TWIN CIRCLES IN A TRIANGLE AND ITS APPLICATION TO AN ARBELOS

Hiroshi Okumura, Masayuki Watanabe

ABSTRACT

In this paper, we will show the existence of a new pair of congruent circles in a triangle and use it to generalize the results in (OKUMURA, 2008) and (OKUMURA, 2009) which showed the existence of non-Archimedean twins of an arbelos. The new results can be used also in Informatics education by providing a learning material for computer programming.

Keywords: arbelos, twins circles, isogonal conjugate.

1. Twin Circles in a Triangle

For a triangle ABC, let $\delta_A$ and $\delta'_A$ denote the circles passing through the vertex A and touching the side BC at B and C respectively. The circles $\delta_b$, $\delta'_b$, $\delta_c$ and $\delta'_c$ are defined similarly. Then, we have

**Theorem 1.** Three circles $\delta_A$, $\delta_b$ and $\delta_c$ meet in a single point.

Let S denote the intersection point of the circles. Three circles $\delta'_A$, $\delta'_b$ and $\delta'_c$ also meet in a single point, which we denote by T. Then,

**Theorem 2.** The point T is an isogonal conjugate of S.

Fig. 1.
Now, we show the existence of new pairs of congruent circles which exist in a triangle ABC.

**Theorem 3.** Let $C_A$ be a circle in a triangle $ABC$ touching the sides $AC$ and $BC$ and touching the circle $\delta_A$ externally. Let $C'_A$ be a circle touching the sides $AB$ and $BC$ and touching the circle $\delta'_A$ externally. Then the circles $C_A$ and $C'_A$ are congruent with each other and their radii are given by $2a/(a+b+c)$, where $r$ is an inradius of the triangle $ABC$. The radius is expressed also by

$$\frac{h}{2} \left( \frac{\cos B + \cos C + \cos A - 1}{\sin B \cdot \sin C} \right)^2,$$

where $h$ is the distance from the vertex $A$ to the line $BC$.

Similarly, define $C_B$ and $C_C$ for the vertices $B$ and $C$ respectively. Then

**Corollary** It holds that $r_A + r_B + r_C = 2r$, where $r_A$, $r_B$, and $r_C$ are the radii of $C_A$, $C_B$, and $C_C$.

### 2. An application to an arbelos

First, we consider an usual arbelos formed by three circles $\alpha$, $\beta$, and $\gamma$, where $\alpha$ and $\beta$ touches externally at a point $O$, and $\gamma$ touches the circles $\alpha$ and $\beta$ internally. We denote by $A$ the tangent point of $\alpha$ and $\gamma$, and by $B$ the tangent point of $\beta$ and $\gamma$. Let $J$ be a point different from $O$ such that the line $OJ$ touches both $\alpha$ and $\beta$ at $O$, and let $E$ and $F$ be points on the lines $AJ$ and $BJ$ respectively such that $OEJF$ form a parallelogram. In (OKUMURA, 2008) and (OKUMURA, 2009), we showed
**Theorem 4.** Let $\alpha_j$ and $\beta_j$ be circles touching the line OJ at O and passing through the points E and F respectively. Then the inscribed circle in a curvilinear triangle formed by the segments EJ, OJ and the circle $\alpha_j$ is congruent to the inscribed circle in a curvilinear triangle formed by the segments FJ, OJ and the circle $\beta_j$. If we denote the distance between the points O and J by $2d$, the radii of the congruent circles are given by

$$\frac{ab(a+b)}{(\sqrt{a^2+d^2} - d)(\sqrt{b^2+d^2} - d) - ab},$$

where $a$ and $b$ are the radii of $\alpha$ and $\beta$ respectively.

Theorem 4 is proved again by using Theorem 3, and the similar results hold when the circles $\alpha$ and $\beta$ touch internally. In this case, we consider that the circle $\gamma$ is the one touching one of $\alpha$ and $\beta$ internally and the other externally. Define O, A, B, J, E, F, $\alpha_j$, $\beta_j$ and $d$ as above.

