We propose two problems arising from figures appeared in sangaku problems in Fukushima [1].

**Problem 1.** Let $E$ and $F$ be points on the side $DA$ of a square $ABCD$ (see Figure 1). Let $\delta_1$ be the incircle of the triangle $ABE$ touching $BE$ and $AE$ at points $G$ and $H$. Also let $\delta_2$ be the incircle of the triangle $DCF$ touching $CF$ and $DF$ at points $I$ and $J$. Prove or disprove the followings:

(i) The lines $GH$ and $IJ$ meet in the center of $ABCD$.
(ii) Let $\varepsilon$ be the incircle of the triangle made by the lines $BC$, $BE$ and $CF$. Then $\varepsilon$ touches the remaining external common tangent of $\delta_1$ and $\delta_2$.

**Remark.** Let $d_i$ and $e$ be the radii of $\delta_i$ and $\varepsilon$, respectively, and let $s = |AB|$. There is a sangaku problem stating that the relation

$$\frac{1}{e} = \frac{1}{s - 2d_1} + \frac{1}{s - 2d_2}$$

holds [1]. The problem was proposed by Takagi (or Takaki) (高木貞六) in 1877 [1, p. 44].
Problem 2. Let $\gamma$ be a circle of radius $c$ with diameter $BC$ for a rectangle $ABCD$ with $|AB| > c$ (see Figure 2). The remaining tangent of $\gamma$ from $A$ intersects the side $CD$ in a point $E$. Circles $\delta_1$, $\delta_2$, $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are defined as follows:

- $\delta_1$: incircle of the curvilinear triangle made by $AB$, $AE$ and $\gamma$, where the common internal tangent of $\gamma$ and $\delta_1$ intersects $DA$ and $AE$ at points $F$ and $G$.
- $\delta_2$: incircle of the triangle $AED$ touching $DE$ and $AE$ at points $P$ and $Q$.
- $\varepsilon_1$: incircle of the triangle $AFG$.
- $\varepsilon_2$: incircle of the curvilinear triangle made by $\gamma$, $\delta_1$ and $AB$ touching $AB$ and $\gamma$ at points $R$ and $S$.
- $\varepsilon_3$: incircle of the curvilinear triangle made by $CE$, $AE$ and $\gamma$ touching $CE$ and $\gamma$ at points $T$ and $U$.

We assume that the circles $\delta_1$ and $\delta_2$ are congruent. Prove or disprove the followings:

(i) The circles $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ are congruent.
(ii) The lines $PQ$, $RS$, $TU$ meet in a point $H$.
(iii) The distances from $H$ to $AB$, $CD$ and $DA$ are the same and equals $c$ in the occasion (ii) being true.

![Figure 2.](image)

Remark. Let $d$ be the common radius of $\delta_1$ and $\delta_2$. There are several sangaku problems stating that the radius of $\varepsilon_3$ equals $4d/9$ [1, pp. 263, 310, 315], while $d = c/4$ holds [2, 3, 4]. Notice that (iii) and (i) of Problem 1 assert similar things in a sense.

References