FORCING OVER CHOICELESS MODELS AND GENERIC ABSOLUTENESS

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Abstract. We develop a toolbox for forcing over arbitrary models of set theory without the axiom of choice. In particular, we introduce a variant of the countable chain condition and prove an iteration theorem that applies to many classical forcings such as Cohen forcing and random algebras. Our approach sidesteps the problem that forcing with the countable chain condition can collapse ω_1 by a recent result of Karagila and Schweber.

Using this, we show that adding many Cohen reals and random reals leads to different theories. This result is due to Woodin. Thus one can always change the theory of the universe by forcing, just like the continuum hypothesis and its negation can be obtained by forcing over arbitrary models with choice.

We further study principles stipulating that the first-order theory of the universe remains the same in all generic extension by a fixed class of forcings. Extending a result of Woodin, we show that even for very restricted classes such as the class of all finite support products of Cohen forcing or the class of all random algebras, this principle implies that all infinite cardinals have countable cofinality.

Contents

The authors are grateful to W. Hugh Woodin for discussions and for permission to include his Theorem 1.3 and the joint Theorem 1.4. They further thank Arthur Apter and Peter Koepke for discussions on set theory without the axiom of choice during initial stages of this project, and Asaf Karagila and John Truss for comments.

The first-listed author thanks the Japan Society for the Promotion of Science (JSPS) for its generous support through the grants with JSPS KAKENHI Grant Number 15K17586 and 19K03604. He is also grateful to the Sumitomo Foundation for its generous support through a Grant for Basic Science Research. This research was funded in whole or in part by EPSRC grant number EP/V009001/1 for the second-listed author. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 794020 of the second-listed author.

2020 *Mathematics Subject Classification.* (Primary) 03E25; (Secondary) 03E57, 03E35, 03E17.

Key words and phrases. Choiceless set theory, forcing, generic absoluteness.

1. Introduction

Forcing over $L(\mathbb{R})$ has been important in work of Steel, Van Wesep [22], Woodin [26], Laflamme [18], Di Prisco, Todorcevic [6] and recently Larson and Zapletal [19]. Forcing over arbitrary choiceless models appeared in studies of Monro [20] and recent work of Karagila, Schlicht [15] and Usuba [24]. Most basic results in forcing such as the forcing theorem go through in ZF without use of the axiom of choice.[1] Regarding preservation of cardinals, it is a serious problem that even very simple forcings can collapse cardinals over choiceless models. For instance, both σ-closed forcings and forcings with the countable chain condition can collapse ω_1, the latter by a recent result by Karagila and Schweber [16].[2] While even the the most basic σ-closed forcings collapse ω_1 if ω_1 is singular, various well-known c.c.c. forcings do preserve cardinals. For instance, this is the case for finite support products of Cohen forcing, finite support iterations of random forcing and Hechler forcing and random algebras on all cardinals. We show this in this paper. We choose a general approach by introducing a variant of the countable chain condition that can be iterated. In models of the axiom of choice, this notion is equivalent to the c.c.c. The more general versions are called θ-narrow and uniformly θ-narrow, respectively, for any infinite ordinal θ. We show in Lemma 3.2 and Theorem 3.4:

Theorem 1.1.
(1) *Every θ-narrow forcing preserves all cardinals and cofinalities $>\theta$.*
(2) *Any uniform iteration of θ-narrow forcings with finite support is again uniformly θ-narrow.*

The notion of a uniform iteration of θ-narrow forcings is explained below. As an application, we will see that a mix of any of the above forcings can be iterated with finite support while preserving all cardinals and cofinalities. In order to apply the previous result to random algebras, we show in Theorem 3.21:

Theorem 1.2. *The random algebra on any number of generators is complete.*

A special case of the above iteration theorem is that any uniform iteration of σ-linked forcings preserves all cardinals and cofinalities in Corollary 3.5. Here, uniform means that the iteration comes with a sequence of names for linking functions.

We use these results to study the effect of products and iterations of the above forcings on choiceless models. By a Cohen model, we mean an extension by a finite support product of Cohen forcing of length at least ω_2. A random model is an extension by a random algebra with at least ω_2 generators. We separate Cohen from random models in Theorem 4.5. This result was proved by Woodin.

Theorem 1.3 (Woodin)**.** *Cohen models and random models have different theories.*

In particular, there is a first-order sentence that can be forced true or false over any choiceless model, as you like. Using this, we study the principle $\mathsf{A}_{\mathcal{C}}$ which states V is elementarily equivalent to all its generic extensions by forcings in a class $\mathcal{C}$. The previous result shows that $\mathsf{A}_{\mathcal{C}}$ fails for the class $\mathcal{C}$ of all random algebras and all finite support products of Cohen forcings. In the language of the modal logic of forcing [10], we find a *switch* with respect to this class. Recall that a switch is a sentence that can be forced both true or false over any generic extension of V. For example, the continuum hypothesis and the existence of Suslin trees are switches for models of ZFC.

We then study more restrictive absoluteness principles for each of the classes $\mathbb{C}^*$ of all finite support products of Cohen forcings, $\mathbb{R}_*$ of all random algebras and $\mathbb{H}^{(*)}$ of finite support iterations of Hechler forcing of arbitrary length. If there exists an uncountable regular cardinal, then within each of these classes the generic extension can detect differences in the the length of the product, number of generators or length of the iteration in the ground model. Using

[1] An exception is maximality or fullness, i.e., the statement: if an existential formula $\exists x\ \varphi(x,\tau)$ is forced, then there exists a name σ such that $\varphi(\sigma,\tau)$ is forced.

[2] They find a model and a forcing such that each antichain is countable, but ω_1 is collapsed. In the present paper, c.c.c. means the weaker condition that there exist no antichains of size ω_1.

this, we will show in Theorem 4.14 and Corollary 4.20 that each of these principles implies that all infinite cardinals have countable cofinality. The next result was proved by Woodin for the class $\mathbb{C}^*$.

Theorem 1.4 (joint with Woodin)**.** *Let* A *denote any of the principles* $\mathsf{A}_{\mathbb{C}^*}$, $\mathsf{A}_{\mathbb{R}_*}$ *or* $\mathsf{A}_{\mathbb{H}^{(*)}}$. *If* A *holds, then all infinite cardinals have countable cofinality.*

For random algebras and products of Cohen forcing, this is proved in Section 4.2 using a new cardinal characteristic $\mathbf{j}$. Regarding Hechler forcing, let $\mathbb{H}^{(\kappa)}$ denote the finite support iteration of Hechler forcing of length κ. By the above iteration theorem, we know that $\mathbb{H}^{(\kappa)}$ preserves all cardinals and cofinalities. The case for Hechler forcing of the previous theorem uses the next result that is proved in Theorems 4.18 and 4.19:

Theorem 1.5. *For any uncountable cardinal* κ, $\mathbb{H}^{(\kappa)}$ *forces that the bounding number equals* $\max(\mathrm{cof}(\kappa), \omega_1)$.

In a well known model constructed by Gitik, all infinite cardinals have countable cofinality. However, Theorem 4.32 shows that the above principles do not hold there:

Theorem 1.6. *Let* A *denote any of the principles* $\mathsf{A}_{\mathbb{C}^*}$, $\mathsf{A}_{\mathbb{R}_*}$ *or* $\mathsf{A}_{\mathbb{H}^{(*)}}$. A *fails in Gitik's model from* [7, Theorem I].

It is open whether A is consistent for any of the above choices. Our results in Section 4.4.2 provide properties that a model of A must have. Regarding its consistency strength, note that Gitik's construction starts from a proper class of strongly compact cardinals. Moreover, it follows from the previous theorem and a result of Busche and Schindler [3, Theorem 1.5] that A has at least the consistency strength of $\mathsf{AD}^{L(\mathbb{R})}$. It therefore has at least the consistency strength of infinitely many Woodin cardinals.

2. Preliminaries

2.1. **Notation.** Write $A \leq_i B$ if there exists an injective function $f\colon A \to B$ and $A \leq_s B$ if there exists a surjective function $g\colon B \to A$ or A is empty. A *partial function* from A to B is denoted $f\colon A \rightharpoonup B$. Basic open sets in Cantor space $2^\omega = \{f \mid f\colon \omega \to 2\}$ are denoted $N_t = \{x \in {}^\omega 2 \mid t \subseteq x\}$, where $t \in 2^{<\omega}$.

Let $\mathrm{p}\colon \mathsf{Ord} \times \mathsf{Ord} \to \mathsf{Ord}$ denote the standard pairing function.[3] The *rank* $\mathrm{rank}(x)$ of a set x is the least $\alpha \in \mathsf{Ord}$ with $x \in V_{\alpha+1}$. For ordinals α and β, $\mathcal{P}_\alpha(\beta)$ denotes the set of subsets of β of order type $<\alpha$. A *cardinal* is an ordinal κ that is not the surjective image of any $\alpha < \kappa$. Card, $\mathsf{SucCard}$ and Reg denote the classes of infinite cardinals, infinite successor cardinals and infinite regular cardinals, respectively. κ and λ denote infinite cardinals, unless stated otherwise. κ is called a λ*-strong limit* if for all $\nu < \kappa$, there does not exist a surjection from ν^λ onto κ. κ is called λ*-inaccessible* if it is a λ-strong limit and $\mathrm{cof}(\kappa) > \lambda$.

A *forcing* is a set $\mathbb{P}$ with a *quasiorder*[4] $\leq$ on $\mathbb{P}$ and a largest element $\mathbf{1}_\mathbb{P}$. We often identify $\mathbb{P}$ and $(\mathbb{P}, \leq, \mathbf{1}_\mathbb{P})$. The *discrete partial order* on a set is the one with no relation between distinct elements. Conditions $p, q \in \mathbb{P}$ are *compatible*, denoted $p \parallel q$, if there exists some $r \leq p, q$. Otherwise p and q are *incompatible*, denoted $p \perp q$. If $(\mathbb{P}, \leq)$ and $(\mathbb{Q}, \leq)$ are forcings, a $\parallel$*-homomorphism* from $\mathbb{P}$ to $\mathbb{Q}$ is a function $f\colon \mathbb{P} \to \mathbb{Q}$ such that for all $p, q \in \mathbb{P}$, $p \parallel q$ implies $f(p) \parallel f(q)$. The notions of $\leq$*-homomorphism* and $\perp$*-homomorphism* are defined similarly.

The *Boolean completion* $\mathbb{B}(\mathbb{P})$ of a forcing $\mathbb{P}$ is the set of all regular open subsets of $\mathbb{P}$, ordered by inclusion.[5] It comes with a canonical $\leq$- and $\perp$-homomorphism $\iota := \iota_\mathbb{P}\colon \mathbb{P} \to \mathbb{B}(\mathbb{P})$ with dense range in $\mathbb{B}(\mathbb{P})$, where $\iota(p) = \{q \in \mathbb{P} \mid \forall r \leq q\ r \leq p\}$. We will use the notation $p_\iota := \iota(p)$. For any subset A of $\mathbb{P}$, we call $\sup(A) := \sup(\iota[A])$ the supremum of A.

For any set S, let $\mathrm{Fun}_{<\lambda}(S, \kappa)$ denote the set of partial functions $f\colon S \to \kappa$ of size $<\lambda$, partially ordered by reverse inclusion. For any ordinal α, we write $2^{(\alpha)}$ for $\mathrm{Fun}_{<\omega}(\alpha, 2)$. Let $\mathbb{C}^\alpha := \mathrm{Fun}_{<\omega}(\alpha \times \omega, 2)$ and $\mathbb{C} := \mathbb{C}^1$. $\mathbb{C}^\alpha$ is isomorphic to the product of α many copies of

[3] An ordinal $\alpha > 0$ is closed under p if and only if α is multiplicatively closed, i.e., $\beta \cdot \gamma < \alpha$ for all $\beta, \gamma < \alpha$.

[4] A quasiorder is a transitive reflexive relation.

[5] It is easy check that $\mathbb{B}(\mathbb{P})$ is isomorphic to $\mathbb{B}(\mathbb{P}_{\mathrm{sep}})$, where $\mathbb{P}_{\mathrm{sep}}$ denotes the separative quotient of $\mathbb{P}$.

Cohen forcing $\mathbb{C}$ with finite support. For any cardinal κ, a forcing is called *κ-c.c.* if it does not contain antichains of size κ.

2.2. **Iterated forcing.** A two-step iteration has to be restricted to names of bounded ranks to avoid the use of proper classes, and the same should happen uniformly at every stage of an iterated forcing. For instance, one can restrict the names for elements of the next forcing to the least possible V_α in each step of the iteration. The next lemma gives a clear account of this by providing names with optimal ranks. To state the lemma, we define the *$\mathbb{P}$-rank* $\mathrm{rank}_{\mathbb{P}}(\tau) := \sup\{\mathrm{rank}_{\mathbb{P}}(\sigma)+1 \mid \exists p\ (\sigma,p)\in\tau\}$ of a $\mathbb{P}$-name τ by induction on the rank of τ.[6]

In the proof, we will work with a generic filter over V for convenience. In more detail, one can run the argument in a Boolean-valued model $V^{\mathbb{B}}$, where $\mathbb{B}=\mathbb{B}(\mathbb{P})$. $V^{\mathbb{B}}$ believes that it is of the form $V[G]$ for a $\mathbb{P}$-generic filter G over V. Every statement claimed in the proof holds in $V^{\mathbb{B}}$ with Boolean value $\mathbf{1}_{\mathbb{B}}$.[7]

Lemma 2.1. *There is a formula defining a class function $F\colon V^3\to V$ such that the following hold for any forcing $\mathbb{P}$ and any $\mathbb{P}$-name τ:*

(1) *$F(\mathbb{P},q,\tau)$ is a $\mathbb{P}$-name with $q\Vdash_{\mathbb{P}} F(\mathbb{P},q,\tau)=\tau$.*
(2) *If $\beta\in\mathsf{Ord}$ and $q\Vdash_{\mathbb{P}}\mathrm{rank}(\tau)\leq\beta$, then $\mathrm{rank}_{\mathbb{P}}(F(\mathbb{P},q,\tau))\leq\beta$.*

Proof. We define F by induction on the $\mathbb{P}$-rank of τ by letting

$$F(\mathbb{P},q,\tau)=\{(\nu,s)\mid \exists\sigma,r\ (\sigma,r)\in\tau,\ \nu=F(\mathbb{P},s,\sigma),\ s\leq q,r,\ \exists\gamma\ s\Vdash_{\mathbb{P}}\mathrm{rank}(\sigma)=\gamma\}.$$

(1): It suffices to show $\tau^G=F(\mathbb{P},q,\tau)^G$ for any $\mathbb{P}$-generic filter over V with $q\in G$ by the forcing theorem. First, suppose $x\in\tau^G$. Then $x=\sigma^G$ for some $(\sigma,r)\in\tau$ with $r\in G$. There exist $s\leq q,r$ in G and γ with $s\Vdash_{\mathbb{P}}\mathrm{rank}(\sigma)=\gamma$ by the forcing theorem. Then $x=\sigma^G=F(\mathbb{P},s,\sigma)^G\in F(\mathbb{P},q,\tau)^G$ by the inductive hypothesis (1). Conversely, suppose $x\in F(\mathbb{P},q,\tau)^G$. There exist $(\sigma,r)\in\tau$ and $s\leq q,r$ in G with $x=F(\mathbb{P},s,\sigma)^G$. Then $x=F(\mathbb{P},s,\sigma)^G=\sigma^G\in\tau^G$ by the inductive hypothesis (1).

(2): We prove the claim by induction on β. Suppose that $q\Vdash_{\mathbb{P}}\mathrm{rank}(\tau)\leq\beta$. Take an element (ν,s) of $F(\mathbb{P},q,\tau)$ and witnesses σ, r, γ as in the definition of $F(\mathbb{P},q,\tau)$. It suffices to show $\mathrm{rank}_{\mathbb{P}}(\nu)<\beta$. Since $s\Vdash\mathrm{rank}(\sigma)=\gamma$ and $s\Vdash_{\mathbb{P}}\sigma\in\tau$, we have $\gamma<\beta$. Since $\nu=F(\mathbb{P},s,\sigma)$, we have $\mathrm{rank}_{\mathbb{P}}(\nu)=\mathrm{rank}_{\mathbb{P}}(F(\mathbb{P},s,\sigma))\leq\gamma$ by the inductive hypothesis (2). □

We now fix a definition of two-step iterations.

Definition 2.2. Suppose that $\mathbb{P}$ is a forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing with $\mathbf{1}\Vdash_{\mathbb{P}}\mathrm{rank}(\dot{\mathbb{Q}})\leq\beta+1$. Let $\mathbb{P}*\dot{\mathbb{Q}}$ denote the set of all pairs $(p,\dot{q})$ with $p\Vdash\dot{q}\in\dot{\mathbb{Q}}$, where $p\in\mathbb{P}$ and $\dot{q}$ is a $\mathbb{P}$-name with $\mathrm{rank}_{\mathbb{P}}(\dot{q})\leq\beta$.

It follows from Lemma 2.1 that the $(\mathbb{P}*\dot{\mathbb{Q}})$-generic extensions of V are precisely those of the form $V[G][H]$, where $V[G]$ is a $\mathbb{P}$-generic extension of V and $V[G][H]$ is a $\dot{\mathbb{Q}}^G$-generic extension of $V[G]$.

We define iterated forcing by iterating the above definition of two-step iterations as in [5, Section 7]. Every iteration has finite support.[8] We shall write

$$\vec{\mathbb{P}}=\langle\langle\mathbb{P}_\alpha,\leq_\alpha,\mathbf{1}_{\mathbb{P}_\alpha}\rangle,\langle\dot{\mathbb{P}}_\alpha,\dot{\leq}_\alpha,\dot{\mathbf{1}}_{\dot{\mathbb{P}}_\alpha}\rangle,\langle\mathbb{P}_\gamma,\leq_\gamma,\mathbf{1}_{\mathbb{P}_\gamma}\rangle\mid\alpha<\gamma\rangle$$

for an iteration of length γ.[9] We will abbreviate this by writing $\vec{\mathbb{P}}=\langle\mathbb{P}_\alpha,\dot{\mathbb{P}}_\alpha,\mathbb{P}_\gamma\mid\alpha<\gamma\rangle$ and sometimes just $\mathbb{P}_\gamma$. For any $\mathbb{P}_\gamma$-generic filter G over V, let $G{\restriction}\alpha:=\{p{\restriction}\alpha\mid p\in G\}$ for $\alpha\leq\gamma$. Also let G_α denote the filter induced by G on $\dot{\mathbb{P}}_\alpha^{G\restriction\alpha}$ for $\alpha<\gamma$ and let $\dot{G}_\alpha$ be the

[6] Then $\mathrm{rank}(\sigma)\leq\mathrm{rank}(\mathbb{P})+3\cdot\mathrm{rank}_{\mathbb{P}}(\sigma)+1$ by induction.

[7] The usual argument in ZFC for working with generic filters in V uses countable elementary submodels of some H_θ, but such submodels need not exist in models of ZF.

[8] We do not study iterations with other supports here. One can for example study bounded support iterations of length ω_1, as mentioned after Theorem 3.27 below.

[9] This is standard terminology from [5, Section 7] with the tweak of introducing $\dot{\mathbf{1}}_{\dot{\mathbb{P}}_\alpha}$ due to the absence of the axiom of choice and a minor notational difference. Note that a 2-step iteration has length 1, since it consists of $\mathbb{P}_0$ and $\mathbb{P}_1:=\mathbb{P}_0*\dot{\mathbb{P}}_1$.

canonical $\mathbb{P}_\alpha$-name for G_α. The *support* $\operatorname{supp}(p)$ of a condition $p \in \mathbb{P}_\gamma$ is the set of $\alpha < \kappa$ with $p(\alpha) \neq \mathbf{1}_{\dot{\mathbb{P}}_\alpha}$.

Remark 2.3. An iterated forcing is often given by a definition for the forcing at step α in the $\mathbb{P}_\alpha$-generic extension. Without the axiom of choice, we have to provide names for such forcings in a uniform fashion. The following suffices. If φ is a formula with two free variables and x is a set, let $A_{\varphi,y}$ denote the class $\{x \mid \varphi(x,y)\}$ if this class is a set and $\emptyset$ if it is a proper class. We claim that for any formula φ with two free variables, there is a formula defining a class function $G\colon V^2 \to V$ such that in any generic extension of V, for any forcing $\mathbb{P}$ and any $\mathbb{P}$-name σ, $G(\mathbb{P},\sigma)$ is a $\mathbb{P}$-name with $\mathbf{1} \Vdash_{\mathbb{P}} G(\mathbb{P},\sigma) = A_{\varphi,\sigma}$. To see this, let γ be least with $\mathbf{1} \Vdash_{\mathbb{P}} \operatorname{rank}(A_{\varphi,\sigma}) \leq \gamma$ and define

$$G(\mathbb{P},\sigma) := \{(\nu,q) \mid \exists \beta < \gamma \ \operatorname{rank}_{\mathbb{P}}(\nu) \leq \beta,\ q \Vdash_{\mathbb{P}} \nu \in A_{\varphi,\sigma}\}.$$

It suffices to show $\mathbf{1} \Vdash_{\mathbb{P}} A_{\varphi,\sigma} \subseteq G(\mathbb{P},\sigma)$. To see this, suppose that $p \Vdash \mu \in A_{\varphi,\sigma}$. Then there is some $q \leq p$ and some $\beta < \gamma$ with $q \Vdash \operatorname{rank}(\mu) = \beta$. By Lemma 2.1, $\nu := F(\mathbb{P},q,\mu)$ satisfies $\operatorname{rank}_{\mathbb{P}}(\nu) \leq \beta$ and $q \Vdash \mu = \nu$. Then $(\nu,q) \in G(\mathbb{P},\sigma)$ and thus $q \Vdash_{\mathbb{P}} \mu \in G(\mathbb{P},\sigma)$.

3. Cardinal preserving forcings

In this section, we provide a toolbox for proving that a forcing is cardinal preserving. The main tool is a condition that strengthens the countable chain condition and implies that all cardinals and cofinalities are preserved.

3.1. Narrow forcings.

The following describes a variant of the countable chain condition that we call *narrow*. In models of the axiom of choice, the two are equivalent. We show that all σ-linked forcings are narrow at the beginning of subsection 3.2 using Lemma 3.3 and that all random algebras are narrow in Theorem 3.21. We equip Ord with the discrete partial order where no two distinct elements are comparable.

Definition 3.1. Suppose that $\mathbb{P}$ is a forcing and θ, ν are ordinals, where θ is infinite.

(1) $\mathbb{P}$ is called (θ,ν)-*narrow* if for any ordinal $\mu \leq \nu$ and any sequence $\vec{f} = \langle f_i \mid i < \mu\rangle$ of partial $\|$-homomorphisms $f_i\colon \mathbb{P} \rightharpoonup \mathsf{Ord}$,[10]

$$\Big|\bigcup_{i<\mu} \operatorname{ran}(f_i)\Big| \leq |\max(\theta,\mu)|.$$

(2) $\mathbb{P}$ is called θ-*narrow* if it is (θ,ν)-narrow for all $\nu \in \mathsf{Ord}$. It is called *narrow* if it is ω-narrow.

We further call $\mathbb{P}$ *uniformly θ-narrow* if there exists a function G that sends each partial $\|$-homomorphism $f\colon \mathbb{P} \rightharpoonup \mathsf{Ord}$[11] to an injective function $G(f)\colon \operatorname{ran}(f) \to \theta$. It is called *uniformly narrow* if it is uniformly ω-narrow.

It is easy to see that (θ,θ)-narrow implies (θ,ν)-narrow for all ν, since the condition is already satisfied for all $\mu \geq \theta^+$ if the forcing is $(\theta,1)$-narrow. Moreover, any uniformly θ-narrow forcing is θ-narrow. Note that $(\theta,1)$-narrow is a variant of the θ^+-c.c., since it implies that there are no antichains of size θ^+. Conversely, it is easy to show that any wellordered θ^+-c.c. forcing $\mathbb{P}$ is θ-narrow by working in $\mathsf{HOD}_{\mathbb{P},\vec{f}}$ for any $\vec{f}$ as above, since $\mathbb{P} \cap \mathsf{HOD}_{\mathbb{P},\vec{f}}$ is ν-c.c. in $\mathsf{HOD}_{\mathbb{P},\vec{f}}$ for some $\nu < \theta^+$ if θ^+ is singular in $\mathsf{HOD}_{\mathbb{P},\vec{f}}$.[12]

Note that if θ^+ is regular, then $(\theta,1)$-narrow implies θ-narrow. Moreover, if there exists a sequence of injective functions from all $\alpha < \theta^+$ into θ, then $(\theta,1)$-narrow implies uniformly θ-narrow. We do not know if every $(\theta,1)$-narrow forcing is θ-narrow and whether every θ-narrow

[10] Thanks to Asaf Karagila for his observation in December 2022 that $\|$-homomorphisms $f\colon \mathbb{P} \rightharpoonup \mathsf{Ord}$ correspond to wellordered antichains in the Boolean completion of $\mathbb{P}$. Thus θ-narrow is equivalent to the condition that the Boolean completion is θ^+-c.c. Note that one could translate the following proofs about narrow forcing and its variants to Boolean completions.

[11] We can assume that $\operatorname{ran}(f)$ is an ordinal.

[12] Thanks to Asaf Karagila for sending us a direct proof that wellordered c.c.c. forcings preserve cardinals in October 2022.

forcing is uniformly θ-narrow. Moreover, we do not know if $(\theta, 1)$-narrow forcings preserve θ^+ for all $\theta \in \mathsf{Card}$.[13] However, θ-narrow forcings preserve all cardinals $>\theta$ by the next lemma.

Lemma 3.2.

(1) *Every $(\theta, 1)$-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities $\geq\theta^{++}$.*[14]
(2) *Every θ-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities $\geq\theta^+$.*

Proof. (1): We first show that $\mathbb{P}$ preserves any cardinal $\lambda \geq \theta^{++}$. Suppose that $\mu < \lambda$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f} \colon \mu \to \lambda$ for some $p \in \mathbb{P}$. For each $\alpha < \mu$, let D_α denote the set of all $q \leq p$ in $\mathbb{P}$ that decide $\dot{f}(\alpha)$. Define $f_\alpha \colon D_\alpha \to \lambda$ by sending each q to the unique $\beta < \lambda$ with $q \Vdash \dot{f}(\alpha) = \beta$. Note that each f_α is a partial $\|$-homomorphism on $\mathbb{P}$. Since $\mathbb{P}$ is $(\theta, 1)$-narrow, $\mathrm{otp}(\mathrm{ran}(f_\alpha)) < \theta^+$ for each $\alpha < \mu$. Hence $|\bigcup_{\alpha<\mu} \mathrm{ran}(f_\alpha)| \leq |\max(\theta^+, \mu)| < \lambda$. Hence p forces that $\dot{f}$ is not surjective.

A similar argument works for cofinalities. Suppose that λ is a cardinal with $\mathrm{cof}(\lambda) \geq \theta^{++}$. Suppose that $\mu < \mathrm{cof}(\lambda)$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f} \colon \mu \to \lambda$ for some $p \in \mathbb{P}$. With the same notation as above, $|\bigcup_{\alpha<\mu} \mathrm{ran}(f_\alpha)| \leq |\max(\theta^+, \mu)| < \mathrm{cof}(\lambda)$, so p forces that $\dot{f}$ is not cofinal.

(2): We first show that $\mathbb{P}$ preserves θ^+. Suppose that $\mu < \theta^+$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f} \colon \mu \to \theta^+$ for some $p \in \mathbb{P}$. For each $\alpha < \mu$, let D_α denote the set of all $q \leq p$ in $\mathbb{P}$ that decide $\dot{f}(\alpha)$. Define $f_\alpha \colon D_\alpha \to \theta^+$ by sending each q to the unique $\beta < \theta^+$ with $q \Vdash \dot{f}(\alpha) = \beta$. Note that each f_α is a partial $\|$-homomorphism on $\mathbb{P}$. Since $\mathbb{P}$ is θ-narrow, we have $|\bigcup_{\alpha<\mu} \mathrm{ran}(f_\alpha)| \leq |\max(\theta, \mu)| < \theta^+$. Hence p forces that $\dot{f}$ is not surjective.

A similar argument works for cofinalities. Suppose that λ is a cardinal with $\mathrm{cof}(\lambda) = \theta^+$. Suppose that $\mu < \mathrm{cof}(\lambda)$ is a cardinal, $\dot{f}$ is a $\mathbb{P}$-name, and $p \Vdash_{\mathbb{P}} \dot{f} \colon \mu \to \lambda$ for some $p \in \mathbb{P}$. With the same notation as above, $|\bigcup_{\alpha<\mu} \mathrm{ran}(f_\alpha)| \leq |\max(\theta, \mu)| < \mathrm{cof}(\lambda)$, so p forces that $\dot{f}$ is not cofinal. □

Lemma 3.3. *Suppose that θ, ν are cardinals, where θ is infinite, and $f \colon \mathbb{P} \to \mathbb{Q}$ is a $\perp$-homomorphism.*

(1) *$\mathbb{Q}$ is (θ, ν)-narrow, then $\mathbb{P}$ is (θ, ν)-narrow.*
(2) *$\mathbb{Q}$ is uniformly θ-narrow, then $\mathbb{P}$ is uniformly θ-narrow.*

Proof. (1) Suppose that $\vec{f} = \langle f_i \mid i < \mu \rangle$ is a sequence of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \rightharpoonup \mathsf{Ord}$. Let $D := \mathrm{ran}(f)$ and define $g_i \colon D \to \mathsf{Ord}$ as follows. Note that for all $p, r \in \mathbb{P}$ with $f(p) = f(r)$, we have $f_i(p) = f_i(r)$, since f is a $\perp$-homomorphism and f_i is a $\|$-homomorphism. For $f(p) = q \in D$, we can thus define $g_i(q) = f_i(p)$. We claim that each g_i is a partial $\|$-homomorphism. Suppose that $q, s \in D$ with $f(p) = q$, $f(r) = s$ and $q \parallel s$. Since f is a $\perp$-homomorphism, $p \parallel r$. Since f_i is a $\|$-homomorphism, $g_i(q) = f_i(p) \parallel f_i(r) = g_i(s)$ as desired. Since $\mathrm{ran}(f_i) = \mathrm{ran}(g_i)$ for all $i < \mu$ and $\mathbb{Q}$ is (θ, ν)-narrow, the statement of the lemma follows.

(2) Suppose G witnesses that $\mathbb{Q}$ is uniformly θ-narrow. The proof of (1) defines a function H from G that witnesses $\mathbb{P}$ is uniformly θ-narrow by mapping a partial $\|$-homomorphism $f \colon \mathbb{P} \rightharpoonup \mathsf{Ord}$ to a partial $\|$-homomorphisms g on $\mathbb{Q} \rightharpoonup \mathsf{Ord}$ with $\mathrm{ran}(f) = \mathrm{ran}(g)$. □

We want to avoid collapsing cardinals when iterating θ-narrow forcings. However, precisely this will happen if we are not careful. To see this, recall that both θ-narrow and uniformly θ-narrow are equivalent to the θ^+-c.c. for wellordered forcings. Using this, we argue that an iteration of narrow forcings with finite support can collapse cardinals. Suppose that ω_1 is singular and $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ is cofinal in ω_1. Suppose that $\mathbb{P}_n$ is the discrete partial order on α_n. The finite support iteration of the forcings $\mathbb{P}_n$ has a dense subset isomorphic to the finite support product $\prod_{n\in\omega} \mathbb{P}_n$. It is easy to see that this collapses ω_1. We therefore need to take precautions. Suppose that θ is an infinite ordinal. A *uniform iteration* of θ-narrow forcings with finite support is a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{G}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma \rangle$ such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma \rangle$ is a

[13] However, this holds for $\theta = \omega$ by an argument of Karagila, Schilhan and the second-listed author.
[14] This can also be proved via a result of Karagila and Schweber [16, Proposition 5.7]

finite support iteration and for each $\alpha < \gamma$, $\mathbf{1}_{\mathbb{P}_\alpha}$ forces that $\dot{\mathbb{P}}_\alpha$ is uniformly θ-narrow witnessed by $\dot{G}_\alpha$.

Theorem 3.4. *Suppose that θ is an infinite ordinal. Any uniform iteration of θ-narrow forcings with finite support is again uniformly θ-narrow.*

Proof. We can assume $\theta \in \mathsf{Card}$. Suppose that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{g}_\alpha, \mathbb{P}_\delta \mid \alpha < \delta \rangle$ is such an iteration. We construct a sequence $\vec{G} = \langle G_\gamma \mid \gamma \leq \delta \rangle$ of functions by induction on $\gamma \leq \delta$, where G_γ witnesses that $\mathbb{P}_\gamma$ is uniformly θ-narrow. $\vec{G}$ will be defined from $\vec{\mathbb{P}}$ and θ by recursion. We will assume that $\mathrm{ran}(f)$ is an ordinal for any input f of G_γ to ensure that G_γ is a set function.

First suppose $\gamma = \beta + 1$. The following provides a construction of $G_{\beta+1}$ from G_β that works uniformly for all $\beta < \delta$. Fix a partial $\|$-homomorphism $f\colon \mathbb{P}_\beta * \dot{\mathbb{P}}_\beta \rightharpoonup \mathsf{Ord}$. For $\mathbb{P}_\beta$-names σ and τ, let $\mathrm{pair}(\sigma, \tau)$ denote the canonical $\mathbb{P}_\beta$-name for the ordered pair (σ, τ). Let

$$\dot{f} := \{(\mathrm{pair}(\dot{q}, \check{\alpha}), p) \mid f(p, \dot{q}) = \alpha\}.$$

Claim. $\mathbf{1}_{\mathbb{P}_\beta}$ forces that $\dot{f}$ is a partial $\|$-homomorphism on $\dot{\mathbb{P}}_\beta$.

Proof. Suppose that G is $\mathbb{P}_\beta$-generic over V. In $V[G]$, take $q_0, q_1 \in \dot{\mathbb{P}}_\beta^G$ with $\dot{f}^G(q_i) = \alpha_i$ for $i < 2$ and $\alpha_0 \neq \alpha_1$. We claim $q_0 \perp q_1$. There exist $\dot{q}_i$ with $\dot{q}_i^G = q_i$ and $p_i \in G$ with $(\mathrm{pair}(\dot{q}_i, \check{\alpha}_i), p_i) \in \dot{f}$ for $i < 2$. If q_0 and q_1 were compatible, then some $p \in G$ forces that $\dot{q}_0$ and $\dot{q}_1$ are compatible. Since we can assume $p \leq p_0, p_1$, then $(p_0, \dot{q}_0)$ and $(p_1, \dot{q}_1)$ would be compatible. But $f(p_0, \dot{q}_0) = \alpha_0 \neq \alpha_1 = f(p_1, \dot{q}_1)$ and f is a $\|$-homomorphism. □

By the previous claim, $\mathbf{1}_{\mathbb{P}_\beta}$ forces that $\dot{g}_\beta(\dot{f})$ is an injective function from $\mathrm{ran}(\dot{f})$ into θ. This induces a $\mathbb{P}_\beta$-name $\dot{h}$ for a surjection from θ onto $\mathrm{ran}(\dot{f})$.[15] For each $\alpha < \theta$, let D_α denote the set of all $p \in \mathbb{P}_\beta$ that decide $\dot{h}(\alpha)$. For each $\alpha < \theta$, define a $\|$-homomorphism $h_\alpha\colon D_\alpha \to \mathsf{Ord}$ by letting $h_\alpha(p)$ be the unique δ such that p forces $\dot{h}(\alpha) = \delta$. Since G_β witnesses that $\mathbb{P}_\beta$ is uniformly θ-narrow, $G_\beta(h_\alpha)\colon \mathrm{ran}(h_\alpha) \to \theta$ is injective for each $\alpha < \theta$, and the sequence $(G_\beta(h_\alpha) \mid \alpha < \theta)$ induces an injection from $\bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha)$ to θ. Since $\mathbf{1}_{\mathbb{P}}$ forces $\mathrm{ran}(\dot{f}) \subseteq \bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha)$, we have $\mathrm{ran}(f) \subseteq \bigcup_{\alpha<\theta} \mathrm{ran}(h_\alpha)$ by the definition of $\dot{f}$. Thus G_β induces an injective map $G_\gamma(f)\colon \mathrm{ran}(f) \to \theta$ uniformly in f, and G_γ is definable from $\vec{\mathbb{P}}$, G_β, and ordinals.

Now suppose γ is a limit ordinal. Suppose that $f\colon \mathbb{P}_\gamma \rightharpoonup \mathsf{Ord}$ is a partial $\|$-homomorphism.

Claim. $|\mathrm{ran}(f)|^{\mathsf{HOD}_{\vec{\mathbb{P}},f}} \leq \theta$.

Proof. Fix a wellorder $\leq^*$ of $[\mathsf{Ord}]^{<\omega}$ that is definable without parameters. Define $\vec{s} = \langle s_\alpha \mid \alpha \in \mathrm{ran}(f) \rangle$ as follows. For each $\alpha \in \mathrm{ran}(f)$, let s_α be the $\leq^*$-least element s of $[\gamma]^{<\omega}$ such that there exists some $p \in \mathbb{P}_\gamma$ with support s and $f(p) = \alpha$. We can assume all p with $f(p) = \alpha$ have support s_α for each $\alpha \in \mathrm{ran}(f)$ by restricting f while keeping its range intact.

Towards a contradiction, suppose the claim fails for f. We can assume $|\mathrm{ran}(f)|^{\mathsf{HOD}_{\vec{\mathbb{P}},f}} = \theta^{+\mathsf{HOD}_{\vec{\mathbb{P}},f}}$. By applying the infinite sunflower lemma[16] in $\mathsf{HOD}_{\vec{\mathbb{P}},f}$ and restricting f, we can assume $\vec{s}$ is a sunflower system with centre r. Let $\beta = \max(r) + 1 < \gamma$ if $r \neq \emptyset$ and $\beta = 0$ otherwise. Let $A = \{p{\restriction}\beta \mid p \in \mathrm{dom}(f)\}$ denote the projection of $\mathrm{dom}(f)$ to β. Define $g\colon A \to \mathsf{Ord}$ as follows. For each $p \in A$, let

$$g(p) = \alpha :\Longleftrightarrow \exists s \in \mathrm{dom}(f) \quad (s{\restriction}\beta = p \wedge f(s) = \alpha).$$

To see that g is well-defined, note that for any $p \in A$ and $t, u \in \mathrm{dom}(f)$ with $t{\restriction}\beta = u{\restriction}\beta = p$, we have $f(t) = f(u)$. Otherwise let $f(t) = \zeta$ and $f(u) = \xi$ with $\zeta \neq \xi$. Since $\mathrm{supp}(t) = s_\zeta$, $\mathrm{supp}(u) = s_\xi$, $s_\zeta \cap s_\xi = \mathrm{supp}(p)$ and $\vec{s}$ is a sunflower system with centre r, t and u are compatible. But f is a $\|$-homomorphism and $f(t) \neq f(u)$.

[15] The inverse of $\dot{g}_\beta(\dot{f})$ is partial function from θ onto $\mathrm{ran}(\dot{f})$. One can take $\dot{h}$ to be a nice name for a total function extending it.

[16] We apply Lemma [17, Theorem 1.6], also known as the *Δ-system lemma*, to $\theta^{+\mathsf{HOD}_{\vec{\mathbb{P}},f}}$. Recall that a *sunflower* or *Δ-system* is a collection S of sets such that the intersection $a \cap b = c$, the *centre*, is the same for all $a, b \in S$ with $a \neq b$.

Subclaim. g is a $\|$-homomorphism on $\mathbb{P}_\beta$ with $\mathrm{ran}(f) = \mathrm{ran}(g)$.

Proof. We first show that g is a $\|$-homomorphism. Suppose that $p, q \in A$ with $g(p) \neq g(q)$. By the definition of g, there exist $t, u \in \mathrm{dom}(f)$ with $t{\upharpoonright}\beta = p$, $\mathrm{supp}(t) = s_\zeta$, $u{\upharpoonright}\beta = q$, $\mathrm{supp}(u) = s_\xi$ and $\zeta \neq \xi$. Since $f(t) \neq f(u)$ and f is a $\|$-homomorphism, t and u are incompatible. Since $\vec{s}$ is a sunflower with centre r, this implies $p = t{\upharpoonright}\beta$ and $q = u{\upharpoonright}\beta$ are incompatible.

To see that $\mathrm{ran}(f) = \mathrm{ran}(g)$, take any $\alpha \in \mathrm{ran}(f)$. Pick any p with support s_α and $f(p) = \alpha$. Then $p{\upharpoonright}\beta \in A$ and $g(p{\upharpoonright}\beta) = \alpha$ by the definition of A and g. □

By the inductive hypothesis, G_β witnesses that $\mathbb{P}_\beta$ is uniformly θ-narrow. The previous subclaim yields an injective function $G_\beta(g)\colon \mathrm{ran}(g) \to \theta$. Since G_β and g are definable from $\vec{\mathbb{P}}$, f, and ordinals, we have $|\mathrm{ran}(f)|^{\mathsf{HOD}_{\vec{\mathbb{P}},f}} = |\mathrm{ran}(g)|^{\mathsf{HOD}_{\vec{\mathbb{P}},f}} \leq \theta$, contradicting the assumption. □

Using the previous claim, let $G_\gamma(f)$ be the least injective function $h\colon \mathrm{ran}(f) \to \theta$ in $\mathsf{HOD}_{\vec{\mathbb{P}},f}$. Then the definition of $G_\gamma(f)$ is uniform in f and G_γ is definable from $\vec{\mathbb{P}}$ and ordinals. □

3.2. **Linked forcings.** A forcing $\mathbb{P}$ is called $\mathbb{Q}$-*linked* if there is a $\perp$-homomorphism from $\mathbb{P}$ to $\mathbb{Q}$. We equip each ordinal θ with the discrete partial order. Clearly θ is uniformly θ-narrow. Every θ-linked forcing is uniformly θ-narrow by Lemma 3.3.

Suppose that θ is an ordinal. A *uniform iteration* of θ-linked forcings with finite support is a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{f}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma \rangle$ such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma \rangle$ is a finite support iteration and for each $\alpha < \gamma$, $\mathbf{1}_{\mathbb{P}_\alpha}$ forces that $\dot{\mathbb{P}}_\alpha$ is θ-linked witnessed by $\dot{f}_\alpha$. Uniform products of θ-linked forcings with finite support are defined similarly. Given a uniform iteration of θ-linked forcings, the proof of Lemma 3.3 provides a map that turns a function witnessing θ-linked into a function witnessing θ-narrow. This supplies us with a sequence $\vec{G} = \langle \dot{G}_\alpha \mid \alpha < \gamma \rangle$ such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{G}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma \rangle$ is a uniform iteration of θ-narrow forcings. Note that any uniform product of θ-linked forcings with finite support can also be understood as a uniform iteration of θ-linked forcings. Lemma 3.2 and Theorem 3.4 then have the following corollary.

Corollary 3.5. *Uniform iterations and products of θ-linked forcings with finite support preserve all cardinals and cofinalities $>\theta$.*

For instance, this works for mixed finite support products and iterations in any order of Cohen forcing, Hechler forcing and eventually different forcing. These forcings are ω-linked. To see that products and iterations are uniform, it suffices by Remark 2.3 that there is a definition for a linking function that works uniformly in all models.

Remark 3.6. One can obtain a direct proof that any uniform iteration $\vec{\mathbb{P}}$ of θ-linked forcings preserves cardinals and cofinalities $>\theta$ without going through θ-narrow. Using the above notation, let $\mathbb{Q}$ denote the set of $p \in \mathbb{P}_\gamma$ such that for all $\alpha \in \mathrm{supp}(p)$, $p{\upharpoonright}\alpha$ decides $\dot{f}_\alpha(p(\alpha))$. We will show in Lemma 4.16 below that $\mathbb{Q}$ is dense in $\mathbb{P}_\gamma$. One then constructs a $\perp$-homomorphism from $\mathbb{Q}$ to $\mathrm{Fun}_{<\omega}(\gamma, \theta)$. This can be extended to a $\perp$-homomorphism on $\mathbb{P}_\gamma$. It is now easy to show directly that the existence of a $\perp$-homomorphism from $\mathbb{P}$ to a wellordered θ^+-c.c. forcing implies that $\mathbb{P}$ preserves cardinals $>\theta$. Alternatively we can argue that $\mathrm{Fun}_{<\omega}(\gamma, \theta)$ is θ-narrow, since it is wellordered and θ^+-c.c., so $\mathbb{P}_\gamma$ is θ-narrow by Lemma 3.3 (2) and it thus preserves cardinals and cofinalities $>\theta$ by Lemma 3.2.

3.3. **Locally complete forcings.** We present criteria for showing that a forcing is θ-narrow or uniformly θ-narrow that may be of independent interest. This is useful to show that random algebras are uniformly narrow and thus preserve all cardinals and cofinalities. The notions introduced here are also used to study absoluteness principles in Section 4.2. The next definition is motivated by the property of random forcing that inner models are correct about predense sets.

Lemma 3.7. *The following conditions are equivalent for a forcing $\mathbb{P}$ and a finite set x that contains $\mathbb{P}$:*

(a) *For any regular open $A \subseteq \mathbb{P}$,*[17] $\sup(A \cap \mathsf{HOD}_{x \cup \{A\}}) = \sup(A)$.

[17] A subset A of $\mathbb{P}$ is called *regular open* if $A = \{p \mid p_\iota \leq \sup(A)\}$.

(b) *For any* $B \subseteq \mathbb{P}$ *with* $B \neq \emptyset$, *there is some* $p \in \mathbb{P} \cap \mathsf{HOD}_{x\cup\{B\}}$ *with* $p_\iota \leq \sup(B)$.
(c) *The same as* (b), *but only for regular open subsets* B *of* $\mathbb{P}$.

Proof. (a)⇒(b): Let $A := \{p \mid p_\iota \leq \sup(B)\} \neq \emptyset$. Since A is regular open, $\sup(A \cap \mathsf{HOD}_{x\cup\{A\}}) = \sup(A) \neq \mathbf{0}_{\mathbb{B}(\mathbb{P})}$ by (a). Thus $A \cap \mathsf{HOD}_{x\cup\{A\}} \subseteq A \cap \mathsf{HOD}_{x\cup\{B\}}$ is nonempty. The inclusion holds since $\mathbb{P} \in x$. Any element of $A \cap \mathsf{HOD}_{x\cup\{A\}}$ witnesses (b).

(b)⇒(c) is obvious.

(c)⇒(a): We can assume that A is nonempty. Towards a contradiction, suppose $a := \sup(A) > b := \sup(A \cap \mathsf{HOD}_{x\cup\{A\}})$ in $\mathbb{B}(\mathbb{P})$. $B := \{p \in \mathbb{P} \mid p_\iota \leq a - b\} \neq \emptyset$ is regular open. Since A is regular open, we have $B \subseteq A$. There is some $p \in \mathbb{P} \cap \mathsf{HOD}_{x\cup\{B\}} \subseteq \mathsf{HOD}_{x\cup\{A\}}$ with $p_\iota \leq \sup(B) = a - b$ by (c). The inclusion holds since $\mathbb{P} \in x$. Since B is regular open, $p \in B \subseteq A$. In particular, $p \in A \cap \mathsf{HOD}_{x\cup\{A\}}$. However, $p_\iota \leq a - b$ is incompatible with all elements of $A \cap \mathsf{HOD}_{x\cup\{A\}}$. □

Note that these conditions also hold for any finite superset of x. Any wellorderable forcing $\mathbb{P}$ satisfies them, since one can pick any finite set x that contains a wellorder of $\mathbb{P}$ and thus $\mathbb{P} \in \mathsf{HOD}_x$. Note that (b) is a variant of the existence of suprema. A useful way of proving (b) is to find some $p \in \mathbb{P} \cap \mathsf{HOD}_x$ that is equivalent to $\sup(B)$. We shall do this for random algebras in Section 3.4.2.

Definition 3.8. A forcing $\mathbb{P}$ is called *locally complete* if there exists a finite set x containing $\mathbb{P}$ such that each of the equivalent conditions (a)-(c) in Lemma 3.7 holds.

We now show that for any locally complete forcing $\mathbb{P}$ and x as above, any $\mathbb{P}$-generic filter over V is $(\mathbb{P} \cap \mathsf{HOD}_x)$-generic over HOD_x.

Lemma 3.9. *Suppose that* $\mathbb{P}$ *is locally complete witnessed by* x *and* $y \supseteq x$ *is finite.*
(1) HOD_y *is correct about compatibility of elements of* $\mathbb{P} \cap \mathsf{HOD}_y$.
(2) *Every predense subset* $A \in \mathsf{HOD}_y$ *of* $\mathbb{P} \cap \mathsf{HOD}_y$ *is predense in* $\mathbb{P}$.

Proof. (1) Suppose that $p, q \in \mathbb{P} \cap \mathsf{HOD}_y$ with $p \parallel q$. Then $A := \{r \in \mathbb{P} \mid r \leq p, q\} \neq \emptyset$ is regular open. By (c), there is some $r \in \mathsf{HOD}_{x\cup\{A\}} \subseteq \mathsf{HOD}_y$ with $r_\iota \leq \sup(A) \leq p_\iota, q_\iota$. r witnesses that p and q are compatible in HOD_y.

(2) Towards a contradiction, suppose that $A \in \mathsf{HOD}_y$ is predense in $\mathbb{P} \cap \mathsf{HOD}_y$, but not predense in $\mathbb{P}$. Then $\sup(A) \neq \mathbf{1}_{\mathbb{B}(\mathbb{P})}$. Then $B := \{p \in \mathbb{P} \mid p_\iota \leq -\sup(A)\} \neq \emptyset$ is regular open. By (c), there is some $p \in \mathsf{HOD}_{x\cup\{B\}} \subseteq \mathsf{HOD}_y$ with $p_\iota \leq \sup(B) = -\sup(A)$. This contradicts the assumption that A is predense in $\mathbb{P} \cap \mathsf{HOD}_y$. □

We now study locally complete θ^+-c.c. forcings and a stronger variant. This is motivated by the fact that random forcing is c.c.c. in any inner model ZFC. We say that a property holds for *sufficiently large finite* y if there exists a finite set x such that it holds for all finite sets $y \supseteq x$. Note that a forcing $\mathbb{P}$ is θ^+-c.c. if and only if for sufficiently large finite x, any antichain $A \in \mathsf{HOD}_x$ in $\mathbb{P}$ satisfies $|A|^{\mathsf{HOD}_x} < \theta^+$. The following defines a variant where the successor of θ is calculated in HOD_x, as indicated by the notation $\theta^{[+]}$.

Definition 3.10. For any infinite ordinal θ, a forcing $\mathbb{P}$ is called *locally* $\theta^{[+]}$*-c.c.* if for sufficiently large finite x, any antichain $A \in \mathsf{HOD}_x$ in $\mathbb{P}$ satisfies $|A|^{\mathsf{HOD}_x} \leq \theta$.

Together with local completeness, this notion suffices to show that a forcing is uniformly θ-narrow. This will allow us to show, for example, that random algebras can be iterated without collapsing cardinals.

Lemma 3.11. *Suppose* θ *is an infinite ordinal and* $\mathbb{P}$ *is a locally complete forcing.*
(1) *If* $\mathbb{P}$ *is* θ^+*-c.c., then it is* θ*-narrow.*
(2) *If* $\mathbb{P}$ *is locally* $\theta^{[+]}$*-c.c., then it is uniformly* θ*-narrow.*

Proof. (1) Suppose that $\vec{f} = \langle f_i \mid i < \mu \rangle$ is a sequence of partial $\parallel$-homomorphisms from $\mathbb{P}$ to Ord for some ordinal μ. Let $A_i := \mathrm{dom}(f_i)$ for each $i < \mu$. We partition A_i into $A_{i,\alpha} := \{p \in A_i \mid f_i(p) = \alpha\}$ for $\alpha \in \mathrm{ran}(f_i)$. Let $\vec{A} = \langle A_{i,\alpha} \mid i < \mu,\ \alpha \in \mathrm{ran}(f_i) \rangle$. Take some finite x witnessing $\mathbb{P}$ is locally complete. Let $H := \mathsf{HOD}_{x\cup\{\vec{A}\}}$. For each $i < \mu$ and $\alpha \in \mathrm{ran}(f_i)$, there exists some $p \in \mathbb{P} \cap H$ with $p \leq \sup(A_{i,\alpha})$. Let $p_{i,\alpha}$ be be least such p in the canonical wellorder of H.

Claim. $p_{i,\alpha} \perp p_{i,\beta}$ for any $i < \mu$ and $\alpha \neq \beta$ in $\mathrm{ran}(f_i)$.

Proof. Towards a contradiction, suppose that there exists some $r \leq p_{i,\alpha}, p_{i,\beta}$. Since $r \leq \sup(A_{i,\alpha})$, there exist $s \in A_{i,\alpha}$ and $t \leq r, s$. Since $t \leq \sup(A_{i,\beta})$, t is compatible with some $u \in A_{i,\beta}$. Thus $s \parallel u$. Since f_i is a $\parallel$-homomorphism, $\alpha = f_i(s) = f_i(u) = \beta$. But we assumed $\alpha \neq \beta$. □

First suppose θ^+ is regular in H. By the θ^+-c.c., we have $|\mathrm{ran}(f_i)|^H < \theta^+$ for each $i < \mu$. Since θ^+ is regular in H, we have $|\bigcup_{i<\mu} \mathrm{ran}(f_i)|^H < \theta^+$ if $\mu < \theta^+$. Thus $|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\theta|$ if $\mu < \theta^+$ and $|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\mu|$ if $\mu \geq \theta^+$.

Next suppose θ^+ is singular in H. Let $\lambda \leq \theta^+$ be the chain condition of $\mathbb{P} \cap H$ in H. By a standard fact, the chain condition of a forcing is regular in any model of ZFC. Thus λ is regular in H and $\lambda < \theta^+$. Since H is correct about compatiblity by Lemma 3.9 (1), $|\mathrm{ran}(f_i)|^H < \lambda$ for each $i < \mu$. Thus $|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\theta|$ if $\lambda \geq \mu$ and $|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\mu|$ if $\lambda < \mu$.

(2) We proceed as in (1), but replace $\vec{f}$ by a single $\parallel$-homomorphism f and x by a finite superset that witnesses locally-$\theta^{[+]}$-c.c. The construction in (1) works as above. Since the θ^{+H}-c.c. holds in H by the choice of x, we have $|\mathrm{ran}(f)|^H \leq |\theta|^H$. Let $G(f)$ be the least injective function $F\colon \mathrm{ran}(f) \to \theta$ in H. □

The following is an alternative way of showing cardinal preservation. It uses a notion of capturing similar to the ones studied in [4, 21].

Definition 3.12. Suppose θ is an infinite ordinal. A forcing $\mathbb{P}$ is called *θ-captured* if for any set A of ordinals in any $\mathbb{P}$-generic extension $V[H]$, there are a transitive class model $U \subseteq V$ of ZFC and a generic filter $G \in V[H]$ over U such that:

(a) $A \in U[G]$.
(b) U and $U[G]$ agree on cardinals and cofinalities $>\theta$.

We further say that $\mathbb{P}$ is θ-captured *by* $\mathbb{P}$ if $G = H \cap U$ is $(\mathbb{P} \cap U)$-generic over U in (a).

Lemma 3.13. *Suppose that θ is an infinite ordinal and $\mathbb{P}$ is a forcing.*

(1) *If $\mathbb{P}$ is locally complete and θ^+-c.c., then it is θ-captured by $\mathbb{P}$.*
(2) *If $\mathbb{P}$ is θ-captured, then it preserves cardinals and cofinalities $>\theta$.*

Proof. (1) Suppose that $\mathbb{P}$ is locally complete witnessed by x. Recall that a *nice name* for a subset of λ has elements of the form $(\check{\alpha}, p)$ with $\alpha < \lambda$ and $p \in \mathbb{P}$. Suppose that σ is a nice name for a subset of λ. Let $A_\alpha := \{p \in \mathbb{P} \mid (\check{\alpha}, p) \in \sigma\}$ for each $\alpha < \lambda$ and set $H := \mathsf{HOD}_{x \cup \{\sigma\}}$.

Claim. There exists a name $\dot{g} \in H$ with $\mathbf{1}_{\mathbb{P}} \Vdash \dot{f} = \dot{g}$.

Proof. We construct a sequence $(B_\alpha \mid \alpha < \lambda) \in H$ of antichains with $\sup(A_\alpha) = \sup(B_\alpha)$ for each $\alpha < \lambda$. Then $\dot{g} = \{(\check{\alpha}, p) \mid p \in B_\alpha\}$ is as required. We construct B_α by induction. Let $p_0 \in H$ with $(p_0)_\iota \leq \sup(A_\alpha)$ by (c). Suppose that $\vec{p} = \langle p_i \mid i < \eta \rangle$ has been constructed. If $\sup(\vec{p}) < \sup(A_\alpha)$, then $B := \{p \in \mathbb{P} \mid p_\iota \leq \sup(A_\alpha) - \sup(\vec{p})\} \neq \emptyset$ is regular open. Let $p_\eta \in \mathsf{HOD}_{x \cup \{B\}} \subseteq H$ with $(p_\eta)_\iota \leq \sup(A_\alpha) - \sup(\vec{p})$ by (c). If $\sup(\vec{p}) = \sup(A_\alpha)$, let $B_\alpha = \mathrm{ran}(\vec{p})$. Then B_α is as required. □

Take $\dot{g}$ as in the previous claim and suppose that G is a $\mathbb{P}$-generic filter over V. Then $\dot{f}^G = \dot{g}^G$. Moreover, $G \cap H$ is $(\mathbb{P} \cap H)$-generic over H by Lemma 3.9 (2). This suffices, since $\mathbb{P} \cap H$ is θ^+-c.c. in H.

(2) Suppose that $\lambda > \theta$ is a cardinal, $\gamma < \lambda$ and $f\colon \gamma \to \lambda$ is a function in some $\mathbb{P}$-generic extension $V[H]$. By assumption, f is contained in a model $U[G]$ as above. Since $\lambda > \theta$ is a cardinal in U and therefore in $U[G]$, f is not surjective. A similar argument works for cofinalities. Suppose that $\lambda \in \mathsf{Ord}$ with $\mathrm{cof}(\lambda) > \theta$, $\gamma < \mathrm{cof}(\lambda)$ and $f\colon \gamma \to \lambda$ is a function in $V[H]$. Then f is contained in a model $U[G]$ as above. Since $\gamma < \mathrm{cof}(\lambda) \leq \mathrm{cof}(\lambda)^U = \mathrm{cof}(\lambda)^{U[G]}$, f is not cofinal. □

Recall that $\mathbb{C}^\alpha = \mathrm{Fun}_{<\omega}(\alpha \times \omega, 2)$ is the forcing adding α-many Cohen reals with finite conditions for any ordinal α. Notice that $\mathbb{C}^\kappa$ is wellorderable and locally $\omega^{[+]}$-c.c. for any cardinal κ. Therefore it is uniformly narrow. We will see in the next section that random algebras are locally complete and locally $\omega^{[+]}$-c.c. So they are uniformly narrow as well.

3.4. **Random algebras.** Fix a multiplicatively closed ordinal α.[18] 2^α is equipped with the product topology. Recall that for any ordinal α, we write $2^{(\alpha)}$ for $\mathrm{Fun}_{<\omega}(\alpha, 2)$. The basic open subsets of 2^α are of the form $N_t = \{x \in 2^\alpha \mid t \subseteq x\}$ for $t \in 2^{(\alpha)}$.

3.4.1. *Borel codes.* An *α-Borel code* for a subset of 2^α is a subset of α that codes a set formed from basic open subsets of 2^α via complements and countable unions of length at most α. Its *rank* is the number of steps of this construction. We may fix any standard definition of α-Borel codes such as the one at the end of [13, Section 25] where ω is replaced with α, using the pairing function $\mathrm{p}\colon \mathsf{Ord} \times \mathsf{Ord} \to \mathsf{Ord}$. An *$\alpha$-Borel* subset of 2^α is one with an α-Borel code. For α-Borel codes A_0 and A_1, let $A_0 \vee A_1$, $A_0 \wedge A_1$, $A_0 - A_1$, $-A_0$ denote canonical codes for their union, intersection, difference and complement. The *support* $\mathrm{supp}(A)$ of an α-Borel code A is the union of all $\mathrm{dom}(t)$, where N_t occurs in A. *Borel codes* are by definition α-Borel codes with countable support. A *Borel subset* B of 2^α is one with a Borel code. A *support* of B is by definition a support of a code for B.

Remark 3.14. Any Borel subset B of 2^α has a least support S. We define S as follows. For any $i < \kappa$, let $i \in S$ if there exist some $x \in B$ and $y \notin B$ with $x(j) = y(j)$ for all $j \neq i$. Clearly S is contained in any support of B. To see that S is a support for B, suppose that A is a Borel code for B. Define a Borel code $A{\restriction}S$ by replacing N_t by $N_{t\restriction S}$ for all $t \in 2^{(\alpha)}$. By the definition of S, $A{\restriction}S$ codes the same set as A.

Let μ denote the product measure on 2^α. We sometimes identify a Borel code with the coded set. For instance, we write $\mu(A)$ for $\mu(B)$ if A is a Borel code for the set B. For Borel codes A_0 and A_1, we write $A_0 \leq A_1$ if $\mu(A_0 - A_1) = 0$ and $A_0 =_\mu A_1$ if $A_0 \leq A_1 \wedge A_0 \leq A_1$. Note that for any $x \in 2^\alpha$ and a Borel code A for a set B, the statement "$x \in B$" is absolute to any inner model of ZFC that contains x and A.

3.4.2. *Completeness.* $\bar{\mathbb{R}}_\alpha$ denotes the forcing that consists of all Borel codes for subsets of 2^α ordered by $\leq$. The quotient of $\bar{\mathbb{R}}_\alpha$ by $=_\mu$ with the operations induced by $\vee$, $\wedge$ and $-$ is a Boolean algebra.

A forcing is called *complete* if every subset has a supremum. We will show that $\bar{\mathbb{R}}_\alpha$ is complete. To this end, we associate to every $A \in \bar{\mathbb{R}}_\alpha$ its *footprint* $\mathrm{f}_A = \langle \mathrm{f}_{A,t} \mid t \in 2^{(\alpha)}\rangle$, where

$$\mathrm{f}_{A,t} = \frac{\mu(A \cap N_t)}{\mu(N_t)}$$

denotes the relative measure. Let $\mathrm{f}_A \leq \mathrm{f}_B$ if $\mathrm{f}_{A,t} \leq \mathrm{f}_{B,t}$ for all $t \in 2^{(\alpha)}$. Note that $A \leq B$ if and only if $\mathrm{f}_A \leq \mathrm{f}_B$. If $A \leq B$, then clearly $\mathrm{f}_{A,t} \leq \mathrm{f}_{B,t}$ for all $t \in 2^{(\alpha)}$. If conversely $A \not\leq B$, then for any $\epsilon > 0$ there exists some $t \in 2^{(\alpha)}$ with $\mathrm{f}_{A,t} > 1 - \epsilon$ and $\mathrm{f}_{B,t} < \epsilon$ by Lebesgue's density theorem. Thus $\mathrm{f}_{A,t} > \mathrm{f}_{B,t}$.

We next define density points of subsets of 2^α. For countable ordinals, this definition agrees with the usual one. The quantifiers in the next definition range over $[\alpha]^{<\omega}$.

Definition 3.15. Suppose that $\vec{\mathrm{f}} = \langle \mathrm{f}_s \mid s \in 2^{(\alpha)}\rangle$ is a sequence in $\mathbb{R}$ and $x \in 2^\alpha$.

(1) For any $\epsilon > 0$, x is called an *ϵ-density point* of f if
$$\exists s\ \forall t \supseteq s\ \mathrm{f}_t > 1 - \epsilon.$$
(2) x is called a *density point* of f if it is an ϵ-density point of f for all $\epsilon \in \mathbb{Q}^+$.

The α-Borel code for the set of density points of f induced by (1) and (2) is denoted $D(\mathrm{f})$.

Any α-Borel code can be reduced to a Borel code as follows.

[18] We make this assumption so that an α-Borel code for a subset of 2^α is a subset of α.

Definition 3.16. The *reduct* $\mathrm{red}(A)$ of an α-Borel code A is the following Borel code, defined by induction on the rank:

(1) If A is a code of rank 0, then $\mathrm{red}(A) = A$.
(2) If A_0 is the canonical code for the complement of A_1, then $\mathrm{red}(A_0)$ is the canonical code for the complement of $\mathrm{red}(A_1)$.
(3) If A is the canonical code for the union of $\vec{A} = \langle A_i \mid i < \alpha \rangle$, work in $L[A] = L[\vec{A}]$. Let I be the $L[A]$-least countable subset of α such that for all $j < \alpha$:
$$\mu\big(\bigcup_{i \in I} \mathrm{red}(A_i)\big) = \mu\big(\mathrm{red}(A_j) \cup \bigcup_{i \in I} \mathrm{red}(A_i)\big).$$
$\mathrm{red}(A)$ is the canonical code for $\bigcup_{i \in I} \mathrm{red}(A_i)$.

Note that $\mathrm{red}(A) \in L[A]$. By induction on the rank, we have $\mathrm{red}(A) =_\mu A$ in any outer model where α is countable, for any α-Borel set A. This is used in the next construction. Suppose that X is a subset of $\bar{\mathbb{R}}_\alpha$. To construct a least upper bound, we first form the least upper bound of the footprints: let $\mathrm{f}_{X,t} = \sup_{A \in X} \mathrm{f}_{A,t}$ for each $t \in 2^{(\alpha)}$ and
$$\mathrm{f}_X = \langle \mathrm{f}_{X,t} \mid t \in 2^{(\alpha)} \rangle.$$

Lemma 3.17.

(1) *In any outer model W of V such that α is countable in W, $D(\mathrm{f}_X)$ is a least upper bound for X.*
(2) *$\bar{\mathbb{R}}_\alpha$ is complete. More precisely, for any subset X of $\bar{\mathbb{R}}_\alpha$ the reduct of $D(\mathrm{f}_X)$ is a least upper bound for X.*

Proof. (1): We work in W. Since α is countable, $D(\mathrm{f}_A)$ is Borel. To see that it is an upper bound, suppose that $A \in X$. We have $D(\mathrm{f}_A) =_\mu A$ by Lebesgue's density theorem. Since $\mathrm{f}_A \leq \mathrm{f}_X$, $D(\mathrm{f}_A) \leq D(\mathrm{f}_X)$ as required. To see that $D(\mathrm{f}_X)$ is least, suppose $B \in \bar{\mathbb{R}}_\alpha$ is an upper bound of X. Again $D(\mathrm{f}_B) =_\mu B$ by Lebesgue's density theorem. We have $\mathrm{f}_A \leq \mathrm{f}_B$ for all $A \in X$, since $A \leq B$. Thus $\mathrm{f}_X \leq \mathrm{f}_B$ and therefore, $D(\mathrm{f}_X) \leq D(\mathrm{f}_B)$ as required.

(2): Let A denote the reduct of $D(\mathrm{f}_X)$. In any outer model W where α is countable, $A =_\mu D(\mathrm{f}_X)$. Since $D(\mathrm{f}_X)$ is a least upper bound for X in W by (1), A is a least upper bound for X in V. □

Remark 3.18.

(1) If X is closed under finite unions, then the footprint of the least upper bound of X is precisely f_X. This is clear if X is countable. In general, this can be seen by passing to a generic extension where X is countable.
(2) One can avoid the use of outer models in the proof of Lemma 3.17 by a direct calculation that the reduct of $D(\mathrm{f}_X)$ is a least upper bound for X. If A and B are codes for unions $\bigcup_{i<\alpha} A_i$ and $\bigcup_{i<\alpha} B_i$ with $\mathrm{red}(A_i) \leq \mathrm{red}(B_i)$ for all $i < \alpha$, one shows $\mathrm{red}(\bigcup_{i<\alpha} A_i) \leq \mathrm{red}(\bigcup_{i<\alpha} B_i)$ and a similar statement for intersections.

We now pass to a subforcing $\mathbb{R}_\alpha$ of $\bar{\mathbb{R}}_\alpha$ whose definition is absolute to inner models. This is not the case for $\bar{\mathbb{R}}_\alpha$. This absoluteness is used to show that $\mathbb{R}_\alpha$ is $\omega^{[+]}$-c.c.

Definition 3.19. The *random algebra* $\mathbb{R}_\alpha$ on α generators is the subforcing of $\bar{\mathbb{R}}_\alpha$ that consists of those A such that $\mathrm{supp}(A)$ is countable in $L[A]$.

We also write $\mathbb{R}$ for *random forcing* $\mathbb{R}_\omega$. The next lemma shows that $\mathbb{R}_\alpha$ and $\bar{\mathbb{R}}_\alpha$ are essentially the same forcings, since $\mathbb{R}_\alpha$ meets every equivalence class in $\bar{\mathbb{R}}_\alpha$ with respect to $=_\mu$.

Lemma 3.20. *There is an OD function $F\colon \bar{\mathbb{R}}_\alpha \to \mathbb{R}_\alpha$ that picks a representative in each equivalence class in $\bar{\mathbb{R}}_\alpha$.*

Proof. Let $A \in \bar{\mathbb{R}}_\alpha$ and $[A]$ denote the equivalence class of A. Let $G\colon \bar{\mathbb{R}}_\alpha \to \bar{\mathbb{R}}_\alpha$ denote the function that sends A to the reduct of $D(\mathrm{f}_{[A]})$. There is a formula that defines $G(A)$ in $L[A]$ from A and α by the definition of G. $G(A)$ is equivalent to A, since $G(A)$ is a least upper bound for $[A]$ by Lemma 3.17 (2). Let $A_i := G^i(A)$ for each $i \in \omega$. We have $\omega_1^{L[A_i]} \geq \omega_1^{L[A_{i+1}]}$, since $A_{i+1} \in L[A_i]$. Let $n \in \omega$ be least with $\omega_1^{L[A_n]} = \omega_1^{L[A_{n+1}]}$. By the definition of G, the support of $A_{n+1} = G(A_n)$ is countable in $L[A_n]$ and thus in $L[A_{n+1}]$. Hence $A_{n+1} \in \mathbb{R}_\alpha$ as required. □

Theorem 3.21.

(1) $\mathbb{R}_\alpha$ *is complete.*
(2) $\mathbb{R}_\alpha$ *is locally complete.*
(3) $\mathbb{R}_\alpha$ *is locally* $\omega^{[+]}$*-c.c.*
(4) $\mathbb{R}_\alpha$ *is uniformly narrow.*

Proof. (1): By Lemmas 3.17 and 3.20.

(2): The condition in Lemma 3.7 (b) holds by completeness of $\mathbb{R}_\alpha$ and Lemma 3.20.

(3): $\mathbb{R}_\alpha$ is provably c.c.c. in ZFC. Since the definition of $\mathbb{R}_\alpha$ is absolute to inner models, it is $\omega^{[+]}$-c.c.

(4): By Lemma 3.11. □

3.4.3. *Borel codes versus sets.* We call a subset of 2^ω Borel* if is contained in the σ-algebra generated by basic open sets. In models of DC, the usual definition of random forcing via Borel* sets is isomorphic to ours, since every Borel* set has a Borel code. However, in the model in [12, Theorem 10.6] and in Gitik's model from [7, Theorem I], every set of reals is Borel*. Thus there exist Borel* sets without Borel codes. We now show that it suffices for this that ω_1 is singular.

Remark 3.22. If ω_1 is singular, then there exists a Borel* set without a Borel code. Towards a contradiction, suppose that every Borel* set has a Borel code. Fix a cofinal sequence $\vec{\alpha} = \langle \alpha_n \mid n \in \omega \rangle$ in ω_1. It can be shown by induction that for all countable α, the set WO_α of codes for α is Borel*. Since

$$B = \{\langle 0\rangle^n{}^\frown\langle 1\rangle^\frown x \mid n \in \omega,\ x \in \mathrm{WO}_{\alpha_n}\}$$

is Borel*, it has a Borel code A by assumption. One can then construct a sequence $\vec{A} = \langle A_n \mid n \in \omega\rangle$ of Borel codes for the sets WO_{α_n} from A.

There exists a function that sends each Borel code A' for a Borel set B' to a subtree T of $2^{<\omega} \times \omega^{<\omega}$ with $p[T] = B'$. For instance, take the tree that searches for an assignment of `true` and `false` to each location in the Borel code. We may assume T is pruned by successively removing nodes without successors. The leftmost branch (x, y) of T yields an element x of B'.

By applying this to $\vec{A}$, we obtain a sequence $\vec{x} = \langle x_n \mid n \in \omega\rangle$ with $x_n \in \mathrm{WO}_{\alpha_n}$ for all $n \in \omega$. But this would provide a surjection from ω to ω_1.

A similar argument shows that the existence of Borel codes for all Borel* sets is equivalent to AC_ω for the set of those Borel* sets with a Borel code.

3.5. **Closed forcings.** A forcing $\mathbb{P}$ is called $<\kappa$*-closed* if every decreasing sequence $\vec{p} = \langle p_i \mid i < \alpha\rangle$ in $\mathbb{P}$ with $\alpha < \kappa$ has a lower bound in $\mathbb{P}$. In this section, we analyse some $<\kappa$-closed forcings and the influence of fragments of the axiom of choice on their properties. As an application, we will see that the forcing $\mathbb{C}_{\omega_1} = \mathrm{Col}(\omega_1, 2)$ that adds a Cohen subset of ω_1 collapses ω_1 if $\mathsf{DC}(2^\omega)$ fails, and therefore virtually all bounded support iterations of uncountable length collapse ω_1.

3.5.1. *Dependent choices.* In this section, we call a class R a *relation* on a class A if R is a subclass of $A^{<\mathsf{Ord}} \times A$. An α*-chain* in R is a sequence $\vec{x} = \langle x_i \mid i < \alpha\rangle$ with $(\vec{x}{\restriction}i, x_i) \in R$ for all $i < \alpha$, and R is called $<\gamma$*-extendible* if any α-chain in R for any $\alpha < \gamma$ has a proper end extension.

Definition 3.23. Suppose that A is a class and $\gamma \in \mathsf{Ord}$.

(1) $\mathsf{DC}_\gamma(A)$ denotes the statement:

   Any $<\gamma$-extendible binary relation R on A contains a γ-chain.

(2) For $\delta \leq \mathsf{Ord}$, $\mathsf{DC}_{<\delta}(A)$ denotes the conjunction of $\mathsf{DC}_\alpha(A)$ for all $\alpha < \delta$. $\mathsf{DC}_{\leq\gamma}(A)$ is defined similarly.

(3) $\mathsf{DC}(A)$ denotes $\mathsf{DC}_\omega(A)$.
(4) DC_γ denotes $\mathsf{DC}_\gamma(V)$. $\mathsf{DC}_{<\gamma}$, $\mathsf{DC}_{\leq\gamma}$ and DC are defined similarly.

The next lemma collects some useful facts about variants of DC. In particular, the last claim shows that all axioms are first-order expressible.

Lemma 3.24. *Suppose A and B are classes and $\gamma \in \mathsf{Ord}$.*

(1) (i) $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_{\leq\gamma}(A)$.
 (ii) *If $B \leq_s A$, then* $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_\gamma(B)$.
 (iii) *If $A^\gamma \leq_s A$, then* $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_{<\gamma^+}(A)$.
(2) $\mathsf{DC}_{<\mathsf{Ord}}$ *is equivalent to the axiom of choice.*
(3) $\mathsf{DC}_\gamma(A)$ *follows from* $\mathsf{DC}_\gamma(x)$ *for all sets $x \subseteq A$.*

Proof. (1)(i): Let $\alpha \leq \gamma$ be least such that $\mathsf{DC}_\alpha(A)$ fails for a $<\alpha$-extendible relation R on A. Note that α is a limit ordinal. Since there are no α-sequences in R, the relation is $<\gamma$-extendible. Applying $\mathsf{DC}_\gamma(A)$ yields a contradiction.

(ii): Apply $\mathsf{DC}_\gamma(A)$ to $F^{-1}(R)$, where R is a relation on B and $F\colon A \to B$ is surjective.

(iii): For any $\alpha < \gamma^+$ and any $<\alpha$-extendible relation R on A, let T denote the tree of α-sequences on A in R. T is closed at all levels $<\alpha$ in the sense that any sequence $\vec{t} = \langle t_i \mid i < j\rangle$ for $j < \alpha$ and $t_i \in \mathrm{Lev}_i(T)$ has an upper bound in T, where $\mathrm{Lev}_i(T)$ denotes the ith level of T for $i \in \mathsf{Ord}$. Take any cofinal $\mathrm{cof}(\alpha)$-sequence of levels in T. We can translate the restriction of the tree to these levels to a relation on $A^{<\alpha}$. Note that $A^{<\alpha} \leq_s A^\gamma$. We thus obtain an α-sequence in T from $\mathsf{DC}_\gamma(A^{<\alpha})$ using (i) and (ii). Hence there is an α-sequence in R.

(2): See [12, Theorem 8.1].

(3): Suppose γ is least such that $\mathsf{DC}_\gamma(A)$ fails for some $<\gamma$-extendible relation R on A. We can replace R by a tree T that is closed at all levels $<\gamma$ as in (1)(iii). We construct $\vec{\alpha} = \langle \alpha_j \mid j < \gamma\rangle$ by induction letting α_j be least such that $\mathrm{Lev}_j(T) \cap V_{\alpha_i}$ extends all branches in $T_{<j} := \bigcup_{i<j} \mathrm{Lev}_i(T) \cap V_{\alpha_j}$ using $\mathsf{DC}_j(A \cap V_{\sup_{i<j}\alpha_i})$. $\mathsf{DC}_\gamma(A \cap V_{\sup_{i<\gamma}\alpha_i})$ yields an element of $\mathrm{Lev}_\gamma(T)$ and thus a γ-sequence in R. □

3.5.2. *Cohen subsets and collapses.* We study the forcing $\mathbb{C}_\kappa = \mathrm{Col}(\kappa, 2)$ for adding a Cohen subset to κ. This is the special case of the standard collapse forcing $\mathrm{Col}(\kappa, \lambda)$ for $\lambda = 2$. Since $\mathrm{Col}(\kappa, \lambda)$ is not $<\kappa$-closed if κ is singular, we introduce the following variant.

Definition 3.25.

(1) $\mathrm{Col}(\kappa, \lambda) := \{p\colon \alpha \to \lambda \mid \alpha < \kappa\}$.
(2) $\mathrm{Col}_*(\kappa, \lambda) := \{(f, g) \mid f \in \mathrm{Col}(\kappa, \lambda),\ g\colon \mathrm{dom}(f) \to |\mathrm{dom}(f)| \text{ is bijective}\}$.

$\mathrm{Col}(\kappa, \lambda)$ is ordered by reverse inclusion, while $\mathrm{Col}_*(\kappa, \lambda)$ is ordered by reverse inclusion in the first coordinate.[19]

Any $\mathrm{Col}(\kappa, \lambda)$-generic filter over V induces a $\mathrm{Col}_*(\kappa, \lambda)$-generic filter and conversely, with identical generic extensions. However, $\mathrm{Col}_*(\kappa, \lambda)$ is $<\kappa$-closed for any successor cardinal $\kappa = \nu^+$. To see this, suppose $\vec{p} = \langle (f_\alpha, g_\alpha) \mid \alpha < \lambda\rangle$ is a decreasing sequence in $\mathrm{Col}_*(\kappa, \lambda)$ for some $\lambda \leq \nu$. Let $f = \bigcup_{\alpha<\lambda} f_\alpha$. Since $\langle g_\alpha \mid \alpha < \lambda\rangle$ yields a bijection $\mathrm{dom}(f) \to \mu$ for some $\mu \leq \nu$, $\mathrm{dom}(f) < \kappa$. Thus (f, g) is a lower bound for $\vec{p}$.

Recall that a forcing $\mathbb{P}$ is called λ-*distributive* if for any sequence $\vec{U} = \langle U_\alpha \mid \alpha < \lambda\rangle \in V$ of dense open subsets, $\bigcap_{\alpha<\lambda} U_\alpha$ is dense.[20] $\mathbb{P}$ is called $<\kappa$-*distributive* if it is λ-*distributive* for all $\lambda < \kappa$. We will characterise $<\kappa$-distributivity of $\mathbb{C}_\kappa = \mathrm{Col}(\kappa, 2)$ for successor cardinals κ. We first provide new criteria for λ-distributivity via properties of the generic filter. A characterisation via games in the ground model is known [11, Section 6]. For an infinite cardinal λ, we say that a filter G on $\mathbb{P}$ is $(\mathbb{P}, \lambda)$-*generic* over V if for any sequence $\vec{U} = \langle U_\alpha \mid \alpha < \lambda\rangle \in V$ of dense open sets, $G \cap \bigcap_{\alpha<\lambda} U_\alpha \neq \emptyset$.

[19] The second coordinate is irrelevant for the order.

[20] In models of ZFC, a forcing is λ-distributive if and only if it does not add new λ-sequences of element of V. However, this equivalence can fail in models of ZF by a result of Karagila and Schilhan [14].

Lemma 3.26. *Suppose $\mathbb{P}$ is a forcing and $\lambda \in \mathsf{Card}$. The following conditions are equivalent, where only (b)⇒(c) and therefore also (b)⇒(a) use the additional assumption that G is wellorderable in $V[G]$.*

(a) *$\mathbb{P}$ is λ-distributive.*
(b) *Any $\mathbb{P}$-generic filter G over V is λ-closed in $V[G]$.*
(c) *Any $\mathbb{P}$-generic filter G over V is $(\mathbb{P}, \lambda)$-generic over V.*

Proof. (a)⇒(b): We work in $V[G]$ and show that G is λ-closed. It is easy to see that λ-distributive forcings do not add new λ-sequences. If $\vec{p} = \langle p_i \mid i < \lambda\rangle$ is any sequence in G, then $\vec{p} \in V$. Since $D_i := \{q \in \mathbb{P} \mid q \leq p_i \vee q \perp p_i\}$ is dense in $\mathbb{P}$ for each $i < \lambda$ and the sequence $\langle D_i \mid i < \lambda\rangle$ is in V, the set $\bigcap_{i<\lambda} D_i$ is dense in V. Since G is $\mathbb{P}$-generic, G contains some p in $\bigcap_{i<\lambda} D_i$ and such a p satisfies that $p \leq p_i$ for all $i < \lambda$, as required.

(b)⇒(c): Suppose $\vec{U} = \langle U_i \mid i < \lambda\rangle \in V$ is a sequence of dense open subsets of $\mathbb{P}$ and G is a $\mathbb{P}$-generic filter over V. Since G is wellorderable in $V[G]$, we can construct a decreasing sequence $\langle p_i \mid i < \lambda\rangle$ with $p_i \in G \cap U_i$ in $V[G]$. By assumption, there exists some $p \in G$ with $p \leq p_i$ for all $i < \lambda$. Then $p \in G \cap \bigcap_{i<\lambda} U_i$ as required.

(c)⇒(a): Suppose $\vec{U} = \langle U_\alpha \mid \alpha < \lambda\rangle$ is a sequence of dense open subsets of $\mathbb{P}$. For any $p \in \mathbb{P}$, let G be a $\mathbb{P}$-generic filter over V that contains p. Fix a condition $q \in G \cap \bigcap_{\alpha<\lambda} U_\alpha$ by assumption. Since $p \parallel q$, let $r \leq p, q$. Since each U_α is open, $r \in \bigcap_{\alpha<\lambda} U_\alpha$.

Note that (a)⇒(c) is clear and does not need the extra assumption. □

For instance, if T is a pruned[21] κ-Aronszajn tree and the associated forcing $\mathbb{P}_T$[22] preserves regularity of κ, then T is $<\kappa$-distributive by (b)⇒(a).

Theorem 3.27. *Suppose that A is any set with $|A| \geq 2$, $\lambda \in \mathsf{Card}$ and $\mathbb{P} = \mathrm{Col}(\lambda^+, A)$. The following conditions are equivalent:*

(a) $\mathsf{DC}_\lambda(A^\lambda)$.
(b) *$\mathbb{P}$ is λ-distributive.*
(c) *$\mathbb{P}$ does not change V^λ.*
(d) *$\mathbb{P}$ preserves size and cofinality of all ordinals $\alpha \leq \lambda^+$.*
(e) *$\mathbb{P}$ preserves λ^+ as a cardinal.*
(f) *$\mathbb{P}$ forces that λ^+ is regular.*

The same equivalences hold for $\mathrm{Col}_*(\lambda^+, A)$. In both cases, A may be replaced by any set B with $A \leq_s B \leq_s A^{<\lambda^+}$.

Proof. The following arguments prove the equivalence of (a)-(f) for both $\mathrm{Col}(\lambda^+, A)$ and $\mathrm{Col}_*(\lambda^+, A)$.

(a) ⇒ (b): Since $A^{<\lambda^+} \leq_s A^\lambda$, we have $\mathsf{DC}_{\leq\lambda}(A^{<\lambda^+})$ by Lemma 3.24(1)(ii) and (i). Suppose $\vec{U} = \langle U_i \mid i < \lambda\rangle$ is a sequence of open dense subsets of $\mathbb{P}$. For any $p \in \mathbb{P}$, let T denote the tree of decreasing sequences $\vec{p} = \langle p_i \mid i < \alpha\rangle$ in $\mathbb{P}$ with $\alpha < \lambda$, $p_0 \leq p$ and $p_i \in U_i$ for all $i < \alpha$. By $\mathsf{DC}_{\leq\lambda}(A^{<\lambda^+})$, T is $<\lambda$-extendible and has a branch of length λ. Since λ^+ is regular by $\mathsf{DC}_{\leq\lambda}(A^{<\lambda^+})$, this branch has a lower bound $q \in \mathbb{P}$. Then $q \in \bigcap_{i<\lambda} U_i$.

(b) ⇒ (c): This is a standard argument.

(c) ⇒ (a): It is easy to see that $\mathbb{P}$ adds a wellorder of $A^{<\lambda^+}$. One can thus find the required λ-chain in the generic extension. Since $\mathbb{P}$ does not change $(A^{<\lambda^+})^\lambda$, it exists in V.

(c) ⇒ (d) ⇒ (e): These implications are clear.

(e) ⇒ (f): Let G be $\mathbb{P}$-generic over V and work in $V[G]$. Suppose that $\vec{\alpha} = \langle \alpha_i \mid i < \lambda\rangle$ is cofinal in λ^{+V} for some $\nu \leq \lambda$. Since $\mathbb{P}$ adds a wellorder of $\mathcal{P}(\lambda)^V$, we obtain a sequence $\langle f_i \mid i < \nu\rangle$ of surjections $f_i \colon \lambda \to \alpha_i$ for $i < \nu$. These can be combined to a surjection $f \colon \lambda \to \lambda^{+V}$, contradicting the assumption.

(f) ⇒ (b): Let G be $\mathbb{P}$-generic over V and work in $V[G]$. Since $\mathbb{P}$ adds a wellorder of $A^{<\lambda^+}$, both $\mathrm{Col}(\lambda^+, A)$ and $\mathrm{Col}_*(\lambda^+, A)$ are wellorderable in $V[G]$. Since λ^{+V} is regular in $V[G]$ by assumption, G is $<\lambda^{+V}$-closed. Then $\mathbb{P}$ is λ-distributive by Lemma 3.26.

[21] I.e. $T_s = \{t \in T \mid s \subseteq t \vee t \subseteq s\}$ has height λ for all $s \in T$.
[22] T with its reverse order.

For the additional claim, note that $A^{<\lambda^+} \leq_s A^\lambda$. Thus $\mathsf{DC}_\lambda(A^\lambda)$, $\mathsf{DC}_\lambda(B^\lambda)$ and $\mathsf{DC}_\lambda(A^{<\lambda^+})$ are equivalent by Lemma 3.24(1)(ii). The equivalence of (a)-(f) holds for $\mathsf{DC}_\lambda(A^{<\lambda^+})$ and $\mathbb{P} = \mathrm{Col}(\lambda^+, A^{<\lambda^+})$. Since $A \leq_s B \leq_s A^{<\lambda^+}$, there exist projections $\mathrm{Col}(\lambda^+, A^{<\lambda^+}) \to \mathrm{Col}(\lambda^+, B^\lambda) \to \mathrm{Col}(\lambda^+, A^\lambda)$. We thus obtain equivalences of (a)-(f) for $\mathsf{DC}_\lambda(B^\lambda)$ and $\mathbb{P} = \mathrm{Col}(\lambda^+, B^\lambda)$ from the previous ones. □

For $\lambda = \omega$, regularity of ω_1 is not sufficient to obtain the above conditions. For instance, in Cohen's first model ω_1 is regular while $\mathsf{DC}(2^\omega)$ fails. By Theorem 3.27, virtually any bounded support iteration of length ω_1 collapses ω_1 over models where $\mathsf{DC}(2^\omega)$ fails. Note that the result does not have an analogue for singular limit cardinals, since $\mathrm{Col}(\aleph_\omega, 2)$ forces $\aleph_\omega^V$ to be countable in ZFC. We finally add a further characterisation via forcing axioms to Theorem 3.27.

Remark 3.28. DC_κ can be characterised via the forcing axiom $\mathsf{FA}_\kappa(<\kappa\text{-closed})$ for any $\kappa \in \mathsf{Card}$. This axiom states that for any sequence $\vec{D} = \langle D_i \mid i < \kappa\rangle$ of predense subsets of a $<\kappa$-cosed forcing $\mathbb{P}$, there exists a filter g on $\mathbb{P}$ with $g \cap D_i \neq \emptyset$ for all $i < \kappa$. $\mathsf{FA}_\kappa(<\kappa\text{-closed}, A)$ for a set A denotes $\mathsf{FA}_\kappa(<\kappa\text{-closed})$ restricted to forcings $\mathbb{P} \subseteq A$. One can show the following equivalences for any set A with $A^{<\kappa} \leq_s A$:

(1) $\mathsf{DC}_\kappa(A) \Longleftrightarrow \mathsf{DC}_{<\kappa}(A) + \mathsf{FA}_\kappa(<\kappa\text{-closed}, A)$
(2) $\mathsf{AC} \Longleftrightarrow \forall\lambda \in (\mathsf{SucCard} \cup \mathsf{Reg})\ \mathsf{FA}_\lambda(<\lambda\text{-closed})$

For instance, $\mathsf{DC}(2^\omega)$ is equivalent to $\mathsf{FA}_\omega(\sigma\text{-closed}, 2^\omega)$ and thus to the remaining conditions in Theorem 3.27 for $A = 2$ and $\lambda = \omega$. Viale proved a related result in [25, Theorem 1.8].

(1): Suppose that $\mathsf{DC}_\kappa(A)$ holds. To show $\mathsf{FA}_\kappa(<\kappa\text{-closed}, A)$, suppose that $\vec{D} = \langle D_i \mid i < \kappa\rangle$ is a sequence of predense subsets of a $<\kappa$-closed forcing $\mathbb{P} \subseteq A$. Let $T \subseteq A^{<\kappa}$ be the tree of sequences $\langle p_i \mid i < \alpha\rangle$ in $\mathbb{P}$ with $\alpha < \kappa$, $p_i \in D_i$ and $p_j \leq p_i$ for all $i < j < \alpha$. An application of $\mathsf{DC}_\kappa(T)$ yields a sequence of length κ and thus a filter on $\mathbb{P}$ as required.

Conversely, suppose that $\mathsf{DC}_{<\kappa}A)$ and $\mathsf{FA}_\kappa(<\kappa\text{-closed}, A)$ hold. Suppose that R is a $<\kappa$-extendible relation on A. Let $T \subseteq A^{<\kappa}$ denote the tree of $<\kappa$-chains in R ordered by end extension. T is $<\kappa$-closed by $\mathsf{DC}_{<\kappa}(T)$. For any $\alpha < \kappa$, let D_α denote the set of chains in R of length at least α. $\mathsf{FA}_\kappa(T)$ yields a branch in T that meets each D_α, inducing a κ-chain in R.

(2): Since $\mathsf{DC}_{<\kappa}$ implies DC_κ at singular limits, the claim follows from (1) by induction.

4. Absoluteness principles

We begin with a definition of the absoluteness principles described in the introduction. Let $M \equiv N$ denote that M and N are elementarily equivalent.

Definition 4.1. The *unrestricted absoluteness principle* $\mathsf{A}_\mathcal{C}$ for a class $\mathcal{C}$ of forcings states that $V \equiv V[G]$ for any generic extension of V by a forcing in $\mathcal{C}$.

More precisely, $\mathsf{A}_\mathcal{C}$ is the scheme of all formulas $\forall\mathbb{P} \in \mathcal{C}\ (\varphi \Longleftrightarrow \mathbf{1} \Vdash_\mathbb{P} \varphi)$, where φ ranges over all sentences. Our main goal is to understand the consequences of unrestricted absoluteness for various classes $\mathcal{C}$. Note that $\mathsf{A}_\mathbb{C}$ holds in any Cohen extension and $\mathsf{A}_\mathbb{R}$ holds in any random extension of a model of ZFC. Recall $\mathbb{C}^*$, $\mathbb{R}_*$ and $\mathbb{H}^{(*)}$ denote the class of finite support products of Cohen forcings, random algebras, and finite support iterations of Hechler forcings, respectively.

4.1. **Cohen versus random models.** We show that for any sufficiently large cardinal κ, extensions by $\mathbb{C}^\kappa$ and $\mathbb{R}_\kappa$ have different theories. The argument is due to Woodin.

Lemma 4.2 (Truss [23])**.** *If x is a Cohen real over $L[y]$ where y is a real, then y is not a random real over $L[x]$.*

Proof. The proof relies on the argument showing that a Cohen real x adds a Borel code A for a null set containing all ground model reals. It suffices to show $A \in L[x]$, since this implies ground model reals are not random over $L[x]$. To see this, let $\vec{B} = \langle B^k_m \mid k, m < \omega\rangle$ be a constructible list of codes for all basic open sets with measure at most 2^{-k} for each $k \in \omega$. Suppose x is Cohen generic over $L[y]$. Let $z \in \omega^\omega$ list the distances of successive $n \in \omega$ with

$x(n) = 1$.[23] Let C_n be a Borel code for $\bigcup_{k\geq n} B^k_{z(k)}$ for each $n \in \omega$. Since $\mu(C_n) \leq 2^{-(n-1)}$, $\bigcap_{n\in\omega} C_n$ is a null set. Let A be a Borel code for $\bigcap_{n\in\omega} C_n$ in $L[x]$. Since x is Cohen generic over $L[y]$, it is easy to show that for every $n \in \omega$, every real $w \in L[y]$ is an element of C_n as required. □

We will use that a random real over V is also random over any inner model M. To see this, note that every maximal antichain $B \in M$ of $\mathbb{R} \cap M$ is maximal in $\mathbb{R}$, since B is maximal if and only if $\mu(\bigcup B) = 1$. Note that this holds for $\mathbb{R}_\kappa$ as well. For any $\mathbb{C}^\kappa$-generic filter G over V and subset S of κ, let G_S denote the set of all $p \in G$ with $\operatorname{supp}(p) \subseteq S$. We use the same notation for $\mathbb{R}_\kappa$.

Lemma 4.3. *Suppose $\lambda \leq \kappa$ are infinite cardinals.*

1. *(Woodin) If H is $\mathbb{C}^\kappa$-generic over V then in $V[H]$, for any subset A of κ of size $<\lambda$, H_λ adds a Cohen real over $L[A]$.*
2. *As in (1), but for $\mathbb{R}_\kappa$ and random reals.*

Proof. We can assume $\lambda \geq \omega_1$. We have shown above that $\mathbb{C}^\kappa$ and $\mathbb{R}_\kappa$ preserve cardinals.

(1): Since $\mathbb{C}^\kappa$ is wellorderable, there is a transitive class model $U \subseteq V$ of ZFC such that H is $\mathbb{C}^\kappa$-generic over U and $A \in U[G]$. By the ω_1-c.c. of $\mathbb{C}^\kappa$ in U, $A \in U[H_S]$ for some subset $S \in U$ of κ with $|S|^U < \lambda$. Any coordinate in $\lambda \setminus S$ induces a Cohen real x over $U[H_S]$. Since $A \in U[H_S]$, x is a Cohen real over $L[A]$.

(2): By Lemmas 3.13 and 3.21, there is a transitive class model $U \subseteq V$ of ZFC such that $G := H \cap U$ is $(\mathbb{R}_\kappa \cap U)$-generic over U and $A \in U[G]$. By the ω_1-c.c. of $\mathbb{R}_\kappa \cap U$ in U, $A \in U[G_S]$ for some subset $S \in U$ of κ with $|S|^U < \lambda$. Any countably infinite set of coordinates in $\lambda \setminus S$ induces a random real x over $U[G_S]$ by the proofs of [2, Lemmas 3.1.5, 2.1.6 & 3.2.8].[24] Since $A \in U[G_S]$, x is a random real over $L[A]$. □

Lemma 4.4. *Suppose κ is an uncountable cardinal.*

1. *(Woodin) If H is $\mathbb{C}^\kappa$-generic over V then in $V[H]$, there exists an subset A of ω_1 such that there exists no random real over $L[A]$.*
2. *As in (1), but for $\mathbb{R}_\kappa$ and Cohen reals.*

Proof. (1): Suppose that $y \in V[H]$ is a random real over $L[H_{\omega_1}]$. Since y is a countable subset of κ in $V[H]$, by Lemma 4.3 (1), H_{ω_1} adds a Cohen real x over $L[y]$. By Lemma 4.2, y is not random over $L[x]$. Therefore y cannot be a random real over $L[H_{\omega_1}]$.

(2): Suppose that $y \in V[H]$ is a Cohen real over $L[H_{\omega_1}]$. Since y is a countable subset of κ in $V[H]$, by Lemma 4.3 (2), H_{ω_1} adds a random real x over $L[y]$. By Lemma 4.2, y is not a Cohen real over $L[x]$. Therefore y cannot be a Cohen real over $L[H_{\omega_1}]$, □

In fact, the proof of (1) shows that in $V[H]$, for any cardinal λ with $\omega_1 \leq \lambda \leq \kappa$, there exists an subset A of λ of size λ such that there exists no random real over $L[A]$. A similar claim holds for (2).

In the next theorem, let $\mathcal{C}$ denote the class of all forcings of the form $\mathbb{C}^\kappa$ or $\mathbb{R}_\kappa$ for any $\kappa \in \mathsf{Card}$.

Theorem 4.5 (Woodin)**.** *$\mathsf{A}_\mathcal{C}$ fails. In fact, there is a single switch that works for all models of* ZF*.*

Proof. After forcing with $\mathbb{R}_\kappa$ for any $\kappa \geq \omega_2$, for any subset A of ω_1 there exists a random real over $L[A]$ by Lemma 4.3 (2). However, this statement is false after forcing with $\mathbb{C}^\lambda$ for any $\lambda \geq \omega_1$ by Lemma 4.4 (1). An alternative proof works for $\mathbb{C}^\kappa$ and $\mathbb{R}_\lambda$ using Lemmas 4.3 (1) and 4.4 (2). □

[23] Thus $\sum_{i<k} z(i)$ is the kth $n \in \omega$ with $x(n) = 1$.

[24] These results are formulated for 2^ω but work as well for 2^κ.

4.2. **The continuum.** Recall that $\mathbb{C}^*$ denotes the class of finite support products of Cohen forcings and $\mathbb{R}_*$ denotes the class of all random algebras. The previous section suggests to study the classes $\mathbb{C}^*$ and $\mathbb{R}_*$ separately. We will see that each of $\mathsf{A}_{\mathbb{C}^*}$ and $\mathsf{A}_{\mathbb{R}_*}$ implies that all limit ordinals have countable cofinality. To this end, we analyse the following characteristic in generic extensions:

$$\mathbf{c} := \sup\{\lambda \in \mathsf{Card} \mid \lambda \leq_i 2^\omega\}.$$

We say that $\mathbf{c}$ is *attained* if $\mathbf{c} \leq_i 2^\omega$. Recall that the *Hartogs number* $\aleph(x)$ of a set x is defined as the least ordinal α such that $\alpha \not\leq_i x$. If $\mathbf{c}$ is attained, then $\mathbf{c}^+ = \aleph(2^\omega)$. If $\mathbf{c}$ is not attained, then $\mathbf{c}$ is a limit cardinal with $\mathbf{c} = \aleph(2^\omega)$. Woodin proved Theorem 4.14 (2) for the class $\mathbb{C}^*$ in response to the authors' Remark 4.6. The proofs in this section are extensions of Woodin's argument.

Remark 4.6. $\mathsf{A}_{\mathbb{C}^*}$ implies that there cannot exist two distinct uncountable regular cardinals. To see this, suppose that $\kappa < \lambda$ are the first two uncountable regular cardinals. We claim that $\mathbb{C}^\nu$ forces $\mathbf{c} = \nu$ for any ω-strong limit cardinal ν of uncountable cofinality. Then we would have $\mathrm{cof}(\mathbf{c}) = \kappa$, the first uncountable regular cardinal, in generic extensions by $\mathbb{C}^\nu$ when ν is an ω-strong limit cardinal of cofinality κ while we would have $\mathrm{cof}(\mathbf{c}) = \lambda$, the second uncountable regular cardinal, in generic estensions by $\mathbb{C}^\nu$ when ν is an ω-strong limit cardinal of cofinality λ, contradicting $\mathsf{A}_{\mathbb{C}^*}$. If the claim fails, then one can obtain $\nu^+ \leq_i \mathcal{P}_{\omega_1}(\nu)$ by working with nice names for reals. Since $\mathrm{cof}(\nu) \geq \omega_1$, we have $\mathcal{P}_{\omega_1}(\nu) = \bigcup_{\mu<\nu} \mathcal{P}_{\omega_1}(\mu)$, and then $\nu \leq_i \mathcal{P}_{\omega_1}(\mu)$ for some $\mu < \nu$. But this contradicts the fact that ν is an ω-strong limit.

Moreover, note that at least two uncountable regular cardinals exist if there exists at least one and $\mathsf{A}_{\mathcal{C}}$ holds for the class $\mathcal{C}$ of all forcings of the form $\mathrm{Col}(\omega, \kappa)$ or $\mathrm{Col}(\omega, {<}\kappa)$, where $\kappa \in \mathsf{Card}$. If κ is an uncountable regular cardinal, then $\mathrm{Col}(\omega, {<}\kappa)$ forces that ω_1 is regular and thus ω_1 is regular in V by $\mathsf{A}_{\mathcal{C}}$. Then any infinite successor cardinal λ^+ is regular by $\mathsf{A}_{\mathcal{C}}$, since otherwise $\mathrm{Col}(\omega, \lambda)$ would force that ω_1 is singular.

We now proceed with a more general argument that works for instance for the classes $\mathbb{C}^*$ and $\mathbb{R}_*$. This is based on a proof of Woodin for $\mathbb{C}^*$.

Definition 4.7. $\mathbb{P}$ is called *nice* if for all ordinals ν, if $p \Vdash_{\mathbb{P}} \nu \leq_i 2^\omega$ for some $p \in \mathbb{P}$ then $\nu \leq_i \mathbb{P}^\omega$.

The idea for this definition is that the required function $\nu \to \mathbb{P}^\omega$ sends each $\alpha < \nu$ to a nice name for the αth real as in the proof of the next lemma.

Lemma 4.8. *Every locally complete locally $\omega^{[+]}$-c.c. forcing $\mathbb{P}$ is nice.*[25]

Proof. Suppose $p \Vdash_{\mathbb{P}} \lambda \leq_i 2^\omega$. Then there exists some $q \leq p$ and a sequence $\vec{\sigma} = \langle \sigma_\alpha \mid \alpha < \lambda\rangle$ of $\mathbb{P}$-names for reals with $q \Vdash \sigma_\alpha \neq \sigma_\beta$ for all $\alpha < \beta < \lambda$. Suppose x witnesses that $\mathbb{P}$ is locally complete. We can assume x also witnesses that $\mathbb{P}$ is locally $\omega^{[+]}$-c.c. and contains $\mathbb{P}$, q and $\vec{\sigma}$. For each $\alpha < \lambda$ and $n < \omega$, let $A_{\alpha,n}$ denote the set of all conditions forcing $n \in \sigma_\alpha$. Since $\sup(A_{\alpha,n} \cap \mathsf{HOD}_x) = \sup(A_{\alpha,n})$ by (a) in Lemma 3.7, there exists an antichain $A'_{\alpha,n}$ in HOD_x with supremum $\sup(A_{\alpha,n})$. Let $A'_{\alpha,n}$ be least in HOD_x. It is countable in HOD_x by the $\omega^{[+]}$-c.c. Let $\vec{p}_{\alpha,n} = \langle p_{\alpha,n,i} \mid i \in \omega\rangle$ be the least enumeration in HOD_x of $A'_{\alpha,n}$ of order type ω. Let

$$\tau_\alpha = \{(\check{n}, p_{\alpha,n,i}) \mid n, i < \omega\}.$$

Then $q \Vdash_{\mathbb{P}} \sigma_\alpha = \tau_\alpha$. Therefore, the map sending $\alpha < \lambda$ to $\vec{p}_\alpha := \langle p_{\alpha,n,i} \mid n, i \in \omega\rangle \in \mathbb{P}^{\omega\times\omega}$ is injective as required. □

Let $\mathcal{P}_{\omega^{[+]}}(\nu) = \{x \in \mathcal{P}_{\omega_1}(\nu) \mid \mathsf{HOD}_{\{x\}} \models |x| \leq \omega\}$ denote the set of *locally countable* subsets of an ordinal ν. It is easy to see that $\mathcal{P}_{\omega^{[+]}}(\nu)^\omega \leq_i \mathcal{P}_{\omega^{[+]}}(\nu)$ if $\omega \cdot \nu = \nu$.

Assumption 4.9. $\vec{\mathbb{P}} = \langle \mathbb{P}_\kappa \mid \kappa \in \mathsf{Card}\rangle$ denotes a sequence of forcings with the properties:

[25] The ω_1-c.c. suffices for Lemma 4.8 if we work with a different definition of nice. In Definition 4.7, replace 2^ω by $\mathcal{P}_{\omega_1}(\omega_1)$ and $\mathbb{P}^\omega$ by $\mathbb{P}^{<\omega_1}$. If in addition 2^ω is replaced by $\mathcal{P}_{\omega_1}(\omega_1)$ in the definition of $\mathbf{c}$, all remaining proofs in this section go through.

(a) $\mathbb{P}_\kappa \leq_i \mathcal{P}_{\omega^{[+]}}(\kappa)$.
(b) $\mathbb{P}_\kappa$ adds a κ-sequence of distinct reals.
(c) $\mathbb{P}_\kappa$ is nice.
(d) $\mathbb{P}_\kappa$ is ω-narrow.
(e) $\mathbb{P}_\kappa$ preserves all cardinals $\leq\kappa$.