**Theorem 5.** There exists a pair of congruent circles in the parallelogram OEJF as in the previous Theorem, and the radii of the congruent circles are given by

$$\frac{ab(b-a)}{(\sqrt{a^2+d^2} - d)(\sqrt{b^2+d^2} + d) - ab}$$

if $\alpha$ lies inside $\beta$,

$$\frac{ab(a-b)}{(\sqrt{a^2+d^2} + d)(\sqrt{b^2+d^2} - d) - ab}$$

if $\beta$ lies inside $\alpha$.

Fig. 3.
We have the similar results even when one of $\alpha$ and $\beta$ is a line, which is also the consequence form Theorem3. When $\beta$ is a line, it holds that $A=E$, $\alpha=\alpha_1$ and $\gamma$ is a tangent line of $\alpha$ at A. When $\alpha$ is a line, it holds that $B=F$, $\beta=\beta_1$ and $\gamma$ is a tangent line of $\beta$ at B. Then,

**Theorem 6.** There exists a pair of congruent circles in the parallelogram OEJF as in the previous Theorem, and the radii of the congruent circles are given by

\[
\left(\sqrt{a^2 + d^2} - d - a\right)/a \quad \text{if } \beta \text{ is a line},
\]
\[
\left(\sqrt{a^2 + d^2} - d - b\right)/b \quad \text{if } \alpha \text{ is a line}.
\]

![Diagram](image)

Fig. 4.

Note that the results in Theorem6 is the limit of the results in Theorem5 when $b \to \infty$ or $a \to \infty$.

3. An animation of twin circles of an arbelos programmed by Java

The above theorems provide a new learning material for a computer programming in a Informatics Education. We gave the following assignment to a student in a informatics course of a university as a part of his graduation thesis.

Consider an usual arbelos formed by $\alpha$, $\beta$ and $\gamma$. For any point $J$ not lying on the line $AB$, take points $E$ and $F$ on the lines $JA$ and $JB$ respectively such that $OEJF$ make a parallelogram, and let $\alpha_1$ and $\beta_1$ be circles touching the line $OJ$ at $O$ and passing through the points $E$ and $F$ respectively. Make a program to describe twin circles in a parallelogram $OEJF$ inscribed to curvilinear triangles.
OEJ and OFJ, and to move them in accordance with the change of the radii of $\alpha$ and $\beta$.

It was realized by JAVA programming under the following consideration.

Let $l$ a line perpendicular to OJ passing through O, $A'$ be the intersection point of $l$ and the line JA, $B'$ be the intersection point of $l$ and the line JB, and let $\alpha'$ and $\beta'$ be the circles with diameters OA' and OB' respectively. Note that the points $A'$ and $B'$ are on different sides of the line OJ when the point J is outside $\alpha$ and $\beta$, and are on the same side when the point J is inside $\alpha$ or $\beta$. When the point J is on $\beta$, the line JB is parallel to $l$, so we consider $B'$ to be a point at infinity and $\beta'$ to be a line. Similarly, we consider $A'$ to be a point at infinity and $\alpha'$ to be a line when the point J is on $\alpha$. Then the required twin circles can be considered the congruent circles with respect to $\alpha'$ and $\beta'$ in Theorem 4 if the point J is outside $\alpha$ and $\beta$, those in Theorem 5 if the point J is inside $\alpha$ or $\beta$, and those in Theorem 6 if the point J lies on $\alpha$ or $\beta$. Then the results in Theorem 4, 5 and 6 gave us the sufficient information to describe the circles.

Fig. 5.

The student could develop his ability of programming through this assignment, and conversely, to make the program helped him understand the geometric background of the theorems better. It can be concluded that the learning of arbelos, especially, to find new properties, and a programming to realize them in computer graphics motivate students to study both geometry and programming synergistically, and make their understanding deeper.
REFERENCES


Hiroshi Okumura
Department of Life Science and Information
Maebashi Institute of Technology
460-1 Kamisadori Maebashi Gunma 371-0816, Japan
ok@maebashi-it.ac.jp:

Masayuki Watanabe
Department of Integrated Design Engineering
Maebashi Institute of Technology
460-1 Kamisadori Maebashi Gunma 371-0816, Japan
mwatanabe@maebashi-it.ac.jp: