2}\mathbf{x}_{(p-2)}\sigma^*_{p,p-1}x_{p-1} \\
&\quad + \mathrm{i}t\left(\prod\nolimits_{i=1}^{p-2}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\sigma^*_{p-1,p-1}\sigma^*_{p,p-1}x^2_{p-1} - \frac{x^2_{p-1}}{2} \\
&\quad\left. + \left\{\mathrm{i}t\left(\prod\nolimits_{i=1}^{p-2}\boldsymbol{\sigma}^{\mathrm{T}/2}_{(i)}\mathbf{x}_{(i)}\right)\left(\boldsymbol{\sigma}^{\mathrm{T}/2}_{(p-1)p-2}\mathbf{x}_{(p-2)} + \sigma^*_{p-1,p-1}x_{p-1}\right)\sigma^*_{pp}\right\}^2(1/2)\right]\mathrm{d}x_{p-1}\mathrm{d}\mathbf{x}_{(p-2)},
\end{aligned}$$

where the following expansion of the first term in the exponential on the left-hand side of the last equation is used:

$$\mathrm{i}t\left(\prod_{i=1}^{p-1}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p)p-1}^{\mathrm{T}/2}\mathbf{x}_{(p-1)}$$

$$=\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\left(\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}+\sigma_{p-1,p-1}^{*}x_{p-1}\right)\left(\boldsymbol{\sigma}_{(p)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}+\sigma_{p,p-1}^{*}x_{p-1}\right).$$

Then, again using the method of completing the square we have

$$\varphi_Z(t)$$

$$=\int_{-\infty}^{\infty}(2\pi)^{-(p-2)/2}\exp\left[\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\boldsymbol{\sigma}_{(p)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}-\frac{\mathbf{x}_{(p-2)}^{\mathrm{T}}\mathbf{x}_{(p-2)}}{2}\right.$$
$$\left.+\frac{1}{2}\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{pp}^{*}\right\}^{2}\right]$$

$$\times\int_{-\infty}^{\infty}(2\pi)^{-1/2}$$
$$\times\exp\left(-\frac{1}{2}\left[1-2\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{p,p-1}^{*}-\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{pp}^{*}\right\}^{2}\right]x_{p-1}^{2}\right.$$
$$+\left[\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\boldsymbol{\sigma}_{(p)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}+\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{p,p-1}^{*}\right.$$
$$\left.\left.+\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{pp}^{*}\right\}^{2}\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{p-1,p-1}^{*}\right]x_{p-1}\right)\mathrm{d}x_{p-1}\mathrm{d}\mathbf{x}_{(p-2)}$$

$$=\int_{-\infty}^{\infty}(2\pi)^{-(p-2)/2}\exp\left[\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\boldsymbol{\sigma}_{(p)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}-\frac{\mathbf{x}_{(p-2)}^{\mathrm{T}}\mathbf{x}_{(p-2)}}{2}\right.$$
$$\left.+\frac{1}{2}\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{pp}^{*}\right\}^{2}\right]$$
$$\times\left[1-2\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{p,p-1}^{*}-\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{pp}^{*}\right\}^{2}\right]^{-1/2}$$
$$\times\exp\left(\frac{1}{2}\left[1-2\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{p,p-1}^{*}-\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\sigma_{pp}^{*}\right\}^{2}\right]^{-1}\right.$$
$$\times\left[\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{p-1,p-1}^{*}\boldsymbol{\sigma}_{(p)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}+\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{p,p-1}^{*}\right.$$
$$\left.\left.+\left\{\mathrm{i}t\left(\prod_{i=1}^{p-2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\sigma_{pp}^{*}\right\}^{2}\boldsymbol{\sigma}_{(p-1)p-2}^{\mathrm{T}/2}\mathbf{x}_{(p-2)}\sigma_{p-1,p-1}^{*}\right]^{2}\right)\mathrm{d}\mathbf{x}_{(p-2)}.$$

Note that Result 1 gives the reduction of the initial dimensionality of multiple integral by 2. Consequently, when $p = 2$, the last result yields the known closed-form formula of the cf without integration as will be repeated in Example 1.

Example 1 (the cf). For confirmation, the known cf of the case with p = 2 and $\mathbf{Y} \sim \mathrm{N}_2(\mathbf{0}, \boldsymbol{\Sigma})$ (e.g., Ogasawara, 2023b, Remark S.2 when $\sigma_{11} = \sigma_{22} = 1$) for the cf is shown using Result 1:

$$\begin{aligned}
\varphi_Z(t) &= \mathrm{E}\left\{\exp\left(\mathrm{i}t\prod\nolimits_{i=1}^{2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{X}_{(i)}\right)\right\} = \int_{-\infty}^{\infty}\exp\left(\mathrm{i}t\prod\nolimits_{i=1}^{2}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\phi_2(\mathbf{x}\,|\,\mathbf{0},\mathbf{I}_2)\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-1}\exp\left\{\mathrm{i}t\sigma_{11}^{*}x_1(\sigma_{21}^{*}x_1 + \sigma_{22}^{*}x_2) - \frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}}{2}\right\}\mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}x_1^2 - (x_1^2/2)\right\} \\
&\quad \times\int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*}x_1x_2 - (x_2^2/2)\right\}\mathrm{d}x_2\mathrm{d}x_1 \\
&= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}x_1^2 - (x_1^2/2) + (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*}x_1)^2(1/2)\right\}\mathrm{d}x_1 \\
&= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left[\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (1/2) + (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2(1/2)\right\}x_1^2\right]\mathrm{d}x_1.
\end{aligned}$$

Noting that the integrand in the last expression is an even function of x_1 and employing the variable transformation $u = x_1^2$ with $\mathrm{d}x_1/\mathrm{d}u = 1/(2u^{1/2})$ and the gamma function with the complex scale parameter $\left\{(1/2) - \mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2(1/2)\right\}^{-1}$ and $\Gamma(1/2) = \pi^{1/2}$, we obtain

$$\begin{aligned}
\varphi_Z(t) &= 2\int_{0}^{\infty}(2\pi)^{-1/2}(2u^{1/2})^{-1}\exp\left[\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (1/2) + (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2(1/2)\right\}u\right]\mathrm{d}u \\
&= 2^{-1/2}\left\{(1/2) - \mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2(1/2)\right\}^{-1/2} \\
&= \left\{1 - 2\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2\right\}^{-1/2},
\end{aligned}$$

which is equal to Wishart and Bartlett (1932, Equation (9)), when $\sigma_{21} = \sqrt{\sigma_{11}\sigma_{22}}\,\rho$ with $\sigma_{11}^{*} = \sqrt{\sigma_{11}}$, $\sigma_{21}^{*} = \sqrt{\sigma_{22}}\,\rho$ and $\sigma_{22}^{*} = \sqrt{\sigma_{22}(1-\rho^2)}$, and is algebraically equal to $(1-\rho^2)^{1/2}/[1 - \{\rho + (1-\rho^2)\mathrm{i}t\}^2]^{1/2}$ when $\sigma_{11} = \sigma_{22} = 1$ (Ogasawara, 2023b, Remark 2).

Example 2 (**the cf**; Craig, 1936, Equation (10); Ware & Lad, 2003, Subsection (3.3.2),

Appendix A; Oliveira, Oliveira & Seijas-Macías, 2016, Proposition 2.1; Seijas-Macías, Oliveira & Oliveira, 2023, Equation (1)). In this example, the case of p = 2 with a relaxed condition of non-zero means i.e., $\mathbf{Y} \sim \mathrm{N}_2(\boldsymbol{\mu}_\mathbf{Y}, \boldsymbol{\Sigma})$ is dealt with. Using the same decomposition $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{\mathrm{T}/2}$ as in Example 1, $\mathbf{X} = \boldsymbol{\Sigma}^{-1/2}\mathbf{Y} \sim \mathrm{N}_p(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}_\mathbf{Y}, \mathbf{I}_p)$ $\equiv \mathrm{N}_p(\boldsymbol{\mu}_\mathbf{X}, \mathbf{I}_p)$. Let $\boldsymbol{\mu}_\mathbf{X} \equiv \boldsymbol{\mu} = (\mu_1, ..., \mu_p)^\mathrm{T}$ for simplicity of notation. Then, the cf is

$$\varphi_Z(t) = \int_{-\infty}^{\infty} \exp\left(\mathrm{i}t\prod\nolimits_{i=1}^{2} \boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\phi_2(\mathbf{x} \mid \boldsymbol{\mu}, \mathbf{I}_2)\mathrm{d}\mathbf{x}$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-1} \exp\left\{\mathrm{i}t\sigma_{11}^* x_1(\sigma_{21}^* x_1 + \sigma_{22}^* x_2) - \frac{(\mathbf{x}-\boldsymbol{\mu})^\mathrm{T}(\mathbf{x}-\boldsymbol{\mu})}{2}\right\}\mathrm{d}\mathbf{x}$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left[\mathrm{i}t\sigma_{11}^*\sigma_{21}^* x_1^2 - \left\{(x_1-\mu_1)^2/2\right\}\right]$$