Conditions (c)-(e) hold if $\mathbb{P}_\kappa$ is locally complete and $\omega^{[+]}$-c.c. by Lemmas 3.2, 3.11 (2) and 4.8. For example, all conditions hold for $\mathbb{C}^\kappa$ and $\mathbb{R}_\kappa$.

Definition 4.10.

(1) I *covers* J if for each $y \in J$, there exists some $x \in I$ with $x \supseteq y$.
(2) For any cardinal ν, a subset J of $\mathcal{P}_{\omega_1}(\nu)$ of size $\mathbf{c}$ is called *jumbled* if it is not covered by any subset I of $\mathcal{P}_{\omega_1}(\nu)$ of size $<\mathbf{c}$.
(3) $\mathbf{j}$ denotes the least cardinal ν such that there exists a jumbled subset of $\mathcal{P}_{\omega_1}(\nu)$, if this exists.

Note that $\mathbf{j}$ is uncountable and $\mathbf{j} \leq \max(\mathbf{c}, \omega_1)$ if $\mathbf{j}$ exists. Using this, it is easy to see that $\mathbf{j}$ does not exist if $\mathbf{c} = \omega$ and ω_1 is regular. However, $\mathbf{j}$ exists in many interesting cases, for example $\mathbf{j} \leq \mathbf{c}$ if $\mathbf{c} \geq \omega_2$.

We will analyze the circumstances in which $\mathbf{j} \geq \mathbf{c}$ holds in $\mathbb{P}_\kappa$-generic extensions. Note that for any uncountable regular cardinal λ, there exist arbitrarily large ω-inaccessible cardinals of cofinality λ.[26] The next lemma was proved by Woodin for $\mathbb{C}^\kappa$.

Lemma 4.11.

(1) $\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \mathbf{c} = \kappa$ *for any ω-inaccessible cardinal κ.*
(2) $\mathbf{1}_{\mathbb{P}_\kappa} \Vdash (\mathbf{c} = \kappa \Rightarrow \mathbf{j} \geq \mathbf{c})$ *for any ω-strong limit cardinal κ.*

Proof. (1): Otherwise we have $p \Vdash_{\mathbb{P}_\kappa} \mathbf{c} > \kappa$ for some $p \in \mathbb{P}_\kappa$ by (b). Since $\mathbb{P}_\kappa$ is nice by (c), $\kappa^+ \leq_i \mathbb{P}_\kappa^\omega$. Since $\mathbb{P}_\kappa^\omega \leq_i \mathcal{P}_{\omega^{[+]}}(\kappa)^\omega \leq_i \mathcal{P}_{\omega^{[+]}}(\kappa)$ by (a), $\kappa^+ \leq_i \mathcal{P}_{\omega_1}(\kappa)$. Since $\operatorname{cof}(\kappa) > \omega$ by assumption, we have $\mathcal{P}_{\omega_1}(\kappa) = \bigcup_{\nu<\kappa} \mathcal{P}_{\omega_1}(\nu)$ and hence $\kappa \leq_i \mathcal{P}_{\omega_1}(\nu)$ for some $\nu < \kappa$, contradicting that κ is an ω-strong limit.

(2): Let $V[G]$ be a $\mathbb{P}_\kappa$-generic extension of V. We work in $V[G]$. Suppose that $\nu < \kappa = \mathbf{c}$ and B is a subset of $\mathcal{P}_{\omega_1}(\nu)$ of size κ. We claim that B is not jumbled. It suffices to find a wellorderable subset $A \in V$ of $\mathcal{P}_{\omega_1}(\nu)$ that covers B. Since κ is an ω-strong limit in V, $|A| < \kappa$ follows. Fix a bijective function $f \colon \kappa \to B$. Let $\dot{g}$ be a $\mathbb{P}_\kappa$-name for the function $g \colon \kappa \times \omega_1 \to \nu$ that sends (α, β) to the βth element of $f(\alpha)$ if $f(\alpha)$ has order type $>\beta$ and 0 otherwise. Let $\dot{f}$ be a $\mathbb{P}$-name for the bijection f and $p \in \mathbb{P}$ force that $\dot{g}$ satisfies the definition of g with $\dot{f}$ described in the last sentence. For each $(\alpha, \beta) \in \kappa \times \omega_1$, let $D_{\alpha,\beta}$ denote the set of all conditions $\leq p$ in $\mathbb{P}_\kappa$ that decide $\dot{g}(\alpha)(\beta)$. Define $g_{\alpha,\beta} \colon D_{\alpha,\beta} \to \nu$ such that $r \Vdash \dot{g}(\alpha)(\beta) = g_{\alpha,\beta}(r)$ for each $r \in D_{\alpha,\beta}$. Then $g_{\alpha,\beta}$ is a $\|$-homomorphism. Further define $h \colon \kappa \times \omega_1 \to \mathcal{P}_{\omega_1}(\nu)$ by letting

$$h(\alpha, \beta) = \bigcup_{\beta' < \beta} \operatorname{ran}(g_{\alpha,\beta'}).$$

Since $\mathbb{P}_\kappa$ is ω-narrow by (d), $h(\alpha, \beta)$ is countable in V. Finally, $A := \operatorname{ran}(h)$ covers $\operatorname{ran}(f)$, since $f(\alpha) \subseteq h(\alpha, \beta)$ if $f(\alpha)$ has order type $\beta < \omega_1$. □

Recall that $\mathbf{c}$ *is attained* if there exists an injective function from $\mathbf{c}$ into the reals. If some $p \in \mathbb{P}$ decides the value of $\mathbf{c}$ in a $\mathbb{P}$-generic extension, then we write $\mathbf{c}^{\mathbb{P}/p}$ for this value. The next lemma is a stronger version of a lemma of Woodin for $\mathbb{C}^\kappa$.

Lemma 4.12. *If $\nu \in \mathsf{Card}$, $p \in \mathbb{P}_\nu$ forces that $\mathbf{c}$ is attained and $\mathbf{c}^{\mathbb{P}_\nu/p} \geq \mathbf{c}^{++}$, then $p \Vdash_{\mathbb{P}_\nu} \mathbf{j} \leq \nu$.*

Proof. Let $\lambda := \mathbf{c}^{\mathbb{P}_\nu/p}$. Since p forces that $\mathbf{c}$ is attained and $\mathbb{P}$ is nice by (c), $\lambda \leq_i \mathbb{P}_\nu^\omega$. Since $\mathbb{P}_\nu^\omega \leq_i \left(\mathcal{P}_{\omega^{[+]}}(\nu)\right)^\omega \leq_i \mathcal{P}_{\omega^{[+]}}(\nu)$ by (a), $\lambda \leq_i \mathcal{P}_{\omega_1}(\nu)$. We claim that any subset of $\mathcal{P}_{\omega_1}(\nu)$ of size λ in V is jumbled in $V[G]$ for any $\mathbb{P}_\kappa$-generic G over V with $p \in G$. To see this, fix an injective function $f \colon \lambda \to \mathcal{P}_{\omega_1}(\nu)$ in V. If $\operatorname{ran}(f)$ is not jumbled, then there exists some $\mu < \lambda$,

[26]Lemma 4.11 and thus Theorem 4.14(1) use a weaker condition than κ being an ω-strong limit, namely for each $\nu < \kappa$ there exists no injection from κ into $\mathcal{P}_{\omega_1}(\nu)$.

a $\mathbb{P}_\nu$-name $\dot{g}$ for a function $\dot{g}\colon \mu \to \mathcal{P}_{\omega_1}(\nu)$ such that some $q \leq p$ forces that $\mathrm{ran}(\dot{g})$ covers $\mathrm{ran}(f)$. The next step is similar to the proof of Lemma 4.11 (2). For any $x \in \mathcal{P}_{\omega_1}(\nu)$ and $\beta < \omega_1$, let $x(\beta)$ be the βth element of x if the order type of x is $>\beta$ and 0 otherwise. For each $(\alpha, \beta) \in \mu \times \omega_1$, let $D_{\alpha,\beta}$ denote the set of all conditions $\leq q$ in $\mathbb{P}_\kappa$ that decide $\dot{g}(\alpha)(\beta)$. Define $g_{\alpha,\beta}\colon D_{\alpha,\beta} \to \nu$ such that $r \Vdash \dot{g}(\alpha)(\beta) = g_{\alpha,\beta}(r)$. $g_{\alpha,\beta}$ is a $\|$-homomorphism. Further define $h\colon \mu \times \omega_1 \to \mathcal{P}_{\omega_1}(\nu)$ by letting

$$h(\alpha, \beta) = \bigcup_{\beta' < \beta} \mathrm{ran}(g_{\alpha,\beta'}).$$

Since $\mathbb{P}_\nu$ is ω-narrow by (d), each $h(\alpha, \beta)$ is countable. Let $\vec{A} = \langle A_{\alpha,\beta} \mid \alpha < \mu,\ \beta < \omega_1\rangle$, where

$$A_{\alpha,\beta} := \{\gamma < \lambda \mid f(\gamma) \subseteq h(\alpha, \beta)\}.$$

Since $h(\alpha, \beta)$ is countable, $\mathrm{otp}(A_{\alpha,\beta}) < \mathbf{c}^+$ for all $(\alpha, \beta) \in \mu \times \omega_1$. Then $\bigcup \vec{A} = \lambda$ by the choice of $\dot{g}$. We obtain the following contradiction. Recall that $\lambda = \mathbf{c}^{\mathbb{P}_\nu/p} \geq \mathbf{c}^+$. If $\mathbf{c}$ is not attained in V, then $\mathrm{otp}(A_{\alpha,\beta}) < \mathbf{c} < \lambda$ for all $(\alpha, \beta) \in \mu \times \omega_1$ and hence $|\bigcup \vec{A}| < \lambda$. If $\mathbf{c}$ is attained in V and $\lambda \geq \mathbf{c}^{++}$, then similarly $|\bigcup \vec{A}| < \lambda$. □

Remark 4.13. The assumption $\mathbf{c}^{\mathbb{P}_\nu/p} \geq \max(\mathbf{c}^+, \omega_2)$ suffices instead of $\mathbf{c}^{\mathbb{P}_\nu/p} \geq \mathbf{c}^{++}$ in Lemma 4.12 if 2^ω is replaced by $\mathcal{P}_{\omega_1}(\omega_1)$ in the definition of $\mathbf{c}$. For the missing case in the proof of Lemma 4.12 when $\mathbf{c}$ is attained in V and $\lambda = \mathbf{c}^+$, note that there exists an injective function $i\colon \mu \times \omega_1 \to \mathcal{P}_{\omega_1}(\omega_1)$, since $\omega_1, \mu < \lambda = \mathbf{c}^+$. Let $\mathrm{otp}_{\alpha,\beta}\colon h(\alpha, \beta) \to \mathrm{otp}(h(\alpha, \beta))$ denote the transitive collapse and define $j\colon \lambda \to \mathcal{P}_{\omega_1}(\omega_1)^2$ by letting $j(\gamma) = (i(\alpha, \beta), \mathrm{otp}_{\alpha,\beta}[f(\gamma)])$, where $(\alpha, \beta) \in \mu \times \omega_1$ is lexically least with $\gamma \in A_{\alpha,\beta}$. Then j is injective, contradicting $\lambda = \mathbf{c}^+$.

The next theorem was proved by Woodin for the class $\mathbb{C}^*$.

Theorem 4.14. *Suppose* $\mathsf{A}_{\vec{\mathbb{P}}}$ *holds.*

(1) $\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \mathbf{c} > \kappa$ *for any ω-strong limit cardinal κ.*
(2) *All infinite cardinals have countable cofinality.*

Proof. (1): Towards a contradiction, suppose that there exists an ω-strong limit cardinal κ with $p \Vdash_{\mathbb{P}_\kappa} \mathbf{c} = \kappa$ for some $p \in \mathbb{P}_\kappa$. By Lemma 4.11 (2), $p \Vdash_{\mathbb{P}_\kappa} \mathbf{j} \geq \mathbf{c}$. It suffices that $\mathbf{j} < \mathbf{c}$ holds in some $\mathbb{P}_\lambda$-generic extension for some $\lambda \in \mathsf{Card}$, as this would contradict $\mathsf{A}_{\vec{\mathbb{P}}}$. To see this, pick any successor cardinal $\lambda \geq \mathbf{c}^{++}$. Since $\mathbb{P}_\kappa$ preserves all cardinals $\leq\kappa$ by (e), p forces that $\mathbf{c}$ is a limit cardinal and $\mathbf{c}$ is attained. By $\mathsf{A}_{\vec{\mathbb{P}}}$, the same holds for $\mathbb{P}_\lambda$. Since λ is a successor cardinal and $\mathbb{P}_\lambda$ preserves cardinals $\leq \lambda$, $\mathbf{1}_{\mathbb{P}_\lambda} \Vdash \mathbf{c} > \lambda$. Since $\mathbf{1}_{\mathbb{P}_\lambda}$ forces that $\mathbf{c}$ is attained, $\mathbf{1}_{\mathbb{P}_\lambda}$ forces $\mathbf{j} \leq \lambda < \mathbf{c}$ by Lemma 4.12 as required.

(2): Otherwise there exists an ω-inaccessible cardinal. Then (1) and Lemma 4.11 (1) provide contradictory conclusions. □

Corollary 4.15. *There exists some $\kappa \in \mathsf{Card}$ such that there is no elementary embedding $j\colon V \to V[G]$ in any outer model of $V[G]$ for any $\mathbb{P}_\kappa$-generic filter G over V.*

Proof. It the claim fails, then $\mathsf{A}_{\vec{\mathbb{P}}}$ holds in V. Moreover, j necessarily moves an ordinal by (b) if $\kappa > \mathbf{c}$. Thus $\mathrm{crit}(j)$ is regular in V, contradicting Theorem 4.14(2). □

4.3. **The bounding and dominating numbers.** The *bounding number* $\mathbf{b}$ and *dominating number* $\mathbf{d}$ are defined as the least cardinal κ such that there exists an unbounded, respectively dominating, family in ω^ω of size κ. They need not exist in choiceless models.

If $p \in \mathbb{P}$ and σ is a $\mathbb{P}$-name, we write $p \vdash \sigma$ if p decides the value of σ, i.e., $p \Vdash \sigma = \check{x}$ for some $x \in V$.

Lemma 4.16. *Suppose* $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma\rangle$ *is a finite support iteration and* $\vec{f} = \langle \dot{f}_\alpha \mid \alpha < \gamma\rangle$ *is a sequence of $\mathbb{P}_\alpha$-names with* $\mathbf{1}_{\mathbb{P}_\alpha} \Vdash \dot{f}_\alpha\colon \dot{\mathbb{P}}_\alpha \to \check{V}$ *for all $\alpha < \gamma$. Then*

$$\mathbb{Q} := \{p \in \mathbb{P}_\gamma \mid \forall \alpha \in \mathrm{supp}(p)\ p{\restriction}\alpha \vdash \dot{f}_\alpha(p(\alpha))\}$$

is dense in $\mathbb{P}_\gamma$.

Proof. Fix a wellorder $\leq^*$ of $[\gamma]^{<\omega}$ and $p_0 \in \mathbb{P}_\gamma$. We construct the following for some $k \in \omega$:

(i) $\vec{s} = \langle s_n \mid n \leq k \rangle$ with $s_n \in [\gamma]^{<\omega}$.
(ii) $\vec{P} = \langle P_n \mid n < k \rangle$ with $P_n \subseteq \mathbb{P}_\gamma^{(p_0)} := \{p \in \mathbb{P}_\gamma \mid p \leq p_0\}$.
(iii) $\vec{\alpha} = \langle \alpha_n \mid n < k \rangle$ strictly decreasing.

For all $n < k$, we will have $\max(s_n) = \alpha_n$ and for all $p \in P_n$:

(a) $p \leq p_0$
(b) $\operatorname{supp}(p) = s_n \cup \{\alpha_i \mid i < n\}$
(c) $p{\restriction}\alpha_i$ decides $\dot{f}_{\alpha_i}(p(\alpha_i))$ for all $i < n$.

Let $P_0 = \{p_0\}$, $s_0 = \operatorname{supp}(p_0)$ and $\alpha_0 = \max(s_0)$. In the successor step, suppose that s_n, P_n and α_n have been constructed. Let s_{n+1} be the $\leq^*$-least support of a condition $r \leq q{\restriction}\alpha_n$ in $\mathbb{P}{\restriction}\alpha_n$ deciding $\dot{f}_{\alpha_n}(q(\alpha_n))$ for some $q \in P_n$. If $s_{n+1} \neq \emptyset$, let $\alpha_{n+1} = \max(s_{n+1})$ and let P_{n+1} be the set of conditions $q \in \mathbb{P}_\gamma$ such that $\operatorname{supp}(q) \cap \alpha_n = s_{n+1}$, $q{\restriction}\alpha_n$ decides $\dot{f}_{\alpha_n}(q(\alpha_n))$ and there is some $p \in P_n$ with $q{\restriction}\alpha_n \leq p{\restriction}\alpha_n$, $\operatorname{supp}(p) \setminus \alpha_n = \operatorname{supp}(q) \setminus \alpha_n$ and $p(\alpha) = q(\alpha)$ for all $\alpha \geq \alpha_n$. Since $\vec{\alpha}$ is strictly decreasing, there is some $n \in \omega$ with $s_{n+1} = \emptyset$. Let $k = n + 1$. There is some $q \in P_n$ with support $\{\alpha_0, \ldots, \alpha_n\}$ such that $q{\restriction}\alpha_i$ decides $\dot{f}_{\alpha_i}(q(\alpha_i))$ for all $i \leq n$. Thus $q \in \mathbb{Q}$ and $q \leq p_0$. □

Suppose that $\mathbb{P}$ and $\mathbb{S}$ are forcings. Recall that a $\leq$-homomorphism $g \colon \mathbb{P} \to \mathbb{S}$ is called a *projection* if $\operatorname{ran}(g)$ is dense in $\mathbb{S}$ and for all $p \in \mathbb{P}$ and all $t \leq g(p)$, there exists some $p' \leq p$ with $g(p') \leq t$. We call a function $g \colon \mathbb{P} \to \mathbb{S}$ a $\perp$*-projection* if g is simultaneously a $\perp$-homomorphism and projection. For the next lemma, suppose $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{f}_\alpha, \mathbb{P}_\delta \mid \alpha < \delta \rangle$ is a sequence such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \mathbb{P}_\delta \mid \alpha < \delta \rangle$ is a finite support iteration and $\mathbf{1}_{\mathbb{P}_\alpha}$ forces "$\dot{f}_\alpha \colon \dot{\mathbb{P}}_\alpha \to \check{\mathbb{S}}$ is a $\perp$-projection" for all $\alpha < \delta$.