$$\times \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left[\mathrm{i}t\sigma_{11}^*\sigma_{22}^* x_1 x_2 - \left\{(x_2-\mu_2)^2/2\right\}\right]\mathrm{d}x_2\mathrm{d}x_1$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left[\mathrm{i}t\sigma_{11}^*\sigma_{21}^* x_1^2 - \left\{(x_1-\mu_1)^2/2\right\} + \mathrm{i}t\sigma_{11}^*\sigma_{22}^* x_1\mu_2 + (\mathrm{i}t\sigma_{11}^*\sigma_{22}^* x_1)^2(1/2)\right]\mathrm{d}x_1$$

$$= \int_{-\infty}^{\infty} (2\pi)^{-1/2} \exp\left[-\left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}(x_1^2/2)\right.$$
$$\left. + (\mu_1 + \mathrm{i}t\sigma_{11}^*\sigma_{22}^*\mu_2)x_1 - (\mu_1^2/2)\right]\mathrm{d}x_1$$

$$= \left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}^{-1/2} \exp\left[-\frac{\mu_1^2}{2} + \frac{(\mu_1 + \mathrm{i}t\sigma_{11}^*\sigma_{22}^*\mu_2)^2}{2\left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}}\right]$$

$$= \left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}^{-1/2}$$
$$\times \exp\left[\frac{2\mathrm{i}t\sigma_{11}^*(\sigma_{21}^*\mu_1^2 + \sigma_{22}^*\mu_1\mu_2) + (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2(\mu_1^2 + \mu_2^2)}{2\left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}}\right].$$

Noting that

$$\boldsymbol{\mu}_\mathbf{X} = (\mu_1, \mu_2)^\mathrm{T} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}_\mathbf{Y} = \begin{pmatrix} \sigma_{11}^* & 0 \\ \sigma_{21}^* & \sigma_{22}^* \end{pmatrix}^{-1} \boldsymbol{\mu}_\mathbf{Y} = \begin{pmatrix} (\sigma_{11}^*)^{-1} & 0 \\ -\sigma_{21}^*(\sigma_{11}^*\sigma_{22}^*)^{-1} & (\sigma_{22}^*)^{-1} \end{pmatrix}\boldsymbol{\mu}_\mathbf{Y}$$

$$= \begin{pmatrix} \sigma_{11}^{-1/2} & 0 \\ -\sigma_{11}^{-1/2}\rho(1-\rho^2)^{-1/2} & \sigma_{22}^{-1/2}(1-\rho^2)^{-1/2} \end{pmatrix}\begin{pmatrix} \mu_{Y1} \\ \mu_{Y2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{11}^{-1/2}\mu_{Y1} \\ (-\sigma_{11}^{-1/2}\rho\mu_{Y1} + \sigma_{22}^{-1/2}\mu_{Y2})(1-\rho^2)^{-1/2} \end{pmatrix},$$

we also have

$$\varphi_Z(t) = \left\{1 - 2\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2\right\}^{-1/2}$$

$$\times \exp\left[\frac{2\mathrm{i}t\sigma_{11}^{*}(\sigma_{21}^{*}\mu_1^2 + \sigma_{22}^{*}\mu_1\mu_2) + (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2(\mu_1^2 + \mu_2^2)}{2\left\{1 - 2\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*} - (\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*})^2\right\}}\right]$$

$$= \left\{1 - 2\mathrm{i}t\sigma_{11}^{1/2}\sigma_{22}^{1/2}\rho - (\mathrm{i}t)^2\sigma_{11}\sigma_{22}(1-\rho^2)\right\}^{-1/2}$$

$$\times \exp\begin{pmatrix} \Big[2\mathrm{i}t\sigma_{11}^{1/2}\left\{\sigma_{22}^{1/2}\rho(\sigma_{11}^{-1/2}\mu_{Y1})^2\right. \\ \left. +\sigma_{22}^{1/2}(1-\rho^2)^{1/2}\sigma_{11}^{-1/2}\mu_{Y1}(-\sigma_{11}^{-1/2}\rho\mu_{Y1} + \sigma_{22}^{-1/2}\mu_{Y2})(1-\rho^2)^{-1/2}\right\} \\ +(\mathrm{i}t)^2\sigma_{11}\sigma_{22}(1-\rho^2)\left\{(\sigma_{11}^{-1/2}\mu_{Y1})^2 + (-\sigma_{11}^{-1/2}\rho\mu_{Y1} + \sigma_{22}^{-1/2}\mu_{Y2})^2(1-\rho^2)^{-1}\right\}\Big] \\ \times\left[2\left\{1 - 2\mathrm{i}t\sigma_{11}^{1/2}\sigma_{22}^{1/2}\rho - (\mathrm{i}t)^2\sigma_{11}\sigma_{22}(1-\rho^2)\right\}\right]^{-1} \end{pmatrix}$$

$$= \left\{1 - 2\mathrm{i}t\sigma_{11}^{1/2}\sigma_{22}^{1/2}\rho - (\mathrm{i}t)^2\sigma_{11}\sigma_{22}(1-\rho^2)\right\}^{-1/2}$$

$$\times \exp\left[\frac{2\mathrm{i}t\mu_{Y1}\mu_{Y2} + (\mathrm{i}t)^2\sigma_{11}\sigma_{22}\left\{(\sigma_{11}^{-1/2}\mu_{Y1})^2 + (\sigma_{22}^{-1/2}\mu_{Y2})^2 - 2\rho\sigma_{11}^{-1/2}\mu_{Y1}\sigma_{22}^{-1/2}\mu_{Y2}\right\}}{2\left\{1 - 2\mathrm{i}t\sigma_{11}^{1/2}\sigma_{22}^{1/2}\rho - (\mathrm{i}t)^2\sigma_{11}\sigma_{22}(1-\rho^2)\right\}}\right],$$

which is found to be algebraically equal to the results cited at the beginning of this example for the moment generating function (mgf) when it is replaced by t supporting their result.

Note that they used the formulation $Y_1 = X_0 + X_1$ and $Y_2 = X_0 + X_2$, where $X_0 \sim \mathrm{N}_1(0,\rho) \equiv \mathrm{N}(0,\rho)$, $X_1 \sim \mathrm{N}(\mu_1, 1-\rho)$ and $X_2 \sim \mathrm{N}(\mu_2, 1-\rho)$ $(0 < \rho < 1)$ are uncorrelated normal variables. Note also that there are three latent variables for two manifest variables Y_1 and Y_2. It is found that the formulation is a saturated one-factor model with positive ρ in exploratory factor analysis (EFA). The inflated dimensionality over that of manifest variables is known to be a property of latent variable models (see e.g., Ogasawara, 2022a, p. 252; Ogasawara, 2023c, Section 1). Although they did not mention, when Y_1 and Y_2 have negative correlation, $Y_2 = -X_0 + X_2$ (or $Y_1 = -X_0 + X_1$) with unchanged Y_1 (or Y_2, respectively) should be used. On the other hand, in our formulation using the Cholesky decomposition, the dimensionality of $\mathbf{X}$ is the same as that of $\mathbf{Y}$. The EFA model has also been used in the definitions of the multivariate gamma (Cherian, 1941; Prékopa and Szántai, 1978; Mathai and Moschopoulos, 1991, Definition 1; Royen, 1991, 2007) and multivariate power-gamma (Ogasawara, 2023c) distributions.

Example 3 (the cf). Consider the case of p = 3 when $\mathbf{Y} \sim \mathrm{N}_3(\mathbf{0}, \mathbf{\Sigma})$. The cf becomes

$$\varphi_Z(t) = \mathrm{E}\left\{\exp\left(\mathrm{i}t\prod_{i=1}^{3}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{X}_{(i)}\right)\right\} = \int_{-\infty}^{\infty}\exp\left(\mathrm{i}t\prod_{i=1}^{3}\boldsymbol{\sigma}_{(i)}^{\mathrm{T}/2}\mathbf{x}_{(i)}\right)\phi_3(\mathbf{x}\,|\,\mathbf{0},\mathbf{I}_3)\mathrm{d}\mathbf{x}$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-3/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}x_1(\sigma_{21}^{*}x_1+\sigma_{22}^{*}x_2)(\sigma_{31}^{*}x_1+\sigma_{32}^{*}x_2+\sigma_{33}^{*}x_3)-\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}}{2}\right\}\mathrm{d}\mathbf{x}$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-1}\exp\left[\mathrm{i}t\sigma_{11}^{*}\left\{\sigma_{21}^{*}\sigma_{31}^{*}x_1^3+(\sigma_{21}^{*}\sigma_{32}^{*}+\sigma_{22}^{*}\sigma_{31}^{*})x_1^2x_2+\sigma_{22}^{*}\sigma_{32}^{*}x_1x_2^2\right\}-(\mathbf{x}_{(2)}^{\mathrm{T}}\mathbf{x}_{(2)}/2)\right]$$
$$\times\int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}(\sigma_{21}^{*}\sigma_{33}^{*}x_1^2+\sigma_{22}^{*}\sigma_{33}^{*}x_1x_2)x_3-(x_3^2/2)\right\}\mathrm{d}x_3\mathrm{d}\mathbf{x}_{(2)}$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-1}\exp\left[\mathrm{i}t\sigma_{11}^{*}\left\{\sigma_{21}^{*}\sigma_{31}^{*}x_1^3+(\sigma_{21}^{*}\sigma_{32}^{*}+\sigma_{22}^{*}\sigma_{31}^{*})x_1^2x_2+\sigma_{22}^{*}\sigma_{32}^{*}x_1x_2^2\right\}-(\mathbf{x}_{(2)}^{\mathrm{T}}\mathbf{x}_{(2)}/2)\right]$$
$$\times\exp\left[\left\{\mathrm{i}t\sigma_{11}^{*}(\sigma_{21}^{*}\sigma_{33}^{*}x_1^2+\sigma_{22}^{*}\sigma_{33}^{*}x_1x_2)\right\}^2/2\right]\mathrm{d}\mathbf{x}_{(2)}$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{31}^{*}x_1^3+\left(\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{33}^{*}x_1^2\right)^2(1/2)-(x_1^2/2)\right\}$$
$$\times\int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left[\mathrm{i}t\sigma_{11}^{*}\left\{(\sigma_{21}^{*}\sigma_{32}^{*}+\sigma_{22}^{*}\sigma_{31}^{*})x_1^2x_2+\sigma_{22}^{*}\sigma_{32}^{*}x_1x_2^2\right\}\right.$$
$$\left.+(\mathrm{i}t\sigma_{11}^{*})^2\left\{\sigma_{21}^{*}\sigma_{22}^{*}(\sigma_{33}^{*})^2x_1^3x_2+(\sigma_{22}^{*}\sigma_{33}^{*})^2x_1^2x_2^2(1/2)-(x_2^2/2)\right\}\right]\mathrm{d}x_2\mathrm{d}x_1$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left\{\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{31}^{*}x_1^3+\left(\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{33}^{*}x_1^2\right)^2(1/2)-(x_1^2/2)\right\}$$
$$\times\int_{-\infty}^{\infty}(2\pi)^{-1/2}\exp\left[-(1/2)\left\{1-2\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*}\sigma_{32}^{*}x_1-(\mathrm{i}t\sigma_{11}^{*})^2(\sigma_{22}^{*}\sigma_{33}^{*})^2x_1^2\right\}x_2^2\right.$$
$$\left.+\left\{\mathrm{i}t\sigma_{11}^{*}(\sigma_{21}^{*}\sigma_{32}^{*}+\sigma_{22}^{*}\sigma_{31}^{*})x_1^2+(\mathrm{i}t\sigma_{11}^{*})^2\sigma_{21}^{*}\sigma_{22}^{*}(\sigma_{33}^{*})^2x_1^3\right\}x_2\right]\mathrm{d}x_2\mathrm{d}x_1$$

$$= \int_{-\infty}^{\infty}(2\pi)^{-1/2}\left\{1-2\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*}\sigma_{32}^{*}x_1-(\mathrm{i}t\sigma_{11}^{*})^2(\sigma_{22}^{*}\sigma_{33}^{*})^2x_1^2\right\}^{-1/2}$$
$$\times\exp\left[-\frac{x_1^2}{2}+\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{31}^{*}x_1^3+\frac{\left(\mathrm{i}t\sigma_{11}^{*}\sigma_{21}^{*}\sigma_{33}^{*}x_1^2\right)^2}{2}\right.$$
$$\left.+\frac{\left\{\mathrm{i}t\sigma_{11}^{*}(\sigma_{21}^{*}\sigma_{32}^{*}+\sigma_{22}^{*}\sigma_{31}^{*})x_1^2+(\mathrm{i}t\sigma_{11}^{*})^2\sigma_{21}^{*}\sigma_{22}^{*}(\sigma_{33}^{*})^2x_1^3\right\}^2}{2\left\{1-2\mathrm{i}t\sigma_{11}^{*}\sigma_{22}^{*}\sigma_{32}^{*}x_1-(\mathrm{i}t\sigma_{11}^{*})^2(\sigma_{22}^{*}\sigma_{33}^{*})^2x_1^2\right\}}\right]\mathrm{d}x_1.$$

Example 4 (the cf). In this example, we deal with a tri-variate model situated between the independent and unconstrained dependent models. Note that Example 3 is a fully dependent model in that non-singular $\mathbf{\Sigma}$ is unconstrained. Assume that in Example 3, Y_1 is independent of possibly correlated Y_2 and Y_3. Without loss of generality, suppose that

$\sigma_{11} = \sigma_{22} = \sigma_{33} = 1$. Then, under this partially dependent model $\mathbf{\Sigma}$ becomes the correlation matrix:

$$\mathbf{\Sigma} = \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho_{23} \\ 0 & \rho_{32} & 1 \end{pmatrix} \text{ with } \mathbf{\Sigma}^{1/2} = \begin{pmatrix} \sigma_{11}^* & 0 & 0 \\ \sigma_{21}^* & \sigma_{22}^* & 0 \\ \sigma_{31}^* & \sigma_{32}^* & \sigma_{33}^* \end{pmatrix} \equiv \mathbf{P}^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \rho_{32} & (1-\rho_{32}^2)^{1/2} \end{pmatrix}.$$

Noting that this example is a special case of Example 3, using the above expressions, we have

$$\begin{aligned} \varphi_Z(t) &= \mathrm{E}\left\{\exp\left(\mathrm{i}tY_{1*3}\right) \mid \mathbf{\Sigma} = \mathbf{P}\right\} \\ &= \int_{-\infty}^{\infty} (2\pi)^{-1/2} \left\{1 - 2\mathrm{i}t\rho_{32}x_1 - (\mathrm{i}t)^2(1-\rho_{32}^2)x_1^2\right\}^{-1/2} \exp(-x_1^2/2)\mathrm{d}x_1, \end{aligned}$$

which is seen as a mixture of the cf of Example 1 using the normal density.

3. The pdf of the product of correlated normal variables

In this section, the pdf of the product of correlated normal variables is considered.

Result 2. The pdf of $Z = Y_{1*p}$, as defined in Result 1, at z is derived using the inversion theorem (see e.g., Anderson, 2003, Theorem, 2.6.3), which is formally given as:

$$\begin{aligned} f_Z(z) &= (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-\mathrm{i}tz)\varphi_Z(t)\mathrm{d}t = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-\mathrm{i}tz)\mathrm{E}\{\exp(\mathrm{i}tY_{1*p})\}\mathrm{d}t \\ &= (2\pi)^{-1}\int_{-\infty}^{\infty} \exp\{\mathrm{i}t(y_{1*p} - z)\}\,\phi_p(\mathbf{y} \mid \mathbf{0}, \mathbf{\Sigma})\mathrm{d}\mathbf{y}\,\mathrm{d}t, \end{aligned}$$

which is a (p + 1)-fold integral, and tends to be complicated unless $\varphi_Z(t) = \mathrm{E}\{\exp(\mathrm{i}tY_{1*p})\}$ is explicitly given as a tractable function of t. On the other hand, the same pdf is obtained by using the variable transformation from e.g., Y_p to $Z = Y_{1*p}$ and unchanged $Y_1,\ldots,Y_p$ with the Jacobian $|\mathrm{d}y_p/\mathrm{d}z| = 1/|y_{1*(p-1)}|$, which gives

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |y_{1*(p-1)}|^{-1}\,\phi_p\{(y_1,\ldots,y_{p-1}, z/y_{1*(p-1)})^{\mathrm{T}} \mid \mathbf{0}, \mathbf{\Sigma}\}\mathrm{d}y_1\cdots\mathrm{d}y_{p-1} \\ &\equiv \int_{-\infty}^{\infty} |y_{1*(p-1)}|^{-1}\,\phi_p\{(\mathbf{y}_{(p-1)}^{\mathrm{T}}, z/y_{1*(p-1)})^{\mathrm{T}} \mid \mathbf{0}, \mathbf{\Sigma}\}\mathrm{d}\mathbf{y}_{(p-1)}, \end{aligned}$$

where $\mathbf{y}_{(p-1)} \equiv (y_1,\ldots,y_{p-1})^{\mathrm{T}}$ for simplicity of notation. The above representation is an improper $(p-1)$-fold multiple integral whose dimensionality is smaller than that using the cf by 2. The above result is seen as an extension of the Mellin convolution for the bivariate independent cases (Epstein, 1948, Equation (2); Springer & Thompson, 1966, Equation (4), Section 6; Springer, 1979, Section 4.1) or the bivariate correlated cases (Rohatgi, 1976, Section 4.4, Theorem 7, Equation (16); Rohatgi & Saleh, 2015, Section 4.4, Theorem 3, Equation (4); Cui et al., 2016, Theorem 2.1) to the p-dimensional correlated case.

Another convenient integral representation of the pdf of Y_{1*p} is given by the joint pdf of Y_{1*j} $(j = 1, \ldots, p-1)$:

Lemma 1. *The joint pdf of* $\mathbf{Y}_* = (Y_{1*1}, Y_{1*2}, \ldots, Y_{1*p})^{\mathrm{T}}$ *with* $Y_{1*1} = Y_1$ *at* $\mathbf{y}_* = (y_{1*1}, y_{1*2}, \ldots, y_{1*p})^{\mathrm{T}}$ *when* $\mathbf{Y} = (Y_1, \ldots, Y_p)^{\mathrm{T}} \sim \mathrm{N}_p(\mathbf{0}, \mathbf{\Sigma})$ *is*

$$f_{\mathbf{Y}_*}(\mathbf{y}_*) = \left(\prod\nolimits_{j=1}^{p-1} | y_{1*j} |^{-1}\right) \phi_p \left\{ \left(y_{1*1}, \frac{y_{1*2}}{y_{1*1}}, \ldots, \frac{y_{1*p}}{y_{1*(p-1)}} \right)^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma} \right\}.$$

Proof. Employ the variable transformation $\mathbf{Y} \to \mathbf{Y}_*$, where the Jacobian matrix $\mathrm{d}\mathbf{y}_* / \mathrm{d}\mathbf{y}^{\mathrm{T}}$ with $\mathbf{y} = (y_1, \ldots, y_p)^{\mathrm{T}}$ is a lower-triangular one whose diagonal elements are $1, y_{1*1}, \ldots, y_{1*(p-1)}$. Then, the Jacobian becomes $J(\mathbf{y} \to \mathbf{y}_*) = 1 / | \det(\mathrm{d}\mathbf{y}_* / \mathrm{d}\mathbf{y}^{\mathrm{T}}) |$ $= \prod\nolimits_{j=1}^{p-1} | y_{1*j} |^{-1}$. Using the Jacobian we have the pdf of $\mathbf{Y}_* = \mathbf{y}_*$:

$$\begin{aligned} f_{\mathbf{Y}_*}(\mathbf{y}_*) &= \left(\prod\nolimits_{j=1}^{p-1} | y_{1*j} |^{-1}\right) \phi_p \{ (y_1, y_2, \ldots, y_p)^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma} \} \\ &= \left(\prod\nolimits_{j=1}^{p-1} | y_{1*j} |^{-1}\right) \phi_p \left\{ \left(y_{1*1}, \frac{y_{1*2}}{y_{1*1}}, \ldots, \frac{y_{1*p}}{y_{1*(p-1)}} \right)^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma} \right\}, \end{aligned}$$

which is the required result. Q.E.D.

The integral representation of the pdf of Y_{1*p} using Lemma 1 is given as follows:

Theorem 1. *The pdf of* $Z = Y_{1*p}$ *is*

$$\begin{aligned} f_Z(z) = \int_{-\infty}^{\infty} &\left(\prod\nolimits_{j=1}^{p-1} | y_{1*j} |^{-1}\right) \\ &\times \phi_p \left\{ \left(y_{1*1}, \frac{y_{1*2}}{y_{1*1}}, \ldots, \frac{y_{1*(p-1)}}{y_{1*(p-2)}}, \frac{z}{y_{1*(p-1)}} \right)^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma} \right\} \mathrm{d}\mathbf{y}_{1*(p-1)} \end{aligned}$$

where $\mathrm{d}\mathbf{y}_{1*(p-1)} \equiv \mathrm{d}y_{1*1} \mathrm{d}y_{1*2} \cdots \mathrm{d}y_{1*(p-1)}$.

Proof. The pdf of Y_{1*p} is given by the corresponding marginal distribution in the joint pdf of $\mathbf{Y}_*$ obtained by Lemma 1, yielding the required result. Q.E.D.

The pdf of Y_{1*p} using the modified Bessel function of the second kind of order v denoted by $K_v(\cdot)$ (for this function, see the appendix) is derived. Let $\mathbf{\Sigma}^{-1} = \begin{pmatrix} \mathbf{\Sigma}^{(p-1)} & \boldsymbol{\sigma}^{(p-1)p} \\ \boldsymbol{\sigma}^{(p-1)p\mathrm{T}} & \sigma^{pp} \end{pmatrix} = \{\sigma^{ij}\}\ (i, j = 1,..., p)$, where $\mathbf{\Sigma}^{(p-1)} = \begin{pmatrix} \mathbf{\Sigma}^{(p-2)} & \boldsymbol{\sigma}^{(p-2)p-1} \\ \boldsymbol{\sigma}^{(p-2)p-1\mathrm{T}} & \sigma^{p-1,p-1} \end{pmatrix}$ is the $(p-1)\times(p-1)$ submatrix; $\boldsymbol{\sigma}^{(p-1)p}$ is the $(p-1)\times 1$ vector with $\boldsymbol{\sigma}^{(p-1)p\mathrm{T}} = (\boldsymbol{\sigma}^{(p-1)p})^{\mathrm{T}}$. Define $\tilde{\mathbf{y}}_{1*p} = \left(y_{1*1}, \frac{y_{1*2}}{y_{1*1}}, ..., \frac{y_{1*p}}{y_{1*(p-1)}} \right)^{\mathrm{T}}$ for simplicity of notation. Then we obtain the following result.

Theorem 2. The pdf of $Z = Y_{1*p}$ is

$$f_Z(z) = (2\pi)^{-p/2} |\mathbf{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-2} |y_{1*j}|^{-1} \right) \exp\left(-\frac{1}{2} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \mathbf{\Sigma}^{(p-2)} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$
$$\times 2 \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^k \left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{2j-k}$$
$$\times \left(\sigma^{pp} z^2 y_{1*(p-2)}^2 / \sigma^{p-1,p-1} \right)^{(k-j)/2} K_{k-j} \left\{ (\sigma^{p-1,p-1} \sigma^{pp})^{1/2} |z| / |y_{1*(p-2)}| \right\} \mathrm{d}\mathbf{y}_{1*(p-2)}.$$

Proof. Using Theorem 1, we have

$$f_{Y_{1*p}}(z) = f_Z(z)$$
$$= \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-1} |y_{1*j}|^{-1} \right) \phi_p \left\{ \left(\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}}, \frac{y_{1*(p-1)}}{y_{1*(p-2)}}, \frac{z}{y_{1*(p-1)}} \right)^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma} \right\} \mathrm{d}\mathbf{y}_{1*(p-1)}$$
$$= (2\pi)^{-p/2} |\mathbf{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-2} |y_{1*j}|^{-1} \right) \exp\left(-\frac{1}{2} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \mathbf{\Sigma}^{(p-2)} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$
$$\times \int_{-\infty}^{\infty} |y_{1*(p-1)}|^{-1} \exp\left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1} \frac{y_{1*(p-1)}}{y_{1*(p-2)}} - \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} \frac{z}{y_{1*(p-1)}} \right.$$
$$\left. - \frac{1}{2} \frac{\sigma^{p-1,p-1} y_{1*(p-1)}^2}{y_{1*(p-2)}^2} - \frac{1}{2} \frac{\sigma^{pp} z^2}{y_{1*(p-1)}^2} \right) \mathrm{d}y_{1*(p-1)} \mathrm{d}\mathbf{y}_{1*(p-2)}.$$

As in Cui et al. (2015, Equations (6) and (7)) expand the following factor in the last result:

$$\exp\left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1} \frac{y_{1*(p-1)}}{y_{1*(p-2)}} - \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} \frac{z}{y_{1*(p-1)}} \right)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} y_{1*(p-1)} - \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} \frac{z}{y_{1*(p-1)}} \right)^{j}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{j!} \binom{j}{k} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^{k} \left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{j-k} y_{1*(p-1)}^{2k-j}.$$

Then, we obtain

$$f_Z(z) = (2\pi)^{-p/2} |\boldsymbol{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-2} |y_{1*j}|^{-1} \right) \exp\left(-\frac{1}{2} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\Sigma}^{(p-2)} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$

$$\times \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{1}{j!} \binom{j}{k} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^{k} \left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{j-k} y_{1*(p-1)}^{2k-j} |y_{1*(p-1)}|^{-1}$$

$$\times \exp\left(-\frac{1}{2} \frac{\sigma^{p-1,p-1}}{y_{1*(p-2)}^2} y_{1*(p-1)}^2 - \frac{1}{2} \frac{\sigma^{pp} z^2}{y_{1*(p-1)}^2} \right) \mathrm{d}y_{1*(p-1)} \mathrm{d}\mathbf{y}_{1*(p-2)}.$$

In the term by term integral with respect to $y_{1*(p-1)}$, we find that when j is odd, the integral vanishes, which gives

$$f_Z(z) = (2\pi)^{-p/2} |\boldsymbol{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-2} |y_{1*j}|^{-1} \right) \exp\left(-\frac{1}{2} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\Sigma}^{(p-2)} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$

$$\times 2 \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^{k} \left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{2j-k}$$

$$\times \int_{0}^{\infty} y_{1*(p-1)}^{2k-2j-1} \exp\left(-\frac{1}{2} \frac{\sigma^{p-1,p-1}}{y_{1*(p-2)}^2} y_{1*(p-1)}^2 - \frac{1}{2} \frac{\sigma^{pp} z^2}{y_{1*(p-1)}^2} \right) \mathrm{d}y_{1*(p-1)} \mathrm{d}\mathbf{y}_{1*(p-2)}.$$

It is known that the Mellin transform of $\exp(-\alpha x^{h} - \beta x^{-h})$ is

$$\int_{0}^{\infty} x^{s-1} \exp(-\alpha x^{h} - \beta x^{-h}) \mathrm{d}x = 2h^{-1} (\beta/\alpha)^{s/(2h)} K_{s/h}(2\alpha^{1/2}\beta^{1/2}) \quad (\alpha > 0, \beta > 0, h > 0)$$

(Erdélyi, 1954, Section 6.3, Equation 17, p. 313; Zwillinger, 2015, Section 3.478, Equation 4). Using this formula when $x = y_{1*(p-1)}$, $s = 2k - 2j$, $\alpha = \sigma^{p-1,p-1}/(2y_{1*(p-2)}^2)$, $\beta = \sigma^{pp} z^2/2$ and $h = 2$ for the integral with respect to $y_{1*(p-1)}$, we have

$$f_Z(z) = (2\pi)^{-p/2} |\mathbf{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-2} | y_{1*j} |^{-1} \right) \exp\left(-\frac{1}{2} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \mathbf{\Sigma}^{(p-2)} \tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$

$$\times 2 \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} \left(-\frac{\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^k \left(-\tilde{\mathbf{y}}_{1*(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{2j-k}$$

$$\times \left(\sigma^{pp} z^2 y_{1*(p-2)}^2 / \sigma^{p-1,p-1} \right)^{(k-j)/2} K_{k-j} \left\{ (\sigma^{p-1,p-1} \sigma^{pp})^{1/2} | z | / | y_{1*(p-2)} | \right\} \mathrm{d}\mathbf{y}_{1*(p-2)},$$

which is the required result. Q.E.D.

An alternative expression comparable to that of Theorem 2 is obtained by the following corollary.

Corollary 1. *The pdf of* Y_{1*p} *in Theorems 1 and 2 is also given by*

$$f_Z(z) = (2\pi)^{-p/2} |\mathbf{\Sigma}|^{-1/2} \int_{-\infty}^{\infty} | y_{1*(p-2)} |^{-1} \exp\left(-\frac{1}{2} \mathbf{y}_{(p-2)}^{\mathrm{T}} \mathbf{\Sigma}^{(p-2)} \mathbf{y}_{(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$

$$\times 2 \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} \left(-\frac{\mathbf{y}_{(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^k \left(-\mathbf{y}_{(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{2j-k}$$

$$\times \left(\sigma^{pp} z^2 y_{1*(p-2)}^2 / \sigma^{p-1,p-1} \right)^{(k-j)/2} K_{k-j} \left\{ (\sigma^{p-1,p-1} \sigma^{pp})^{1/2} | z | / | y_{1*(p-2)} | \right\} \mathrm{d}\mathbf{y}_{(p-2)}.$$

Proof. For the result of Theorem 2, use the inverse variable transformation $\mathbf{y}_{1*(p-2)} \to \mathbf{y}_{(p-2)}$ with the Jacobian $\prod_{j=1}^{p-3} | y_{1*j} |$. Q.E.D.

Note that in Corollary 1 the scalar variable $y_{1*(p-2)}$ is unchanged since it is the unchanged product $y_1 \times \cdots \times y_{(p-2)}$, while $\tilde{\mathbf{y}}_{1*(p-2)}$ becomes $\mathbf{y}_{(p-2)}$ due to the definition. Recall that Theorem 2 was obtained by fully using Lemma 1 or Theorem 1 whereas Corollary 1 is given by the partial transformation $\mathbf{y} \to (\mathbf{y}_{(p-2)}^{\mathrm{T}}, y_{1*(p-1)}, y_{1*p})^{\mathrm{T}}$.

Example 1 (the pdf). The known pdf using the modified Bessel function of the second kind of zero order i.e., $K_0(\cdot)$ (Wishart & Bartlett, 1932, Equation (12); Springer, 1979, Equation (4.8.22); Grishchuk, 1996, Equation (49); Simon, 2002, Equation (6.15); Nadarajah & Pogány, 2016, Theorem 2.1) is derived in two ways with or without using the cf when $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{21} = \rho$. The first one uses the cf of $Z = Y_1 Y_2$ as derived earlier:

$$
\begin{aligned}
f_Z(z) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\varphi_Z(t)\mathrm{d}t \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\left\{1-2\mathrm{i}t\rho-(\mathrm{i}t)^2(1-\rho^2)\right\}^{-1/2}\mathrm{d}t \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\{1-\mathrm{i}(1+\rho)t\}^{-1/2}\{1+\mathrm{i}(1-\rho)t\}^{-1/2}\mathrm{d}t \\
&= (2\pi)^{-1}(1-\rho^2)^{-1/2}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\left\{t-\frac{1}{\mathrm{i}(1+\rho)}\right\}^{-1/2}\left\{t+\frac{1}{\mathrm{i}(1-\rho)}\right\}^{-1/2}\mathrm{d}t \\
&= (2\pi)^{-1}(1-\rho^2)^{-1/2}\int_{-\infty}^{\infty}\exp\{\mathrm{i}(-z)t\}\left(t+\frac{\mathrm{i}}{1+\rho}\right)^{-1/2}\left(t-\frac{\mathrm{i}}{1-\rho}\right)^{-1/2}\mathrm{d}t.
\end{aligned}
$$

In the above result, using Nadarajah and Pogány (2016, Lemma 3.1) i.e.,

$$
\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}zt)}{\{(t-\mathrm{i}a)(t+\mathrm{i}b)\}^{1/2}}\mathrm{d}t = 2\exp\left(\frac{b-a}{2}z\right)K_0\left(\frac{a+b}{2}|z|\right)
$$
$$
(0<a<\infty,\ 0<b<\infty,\ -\infty<z<\infty),
$$

whose didactic derivation is given in the appendix, when $b = 1/(1+\rho)$ and $a = 1/(1-\rho)$, we obtain

$$
\begin{aligned}
f_Z(z) &= (2\pi)^{-1}(1-\rho^2)^{-1/2}\int_{-\infty}^{\infty}\exp\{\mathrm{i}(-z)t\}\left(t+\frac{\mathrm{i}}{1+\rho}\right)^{-1/2}\left(t-\frac{\mathrm{i}}{1-\rho}\right)^{-1/2}\mathrm{d}t \\
&= \pi^{-1}(1-\rho^2)^{-1/2}\exp\left(\frac{\rho z}{1-\rho^2}\right)K_0\left(\frac{|z|}{1-\rho^2}\right).
\end{aligned}
$$

Note that the Nadarajah-Pogány lemma is of use in that while the integrand is complex-valued, the result is real-valued as expected since the latter corresponds to the pdf of the product. The method of Wishart and Bartlett (1932, (12)) to have the essentially the same pdf is to employ the variable transformation $z = (\mathrm{i}\rho - t)/(1-\rho^2)$ using our notation and the integration over the line from $\mathrm{i}\rho-\infty$ to $\mathrm{i}\rho+\infty$, which reduces to the integral along the real line from $-\infty$ to ∞, whose explicit justification may be given by the Nadarajah-Pogány lemma.

The second method without using the cf to have the pdf is given by Result 2, which is easily obtained by the variable transformation from Y_2 to $Z = Y_1Y_2$ and unchanged Y_1 with the Jacobian $|y_1|^{-1}$ as used by Grishchuk (1996):

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} |y_1|^{-1}\,\phi_2\{(y_1, z/y_1)^{\mathrm{T}} \mid \mathbf{0}, \mathbf{P}\}\mathrm{d}y_1 \\
&= \int_0^{\infty} \frac{1}{\pi(1-\rho^2)^{1/2}|y_1|} \exp\left\{-\frac{y_1^2 + (z^2/y_1^2) - 2\rho z}{2(1-\rho^2)}\right\}\mathrm{d}y_1 \\
&= \frac{1}{\pi(1-\rho^2)^{1/2}} \exp\left(\frac{\rho z}{1-\rho^2}\right) \int_0^{\infty} \frac{1}{|y_1|} \exp\left\{-\frac{y_1^2 + (z^2/y_1^2)}{2(1-\rho^2)}\right\}\mathrm{d}y_1 \\
&= \frac{1}{\pi(1-\rho^2)^{1/2}} \exp\left(\frac{\rho z}{1-\rho^2}\right) \int_0^{\infty} \frac{1}{|u|} \exp\left\{-\frac{u^2 + \{z^2/(1-\rho^2)^2\}u^{-2}}{2}\right\}\mathrm{d}u \\
&= \frac{1}{\pi(1-\rho^2)^{1/2}} \exp\left(\frac{\rho z}{1-\rho^2}\right) K_0\left(\frac{|z|}{1-\rho^2}\right),
\end{aligned}
$$

where $\mathbf{P} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$; and $K_0(z) = \int_0^{\infty}\left(-\frac{u^2}{2} - \frac{z^2}{2u^2}\right)\frac{1}{u}\mathrm{d}u$ $(|\arg(z)| < \pi/4)$ (see the appendix) is used. The last result is equal to that using the cf derived earlier as expected. Note that the integral representation in the second line of the last set of equations is as simple as that of the second last one, whose expression is an integral representation of $K_0(\cdot)$. Note also that though $K_0(\cdot)$ is a known function, for computation of $K_0(\cdot)$ some numerical methods e.g., numerical integration and series expressions are required. So, in this sense, the initial expression

$$
f_Z(z) = \int_{-\infty}^{\infty} |y_1|^{-1}\,\phi_2\{(y_1, z/y_1)^{\mathrm{T}} \mid \mathbf{0}, \mathbf{P}\}\mathrm{d}y_1
$$

is comparable to the last one. One of the advantages of the expression using $K_0(\cdot)$ is that we can use other representations for $K_0(\cdot)$.

Note that Springer (1979, Equation (4.8.22)) obtained the same pdf as above using the Mellin transform though the result includes an error as noted by Gaunt (2022, p. 452). However, the error seems to be a non-fatal one easily corrected by readers when obtained as above: the factor $\exp\{\rho|z|/(1-\rho^2)\}$ should be $\exp\{\rho z/(1-\rho^2)\}$. Note also that Simon (2002, Equation (6.15)) obtained the pdf with an error in that the factor $1/(1-\rho^2)^{1/2}$ was missing as found by Cui et al. (2016, p. 1664).

Example 2 (the pdf). The pdf was obtained by Cui et al. (2016) using a two-fold infinite series with each term including $K_v(\cdot)$. When $\mathbf{Y} = (Y_1, Y_2)^{\mathrm{T}} \sim \mathrm{N}_2\{(\mu_1, \mu_2)^{\mathrm{T}}, \mathbf{P}\}$, the pdf of $Z = Y_1 Y_2$ using the same variable transformation as in Example 1, we have

$$f_Z(z) = \int_{-\infty}^{\infty} | y_1 |^{-1} \phi_2[\{y_1 - \mu_1, (z/y_1) - \mu_2\}^{\mathrm{T}} | \mathbf{0}, \mathbf{P}] \mathrm{d}y_1$$
$$= \int_{-\infty}^{\infty} \frac{1}{2\pi(1-\rho^2)^{1/2} | y_1 |}$$
$$\times \exp\left[-\frac{(y_1 - \mu_1)^2 + \{(z/y_1) - \mu_2\}^2 - 2\rho(y_1 - \mu_1)\{(z/y_1) - \mu_2\}}{2(1-\rho^2)} \right] \mathrm{d}y_1,$$

whose value may be obtained by numerical integration.

Examples 3 (the pdf). The pdf of $Z = Y_{1*3}$ is given by Theorem 1 or Corollary 1 when $p = 3$ with the same expression due to $y_{1*(p-2)} = y_{1*1} = y_1$:

$$f_Z(z) = \int_{-\infty}^{\infty} | y_{1*(p-1)} |^{-1} \phi_p\{(\mathbf{y}_{(p-1)}^{\mathrm{T}}, z / y_{1*(p-1)})^{\mathrm{T}} | \mathbf{0}, \mathbf{\Sigma}\} \mathrm{d}\mathbf{y}_{(p-1)}$$
$$= (2\pi)^{-p/2} | \mathbf{\Sigma} |^{-1/2} \int_{-\infty}^{\infty} | y_{1*(p-2)} |^{-1} \exp\left(-\frac{1}{2} \mathbf{y}_{(p-2)}^{\mathrm{T}} \mathbf{\Sigma}^{(p-2)} \mathbf{y}_{(p-2)}^{\mathrm{T}} - \frac{\sigma^{p-1,p} z}{y_{1*(p-2)}} \right)$$
$$\times 2 \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} \left(-\frac{\mathbf{y}_{(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p-1}}{y_{1*(p-2)}} \right)^k \left(-\mathbf{y}_{(p-2)}^{\mathrm{T}} \boldsymbol{\sigma}^{(p-2)p} z \right)^{2j-k}$$
$$\times \left(\sigma^{pp} z^2 y_{1*(p-2)}^2 / \sigma^{p-1,p-1} \right)^{(k-j)/2} K_{k-j} \left\{ (\sigma^{p-1,p-1} \sigma^{pp})^{1/2} | z | / | y_{1*(p-2)} | \right\} \mathrm{d}\mathbf{y}_{(p-2)}$$
$$= 2^{-1/2} \pi^{-3/2} | \mathbf{\Sigma} |^{-1/2} \int_{-\infty}^{\infty} | y_1 |^{-1} \exp\left(-\frac{\sigma^{23} z}{y_1} - \frac{1}{2} y_1^2 \sigma^{11} \right)$$
$$\times \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \frac{1}{(2j)!} \binom{2j}{k} (-\sigma^{12})^k (-y_1 \sigma^{13} z)^{2j-k}$$
$$\times \left(\sigma^{33} z^2 y_1^2 / \sigma^{22} \right)^{(k-j)/2} K_{k-j} \left\{ (\sigma^{22} \sigma^{33})^{1/2} | z | / | y_1 | \right\} \mathrm{d}y_1.$$

Expanding the exponential, we have

$$f_Z(z) = 2^{-1/2} \pi^{-3/2} | \mathbf{\Sigma} |^{-1/2} \sum_{j=0}^{\infty} \sum_{k=0}^{2j} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(2j)! l! m!} \binom{2j}{k}$$
$$\times (-\sigma^{12})^k (-\sigma^{13} z)^{2j-k} (\sigma^{33} z^2 / \sigma^{22})^{(k-j)/2} (-\sigma^{23} z)^l (-\sigma^{11} / 2)^m$$
$$\times \int_{-\infty}^{\infty} | y_1 |^{-1} y_1^{2j-k-l} (y_1^2)^{(k-j+2m)/2} K_{k-j} \left\{ (\sigma^{22} \sigma^{33})^{1/2} | z | / | y_1 | \right\} \mathrm{d}y_1.$$

Noting that the integrand of the last integral is an odd function when $k + l$ is odd, it follows that

$$f_Z(z) = 2^{1/2}\pi^{-3/2} |\boldsymbol{\Sigma}|^{-1/2} \sum_{j=0}^{\infty}\sum_{k=0}^{2j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(2j)!l!m!}\binom{2j}{k}$$
$$\times(-\sigma^{12})^k(-\sigma^{13})^{2j-k}(\sigma^{33}/\sigma^{22})^{(k-j)/2}(-\sigma^{23})^l(-\sigma^{11}/2)^m z^{-k+l}|z|^{k+j}$$
$$\times\mathbf{1}\{k+l:\text{even}\}\int_0^{\infty} y_1^{j+2m-l-1}K_{k-j}\left\{(\sigma^{22}\sigma^{33})^{1/2}|z|/y_1\right\}\mathrm{d}y_1,$$

where $\mathbf{1}\{\cdot\}$ is the indicator function.

Examples 4 (the pdf). Using the cf for Example 4 derived earlier, the pdf of $Z \equiv Y_{1*3}$ at *z* is given by the inversion formula:

$$f_Z(z) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)\varphi_Z(t)\mathrm{d}t$$
$$= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-\mathrm{i}tz)(2\pi)^{-1/2}\left\{1-2\mathrm{i}t\rho_{32}x_1-(\mathrm{i}t)^2(1-\rho_{32}^2)x_1^2\right\}^{-1/2}\exp(-x_1^2/2)\mathrm{d}x_1\mathrm{d}t,$$

where the factor $\left\{1-2\mathrm{i}t\rho_{32}x_1-(\mathrm{i}t)^2(1-\rho_{32}^2)x_1^2\right\}^{-1/2}$ is equal to that of Example 1 when ρ_{32} and tx_1 are replaced by ρ and *t*, respectively with $\sigma_{11}=\sigma_{22}=\sigma_{33}=1$.

Nadarajah and Pogány (2016, Theorem 2.1) gave the following associated formula for the pdf of Y_{1*2} at *z* in Example 1 using the cf:

$$f_{Y_{1*2}}(z) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}uz)\left\{1-2\mathrm{i}u\rho-(\mathrm{i}u)^2(1-\rho^2)\right\}^{-1/2}\mathrm{d}u$$
$$= \{\pi(1-\rho^2)\}^{-1}\exp\{\rho z/(1-\rho^2)\}K_0\{|z|/(1-\rho^2)\},$$

where $K_0\{\cdot\}$ is the modified Bessel function of the second kind of zero order (see e.g., DLMF, 2023, https://dlmf.nist.gov/10.25) (the right-hand side of the above equation was independently derived by Grishchuk, 1996, Equation (49) without using the cf). Employ the variable transformation $u = tx_1$ with $\mathrm{d}t/\mathrm{d}u = 1/x_1$. Then, we obtain the pdf of $Z = Y_{1*3}$ at *z*:

$$f_Z(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi)^{-1}\exp(-tx_1z/x_1)\left\{1-2\mathrm{i}t\rho_{32}x_1-(\mathrm{i}t)^2(1-\rho_{32}^2)x_1^2\right\}^{-1/2}$$
$$\times\mathrm{d}t\exp(-x_1^2/2)\mathrm{d}x_1$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(2\pi)^{-1}\exp(-uz/x_1)\left\{1-2\mathrm{i}u\rho_{32}-(\mathrm{i}u)^2(1-\rho_{32}^2)\right\}^{-1/2}|x_1|^{-1}$$
$$\times\mathrm{d}u\exp(-x_1^2/2)\mathrm{d}x_1$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \frac{1}{\pi |x_1| (1-\rho_{32}^2)} \exp\left\{\frac{\rho_{32} z}{(1-\rho_{32}^2) x_1}\right\} K_0 \left\{\frac{|z|}{(1-\rho_{32}^2)|x_1|}\right\} \exp(-x_1^2/2) \mathrm{d}x_1$$

$$= \int_0^{\infty} \frac{1}{\pi x_1 (1-\rho_{32}^2)} \left[\exp\left\{\frac{\rho_{32} z}{(1-\rho_{32}^2) x_1}\right\} + \exp\left\{\frac{-\rho_{32} z}{(1-\rho_{32}^2) x_1}\right\}\right] K_0 \left\{\frac{|z|}{(1-\rho_{32}^2) x_1}\right\} \phi(x_1) \mathrm{d}x_1,$$

where $\phi(x_1) \equiv \phi_1(x_1 | 0,1) = (2\pi)^{-1/2} \exp(-x_1^2/2)$. Since x_1 in z/x_1 is seen as a scale parameter when $0 < x_1 < \infty$ is given, the above result is a scale mixture of the Bessel function with an associated factor using the standard normal density.

4. Discussions

(a) Integral representation: As reviewed in Section 1, Nadarajah and Pogány (2016) stated that they solved "a problem that has remained unsolved since 1936" i.e., a closed-form pdf of Example 2. However, the closed-form pdf was derived as early as in 1929 by Pearson et al. Cui et al. (2016) obtained the pdf of the bivariate case with non-zero means i.e., the pdf of Example 2 using an infinite series. Note that these results are for bivariate normal variables.

In Results 1 and 2 of this paper, the cf and pdf of the product of more-than-two correlated normal variables are given by integral representations. Though the results are obtained by elementary integration, they are solutions to the problems of the cf's and pdf's of the products of more-than-two normal variables since the pdf's of the above known cases using the Bessel functions require integral or series expressions for actual computations as mentioned earlier (for the computational aspects and implementation of the pdf' of product distributions, see Glen, Leemis & Drew, 2004). However, if the cf's and pdf's for the more-than-2 cases can be expressed by some known or new functions having e.g., the corresponding differential equations, they are useful and insightful. This is an open problem.

(b) The McKay Bessel distribution: McKay (1932, Equations (1) and (2)) obtained the Bessel function distribution, whose pdf is

$$f_Z(z) = \frac{|1-c^2|^{v+(1/2)} \exp(-cz/b) |z|^v}{\pi^{1/2} 2^v b^{v+1} \Gamma\{v+(1/2)\}} K_v(|z|/b) \quad (b > 0,\ v > -1/2,\ |c| < 1).$$

When $|c| > 1$, the factor $K_v(|z|/b)$ should be replaced by $\pi I_v(|z|/b)$, where $I_v(\cdot)$ is the modified Bessel function of the first kind of order v (McKay, 1932, Equation (1); Erdélyi, 1953, Section 7.2.2; DLMF, 2023, https://dlmf.nist.gov/10.25.E2). As McKay (1932, p. 43) pointed out, it is easily found that the pdf of Example 1 is obtained as a special case of the Bessel distribution when $c = -\rho$ and $v = 0$ (for this equivalence using the cf see also Wishart & Bartlett, 1932, p. 459). Due to the property of generality, the McKay Bessel distribution has been investigated and extended (for historical works and recent researches, see Thabane & Drekic, 2003; Jankov Maširević & Pogány, 2021 and the references therein; Saieed & Altalib, 2023).

(c) The sum or mean of *n* independent products: In practice, we encounter more than single independent products. Suppose that $Z_i \ (i = 1,...,n)$ are independent copies of a random product Z used earlier. Define $Z_+ = Z_1 + ... + Z_n$ and $\bar{Z} = Z_+ / n \ (n = 1, 2, ...)$. Then, from general properties of a cf, the cf's of the sum and mean become

$$\varphi_{Z_+}(t) = \mathrm{E}\{\exp(\mathrm{i}tZ_+)\} = \prod\nolimits_{i=1}^{n} \mathrm{E}\{\exp(\mathrm{i}tZ_i)\} = [\mathrm{E}\{\exp(\mathrm{i}tZ)\}]^n \quad \text{and}$$

$$\varphi_{\bar{Z}}(t) = \mathrm{E}\{\exp(\mathrm{i}tZ_+ / n)\} = \varphi_{Z_+}(t/n) = [\mathrm{E}\{\exp(\mathrm{i}tZ/n)\}]^n,$$

respectively. For instance, the cf's with n independent products of Example 1 are

$$\varphi_{Z_+}(t) = \left\{1 - 2\mathrm{i}t\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}t\sigma_{11}^*\sigma_{22}^*)^2\right\}^{-n/2} \quad \text{and}$$

$$\varphi_{\bar{Z}}(t) = \left\{1 - 2\mathrm{i}tn^{-1}\sigma_{11}^*\sigma_{21}^* - (\mathrm{i}tn^{-1}\sigma_{11}^*\sigma_{22}^*)^2\right\}^{-n/2},$$

respectively. The pdf's using the cf's with the inversion theorem are formally given by

$$f_{Z_+}(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-\mathrm{i}tz)\{\varphi_Z(t)\}^n \mathrm{d}t \quad \text{and}$$

$$f_{\bar{Z}}(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-\mathrm{i}tz)\{\varphi_Z(t/n)\}^n \mathrm{d}t,$$

respectively. Nadarajah and Pogány (2016, Theorem 2.2) gave the pdf of $\bar{Z}$ for Example 1 using the cf and their lemma (see the appendix):

$$f_{\bar{Z}}(z) = \frac{n^{(n+1)/2} 2^{(1-n)/2} |z|^{(n-1)/2}}{\Gamma(n/2)\{\pi(1-\rho^2)\}^{1/2}} \exp\left(\frac{n\rho z}{1-\rho^2}\right) K_{(n-1)/2}\left(\frac{n|z|}{1-\rho^2}\right)$$

with their expression $K_{(1-n)/2}(\cdot)$ replaced by an algebraically equal one $K_{(n-1)/2}(\cdot)$ (see Gaunt, 2022, Equation (2)). The pdf of $Z_+ = n\bar{Z}$ is given by the above density with the

Jacobian $\mathrm{d}\bar{z} / \mathrm{d}z_+ = n^{-1}$ as

$$f_{Z_+}(z) = \frac{n^{(n+1)/2} 2^{(1-n)/2} | n^{-1} z |^{(n-1)/2}}{\Gamma(n/2)\{\pi(1-\rho^2)\}^{1/2}} n^{-1} \exp\left(\frac{\rho z}{1-\rho^2}\right) K_{(n-1)/2}\left(\frac{|z|}{1-\rho^2}\right)$$
$$= \frac{2^{(1-n)/2} | z |^{(n-1)/2}}{\Gamma(n/2)\{\pi(1-\rho^2)\}^{1/2}} \exp\left(\frac{\rho z}{1-\rho^2}\right) K_{(n-1)/2}\left(\frac{|z|}{1-\rho^2}\right).$$

Note that an essentially the same result as the above pdf of the Z_+ was obtained by Wishart and Bartlett (1932, Equation (12)).

(d) The joint pdf of $\mathbf{Y}_* = (Y_{1*1}, Y_{1*2}, ..., Y_{1*p})^{\mathrm{T}}$. In practice, data are more or less non-normally distributed especially in the behavioral and social sciences. Though so far we have been assuming multivariate normality for $\mathbf{Y} = (Y_1, ..., Y_p)^{\mathrm{T}}$, it is found that the pleasantly simple result of Lemma 1 for the pdf of $\mathbf{Y}_*$ holds under arbitrary distributions as long as the distribution of $\mathbf{Y}_*$ is defined since the Jacobian $J(\mathbf{y} \to \mathbf{y}_*) = \prod_{j=1}^{p-1} | y_{1*j} |^{-1}$ does not depend on normality. Then, it can be shown that the result of Theorem 1 for the pdf of $Z = Y_{1*p}$ holds under e.g., elliptical symmetry (see Ogasawara, 2022a) including multivariate normality as a special case:

Theorem 3. *Suppose that* $\mathbf{Y} = (Y_1, ..., Y_p)^{\mathrm{T}}$ *follows the elliptical distribution with the joint pdf*

$$f_{\mathbf{Y}}(\mathbf{y}) = K_p^* | \mathbf{\Lambda} |^{-1/2} g(\mathbf{y}^{\mathrm{T}} \mathbf{\Lambda}^{-1} \mathbf{y}),$$

where K_p^* *is the normalizing constant*; $g(\cdot)$ *is a non-negative scalar function*; *and* $\mathbf{\Lambda}$ is *a non-singular scale matrix. Then, the pdf of* $Z = Y_{1*p}$ *when* $\mathbf{Y}_{1*p} = \mathbf{y}_{1*p}$ *is*

$$f_Z(z) = K_p^* \int_{-\infty}^{\infty} \left(\prod_{j=1}^{p-1} | y_{1*j} |^{-1} \right) g\left\{ (\tilde{\mathbf{y}}_{1*(p-1)}^{\mathrm{T}}, z / y_{1*(p-1)}) \mathbf{\Lambda}^{-1} (\tilde{\mathbf{y}}_{1*(p-1)}^{\mathrm{T}}, z / y_{1*(p-1)})^{\mathrm{T}} \right\} \mathrm{d}\mathbf{y}_{1*(p-1)}.$$

where $\tilde{\mathbf{y}}_{1*(p-1)}$ *was defined before Theorem 2.*

Appendix

A1. An introduction of the modified Bessel function of the second kind

For beginning students or researchers in applied statistics, the Bessel function may not be familiar. So, some introductory explanation associated with this article is given in this appendix. The Bessel functions are defined as solutions of the differential equation (Bessel's equation):

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - v^2) w = 0 ,$$

for a complex variable z and a complex parameter *v* (Abramowitz & Stegun, 1972, Section 9.1.1; DLMF, 2023, https://dlmf.nist.gov/10.2.E1). The modified Bessel function or the Bessel function of imaginary argument is defined by the solutions when *z* in Bessel's equation is replaced by $\mathrm{i}z$:

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z \frac{\mathrm{d}w}{\mathrm{d}z} - (z^2 + v^2) w = 0 ,$$

(Abramowitz & Stegun, 1972, Section 9.6.1; DLMF, 2023, https://dlmf.nist.gov/10.25.E1), which is given as follows. Let $\mathrm{z} = \mathrm{i}z^*$. Then, $\frac{\mathrm{d}w}{\mathrm{d}z^*} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}z^*} = \frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{i}$ yielding $\frac{\mathrm{d}w}{\mathrm{d}z} = \mathrm{i}^{-1}\frac{\mathrm{d}w}{\mathrm{d}z^*}$. Similarly, we have $\frac{\mathrm{d}^2 w}{\mathrm{d}z^{*2}} = \frac{\mathrm{d}}{\mathrm{d}z^*}\frac{\mathrm{d}w}{\mathrm{d}z^*} = \left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{i}\right)\frac{\mathrm{d}z}{\mathrm{d}z^*} = \frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\mathrm{i}^2$, which gives $\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \mathrm{i}^{-2}\frac{\mathrm{d}^2 w}{\mathrm{d}z^{*2}}$. Substituting $\mathrm{z} = \mathrm{i}z^*$, $\frac{\mathrm{d}w}{\mathrm{d}z} = \mathrm{i}^{-1}\frac{\mathrm{d}w}{\mathrm{d}z^*}$ and $\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \mathrm{i}^{-2}\frac{\mathrm{d}^2 w}{\mathrm{d}z^{*2}}$ for the original Bessel function and re-expressing z^* as z , the modified Bessel function follows.

The modified Bessel function of the second kind of order *v* denoted by $K_v(z)$ is given when using e.g., an integral representation:

$$K_v(z) = \frac{(z/2)^v}{2}\int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right)\frac{1}{t^{v+1}}\mathrm{d}t \quad (|\arg(z)| < \pi/4)$$

(Watson, 1944, Section 6.22, Equation (15); Zwillinger, 2015, Section 3.471-Equation 12, Section 8.432-Equation 6; DLMF, 2023, https://dlmf.nist.gov/10.32.E10), which is also given using $t = u^2/2 \ (0 < u < \infty)$ as

$$\begin{aligned} K_v(z) &= \frac{(z/2)^v}{2}\int_0^\infty \exp\left(-\frac{u^2}{2} - \frac{z^2}{2u^2}\right)\frac{1}{(u^2/2)^{v+1}}\frac{\mathrm{d}t}{\mathrm{d}u}\mathrm{d}u \\ &= z^v \int_0^\infty \exp\left(-\frac{u^2}{2} - \frac{z^2}{2u^2}\right)\frac{1}{u^{2v+1}}\mathrm{d}u. \end{aligned}$$

When *v* = 0, we have the modified Bessel function of zero order, whose integral expressions of use in statistics are

$$K_0(z) = \frac{1}{2}\int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right)\frac{1}{t}\mathrm{d}t = \int_0^\infty \exp\left(-\frac{u^2}{2} - \frac{z^2}{2u^2}\right)\frac{1}{u}\mathrm{d}u \ (|\arg(z)| < \pi/4) \text{ and}$$

$$K_0(x) = \int_0^\infty \frac{\cos(xt)}{(t^2+1)^{1/2}}\mathrm{d}t \ (0 < x < \infty),$$

where for $K_0(x)$ $(0 < x < \infty)$ see Watson (1944, Section 6.22, Equation (14)), Abramowitz & Stegun (1972, Section 9.6.21), Zwillinger (2015, Section 3.754, Equation 2) and DLMF (2023, https://dlmf.nist.gov/10.32.E6).

A2. The Nadarajah-Pogány lemma

Another property for $K_0(z)$ useful in our problems is given by Nadarajah and Pogány (2016, Lemma 3.1):

$$\int_{-\infty}^\infty \frac{\exp(\mathrm{i}zt)}{\{(t-\mathrm{i}a)(t+\mathrm{i}b)\}^{1/2}}\mathrm{d}t = 2\exp\left(\frac{b-a}{2}z\right)K_0\left(\frac{a+b}{2}|z|\right)$$
$$(0 < a < \infty,\ 0 < b < \infty,\ -\infty < z < \infty).$$

Their derivation is didactically repeated here. Define $t = y + \mathrm{i}\{(a-b)/2\}$. Then, we obtain

$$\int_{-\infty}^\infty \frac{\exp(\mathrm{i}zt)}{\{(t-\mathrm{i}a)(t+\mathrm{i}b)\}^{1/2}}\mathrm{d}t$$
$$= \int_{-\infty-\mathrm{i}\{(a-b)/2\}}^{\infty-\mathrm{i}\{(a-b)/2\}} \frac{\exp\{\mathrm{i}zy - (1/2)(a-b)z\}}{[\{y+(1/2)\mathrm{i}(a-b)-\mathrm{i}a\}\{y+(1/2)\mathrm{i}(a-b)+\mathrm{i}b\}]^{1/2}}\mathrm{d}y$$
$$= \exp\left(\frac{b-a}{2}z\right)\int_{-\infty-\mathrm{i}\{(a-b)/2\}}^{\infty-\mathrm{i}\{(a-b)/2\}} \frac{\exp(\mathrm{i}zy)}{[\{y-(1/2)\mathrm{i}(a+b)\}\{y+(1/2)\mathrm{i}(a+b)\}]^{1/2}}\mathrm{d}y$$
$$= \exp\left(\frac{b-a}{2}z\right)\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}zt)}{[\{t-(1/2)\mathrm{i}(a+b)\}\{t+(1/2)\mathrm{i}(a+b)\}]^{1/2}}\mathrm{d}t,$$

where the last integral with the real variable from the second last integral is given by applying Cauchy's theorem (e.g., Davies, 2002, Section 1.2) to an associated contour integral. Then, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}zt)}{\{(t-\mathrm{i}a)(t+\mathrm{i}b)\}^{1/2}}\mathrm{d}t \\
&=\exp\left(\frac{b-a}{2}z\right)\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}zt)}{[\{t^2+(1/4)(a+b)^2\}]^{1/2}}\mathrm{d}t \\
&=\exp\left(\frac{b-a}{2}z\right)\int_{-\infty}^{\infty}\frac{\cos(zt)+\mathrm{i}\sin(zt)}{[\{t^2+(1/4)(a+b)^2\}]^{1/2}}\mathrm{d}t \\
&=2\exp\left(\frac{b-a}{2}z\right)\int_{0}^{\infty}\frac{\cos(zt)}{[\{t^2+(1/4)(a+b)^2\}]^{1/2}}\mathrm{d}t \\
&=2\exp\left(\frac{b-a}{2}z\right)\int_{0}^{\infty}\frac{\cos\{u(a+b)z/2\}}{(u^2+1)^{1/2}(a+b)/2}\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right|\mathrm{d}u \\
&=2\exp\left(\frac{b-a}{2}z\right)\int_{0}^{\infty}\frac{\cos\{u(a+b)z/2\}}{(u^2+1)^{1/2}}\mathrm{d}u \\
&=2\exp\left(\frac{b-a}{2}z\right)K_0\{(a+b)|z|/2\},
\end{aligned}
$$

where Euler's formula $\exp(\mathrm{i}u)=\cos u+\mathrm{i}\sin u\ (-\infty<u<\infty)$ and the integral expression of $K_0(x)=\int_0^\infty \cos(xu)(u^2+1)^{-1/2}\mathrm{d}u\ (0<x<\infty)$ shown earlier with the variable transformation $t=u(a+b)/2$ are used.

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