Lemma 4.17. *For each $\gamma \leq \delta$, there exists a dense subset $\mathbb{Q}_\gamma$ of $\mathbb{P}_\gamma$ and a $\perp$-projection $g_\gamma \colon \mathbb{Q}_\gamma \to \mathbb{S}^\gamma$, where $\mathbb{S}^\gamma$ is the finite support product of $\mathbb{S}$ of length γ.*

Proof. For $\gamma \leq \delta$, let $\mathbb{Q}_\gamma$ denote the set of $p \in \mathbb{P}_\gamma$ such that $p{\restriction}\alpha$ decides $\dot{f}_\alpha(p(\alpha))$ for all $\alpha \in \operatorname{supp}(p)$. $\mathbb{Q}_\gamma$ is dense in $\mathbb{P}_\gamma$ by Lemma 4.16. Let $g \colon \mathbb{Q}_\gamma \to \mathbb{S}^\gamma$ with $g(p) = t$, where $\operatorname{dom}(t) = \operatorname{supp}(p)$ and $\forall \alpha \in \operatorname{supp}(p)\ \ p{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \dot{f}_\alpha(p(\alpha)) = \check{t}(\alpha)$. Clearly, g is a $\leq$-homomorphism.

Claim. g is a $\perp$-homomorphism.

Proof. Suppose that $p, q \in \mathbb{Q}_\gamma$ with $g(p) \parallel g(q)$. Let $t = g(p)$ and $u = g(q)$. Let $\alpha_0, \ldots, \alpha_{n-1}$ enumerate $\operatorname{supp}(p) \cup \operatorname{supp}(q)$ in increasing order and let $\alpha_n := \gamma$. It suffices to show $p \parallel q$. To this end, we define a sequence $\langle s_i \mid i \leq n \rangle$ with $\operatorname{dom}(s_i) = \alpha_i$, $s_i \leq p{\restriction}\alpha_i, q{\restriction}\alpha_i$ and $s_j{\restriction}\alpha_i \leq s_i$ for $i \leq j \leq n$. Then $s_n \leq p, q$ witnesses $p \parallel q$. Let $s_0 = \mathbf{1}{\restriction}\alpha_0$. Suppose s_i is defined, where $i < n$. If $\alpha_i \in \operatorname{supp}(p) \setminus \operatorname{supp}(q)$, let $s_{i+1} = s_i^\frown \langle p(\alpha_i) \rangle^\frown \mathbf{1}^{(\alpha_i, \alpha_{i+1})}$. The case $\alpha_i \in \operatorname{supp}(q) \setminus \operatorname{supp}(p)$ is similar. Now suppose $\alpha_i \in \operatorname{supp}(p) \cap \operatorname{supp}(q)$. By the inductive hypothesis, $s_i \Vdash_{\mathbb{P}_{\alpha_i}} \dot{f}_{\alpha_i}(p(\alpha_i)) = \check{t}(\alpha_i) = \check{u}(\alpha_i) = \dot{f}_{\alpha_i}(q(\alpha_i))$. Hence $s_i \Vdash_{\mathbb{P}_{\alpha_i}} p(\alpha_i) \parallel q(\alpha_i)$. Pick a $\mathbb{P}_{\alpha_i}$-name σ and some $s \leq s_i$ with $s \Vdash_{\mathbb{P}_{\alpha_i}} \sigma \leq p(\alpha_i), q(\alpha_i)$. Then $s_{i+1} = s^\frown \langle \sigma \rangle^\frown \mathbf{1}^{(\alpha_i, \alpha_{i+1})}$ is as required. □

Claim. g is a projection.

Proof. To see that g is a projection, let $g(p) = t$ and $s \leq t$. Let $p_0 := p$ and $\alpha := \min(\operatorname{dom}(s))$. Since $\mathbf{1}_{\mathbb{P}_\alpha}$ forces that $\dot{f}_\alpha$ is a projection, there exists $q_\alpha \leq p_0{\restriction}\alpha$ in $\mathbb{Q}_\alpha$ and a $\mathbb{P}_\alpha$-name σ with $q_\alpha \Vdash_{\mathbb{P}_\alpha}$ "$\sigma \leq p(\alpha)$ and $\dot{f}_\alpha(\sigma) \leq s(\alpha)$" and $q_\alpha \vdash \dot{f}_\alpha(\sigma)$. Let $p_\alpha = q_\alpha^\frown \langle \sigma \rangle^\frown \langle p_0(\beta) \mid \alpha < \beta < \gamma \rangle \in \mathbb{Q}_\gamma$. Repeating this process for all other $\alpha \in \operatorname{dom}(s)$ up to $\beta := \max(\operatorname{dom}(s))$ yields some $p_\beta \leq p$ with $g(p_\beta) \leq s$.

It remains to show that $\operatorname{ran}(g)$ is dense in $\mathbb{S}^\gamma$. To see this, take any $t \in \mathbb{S}^\gamma$ and let $\alpha := \min(\operatorname{dom}(t))$. Pick some $q_\alpha \in \mathbb{Q}_\alpha$ and a $\mathbb{P}_\alpha$-name σ_α with $q_\alpha \Vdash \dot{f}_\alpha(\sigma_\alpha) \leq t(\alpha)$ and $q_\alpha \vdash \dot{f}_\alpha(\sigma_\alpha)$. Let $p_\alpha := q_\alpha^\frown \langle \sigma_\alpha \rangle^\frown \mathbf{1}^{(\alpha+1, \gamma)} \in \mathbb{Q}_\gamma$. Repeating this process for all other $\alpha \in \operatorname{dom}(t)$ up to $\beta := \max(\operatorname{dom}(t))$ yields some $p_\beta \in \mathbb{Q}_\gamma$ with $g(p_\beta) \leq t$. □

Thus $g \colon \mathbb{P}_\gamma \to \mathbb{S}^\gamma$ is a $\perp$-projection as required. □

If $\kappa > \omega$ is regular, then one can force $\mathbf{b} = \mathbf{d} = \kappa$:

Theorem 4.18. *Suppose that κ is a cardinal of uncountable cofinality. Then $\mathbb{H}^{(\kappa)}$ forces $\mathbf{b} = \mathbf{d} = \mathrm{cof}(\kappa)$.*

Proof. Suppose that G is $\mathbb{H}^{(\kappa)}$-generic over V. $\mathbb{H}^{(\kappa)}$ does not change the value of $\mathrm{cof}(\kappa)$ by Corollary 3.5. Since the iteration adds a dominating real in each step, it suffices to show that every real $V[G]$ is an element of $V[G{\restriction}\gamma]$ for some $\gamma < \kappa$.

Work in V and note that the usual linking function for Hechler forcing sending a tree to its stem is a $\perp$-projection to the forcing $\mathrm{Fun}_{<\omega}(\omega, \omega)$. Let $\dot{f}_\alpha$ denote the canonical $\mathbb{H}^{(\alpha)}$-name for this linking function. Let $\mathbb{Q}_\kappa$ denote the dense subset of $\mathbb{H}^{(\kappa)}$ and $g\colon \mathbb{Q}_\kappa \to \mathrm{Fun}_{<\omega}(\kappa \times \omega, \omega)$ the $\perp$-projection given by Lemma 4.17.

Let σ be a $\mathbb{Q}_\kappa$-name for a real. We will find some $\gamma < \kappa$ with $\sigma^G \in V[G{\restriction}\gamma]$. For each $n < \omega$, let D_n denote the dense set of all $p \in \mathbb{Q}_\kappa$ that decide whether $n \in \sigma$. Since g is a projection, $g[D_n]$ is a dense subset of $\mathbb{C}^\kappa$. Let A_n be the least maximal antichain in $g[D_n]$ in $\mathsf{HOD}_{\{\sigma\}}$ for each $n < \omega$. Since g is a $\perp$-homomorphism, $\bar{A}_n := D_n \cap g^{-1}[A_n]$ is predense in $\mathbb{Q}_\kappa$. By the c.c.c. of $\mathrm{Fun}_{<\omega}(\kappa \times \omega, \omega)$ in $\mathsf{HOD}_{\{\sigma\}}$, there exists some $\gamma < \kappa$ with

$$\bigcup_{n\in\omega,\ p\in\bar{A}_n} \mathrm{supp}(p) = \bigcup_{n\in\omega,\ t\in A_n} \mathrm{dom}(t) \subseteq \gamma.$$

For any $p \in \mathbb{H}^{(\kappa)}$, write $p|\gamma := (p{\restriction}\gamma)^\frown 1^{(\gamma,\kappa)}$. Fix $n \in \omega$ and let θ_n denote any of the formulas $n \in \sigma$ and $n \notin \sigma$.

Claim. If $p \in \mathbb{Q}_\kappa$ and $p \Vdash_{\mathbb{H}^{(\kappa)}} \theta_n$, then $p|\gamma \Vdash_{\mathbb{H}^{(\kappa)}} \theta_n$.

Proof. Otherwise some $q \leq p|\gamma$ in $\mathbb{Q}_\kappa$ forces $\neg\theta_n$. Since $\bar{A}_n$ is predense in $\mathbb{Q}_\kappa$, there exists some $r \in \bar{A}_n$ with $g(q) \parallel g(r)$. Since g is a $\perp$-homomorphism, $q \parallel r$. Putting this together, we have $r \Vdash \neg\theta_n$ since $q \Vdash \neg\theta_n$, r decides θ_n and $q \parallel r$. Moreover $p \parallel r$, since $\mathrm{supp}(r) \subseteq \gamma$, $q \leq p|\gamma$ and $q \parallel r$. We now obtain $r \Vdash \theta_n$, since $p \Vdash \theta_n$, r decides θ_n and $p \parallel r$. This is a plain contradiction. □

The previous claim yields $\sigma^G \in V[G{\restriction}\gamma]$, as desired. □

The next result shows that if all uncountable cardinals are singular, then any iteration of Hechler forcing of uncountable cardinal length forces $\mathbf{b} = \omega_1$. By a *uniform iteration of nontrivial forcings*, we mean a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{p}_{\alpha,i}, \mathbb{P}_\gamma \mid \alpha < \gamma,\ i < \omega\rangle$ such that $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \mathbb{P}_\gamma \mid \alpha < \gamma\rangle$ is an iteration and $\mathbf{1}_{\mathbb{P}_\alpha} \Vdash \dot{p}_{\alpha,i} \perp_{\dot{\mathbb{P}}_\alpha} \dot{p}_{\alpha,i}$ for all $\alpha < \gamma$ and $i < j < \omega$.

Theorem 4.19. *Suppose $\nu \geq \omega_1$ is multiplicatively closed and has countable cofinality. Any uniform iteration $\mathbb{P}_\nu$ of nontrivial forcings with finite support of length ν forces:*

(1) *$\mathbf{b} = \omega_1$ if $\mathbb{P}_\nu$ preserves ω_1.*
(2) *$\mathbf{d} \geq |\nu|$ if $\mathbb{P}_\nu$ preserves $|\nu|$ and $\mathbf{d}$ exists in the extension.*

Proof. (1): We will construct an unbounded family of size ω_1. Let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{p}_{\alpha,i}, \mathbb{P}_\nu \mid \alpha < \nu,\ i < \omega\rangle$ denote the iteration. We can assume that for each $\alpha < \nu$, $\mathbf{1}_{\mathbb{P}_\alpha}$ forces that $\langle \dot{p}_{\alpha,i} \mid i < \omega\rangle$ is not maximal by omitting a condition. Fix a cofinal strictly increasing sequence $\vec{\alpha} = \langle \alpha_n \mid n \in \omega\rangle$ in ν. Fix an injective function $f\colon \nu \times \omega \to \nu$ such that $f(\alpha, n) \geq \alpha_n$ for all $\alpha < \nu$ and $n < \omega$. Such a function can be obtained by thinning out $\mathrm{p}{\restriction}(\nu \times \nu)\colon \nu \times \nu \to \nu$, using that ν is multiplicatively closed. One can easily write down a sequence $\langle \dot{x}_\alpha \mid \alpha < \nu\rangle$ of $\mathbb{P}_\nu$-names such that $\mathbf{1}_{\mathbb{P}_\alpha} \Vdash \dot{x}_\alpha(n) = i + 1$ if $i < \omega$ is unique with $\dot{p}_{f(\alpha,n),i} \in \dot{G}_{f(\alpha,n)}$ and $\mathbf{1}_{\mathbb{P}_\nu} \Vdash \dot{x}_\alpha(n) = 0$ if no such $i < \omega$ exists. Suppose that G is $\mathbb{P}_\nu$-generic over V and work in $V[G]$. Let $x_\alpha = \dot{x}_\alpha^G$ for all $\alpha < \nu$. Write $y \leq_* z$ if $\exists m\ \forall n \geq m\ y(n) \leq z(n)$ and define the *trace* of $x \in \omega^\omega$ as

$$\mathrm{tr}(x) := \{\alpha < \nu \mid x_\alpha \leq_* x\}.$$

If $\mathrm{tr}(x)$ is countable, then x does not bound $\langle x_\alpha \mid \alpha < \nu\rangle$. Since $\mathbb{P}_\nu$ preserves ω_1, the next claim shows that $\langle x_\alpha \mid \alpha < \omega_1\rangle$ is unbounded.

Claim. $\mathrm{tr}(x)$ is countable for all $x \in \omega^\omega$.

Proof. We will partition $\operatorname{tr}(x)$. Let $\dot{x}$ be a $\mathbb{P}_\nu$-name with $\dot{x}^G = x$. It is easy to write down a $\mathbb{P}_\nu$-name $\dot{g}$ such that $\mathbf{1}_{\mathbb{P}_\nu}$ forces that $\dot{g}(\alpha)$ is the least $i < \omega$ with $\forall j \geq i\ \dot{x}_\alpha(j) \leq \dot{x}(j)$. For all $n, i < \omega$, let

$$A_n^{(i)} := \{\alpha < \nu \mid \exists p \in G{\restriction}\alpha_n \;\; p^\frown \mathbf{1}^\nu \Vdash_{\mathbb{P}_\nu} \dot{g}(\alpha) = i\}.$$

Let $A_n = \bigcup_{i<\omega} A_n^{(i)}$ and note that $\operatorname{tr}(x) = \bigcup_{n<\omega} A_n$.

It suffices to show that each $A_n^{(i)}$ is finite. Towards a contradiction, suppose that there exist $n, i < \omega$ such that $A_n^{(i)}$ is infinite. Work in $V[G{\restriction}\alpha_n]$ and note that $A_n^{(i)} \in V[G{\restriction}\alpha_n]$. Fix a strictly increasing sequence $\langle \beta_k \mid k < \omega\rangle$ in $A_n^{(i)}$. Work in $V[G]$. Since $\beta_k \in A_n^{(i)}$, we have $\dot{g}^G(\beta_k) = i$ and hence $x_{\beta_k}(i') \leq x(i')$ for all $k < \omega$ and all $i' \geq i$. Let $j := \max(i, n)$. Since $f(\beta_k, j) \geq \alpha_j \geq \alpha_n$ for all $k < \omega$ as $j \geq n$, $\langle x_{\beta_k}(j) \mid k < \omega\rangle$ is $\operatorname{Col}(\omega,\omega)$-generic over $V[G{\restriction}\alpha_n]$. Hence $x_{\beta_k}(j) > x(j)$ for some $k < \omega$. This contradicts the fact that $x_\alpha(j) \leq x(j)$ for all $\alpha \in A_n^{(i)}$ as $j \geq i$. □

(2): Suppose $p \in \mathbb{P}_\nu$ forces $\langle \dot{y}_\alpha \mid \alpha < \mu\rangle$ is a dominating family in ω^ω for some cardinal $\mu < |\nu|$. Let G be $\mathbb{P}_\nu$-generic over V with $p \in G$ and work in $V[G]$. Define $\langle x_\alpha \mid \alpha < \nu\rangle$ as in the proof of (1). The proof of (1) provides a function that sends each name $\dot{y}$ for an element of ω^ω to an enumeration of $\operatorname{tr}(\dot{y}^G)$ with order type at most ω. Hence $|\bigcup_{\alpha<\mu} \operatorname{tr}(\dot{y}_\alpha^G)| \leq \mu < |\nu|$. Since $\mathbb{P}_\nu$ preserves $|\nu|$, not every x_α is dominated by $\langle \dot{y}_\alpha^G \mid \alpha < \mu\rangle$. But this contradicts our assumption. □

Note that any iteration of Hechler forcing of length ω_1 with finite support preserves ω_1 by Corollary 3.5. If ω_1 is singular, then the previous theorem shows that $\mathbb{H}^{(\omega_1)}$ forces $\mathbf{b} = \omega_1$. This contrasts the fact that the bounding number is regular in ZFC. If ω_1 is regular, then $\mathbb{H}^{(\omega_1)}$ forces $\mathbf{b} = \omega_1$ by Theorem 4.18. Note that the previous theorem for $\kappa \geq \omega_2$ separates the bounding and dominating numbers. In this case, $\mathbf{b} = \omega_1$ and $\mathbf{d}$ is either at least κ or does not exist.

Recall that $\mathbb{H}^{(*)}$ denotes the class of finite support iterations of Hechler forcing whose length is any infinite cardinal.

Corollary 4.20. $\mathsf{A}_{\mathbb{H}^{(*)}}$ *implies that all infinite cardinals have countable cofinality.*

Proof. Suppose there exists an uncountable regular cardinal κ. Then $\mathbb{H}^{(\kappa)}$ forces $\mathbf{b} = \mathbf{d}$ by Theorem 4.18. On the other hand, since $\mathbb{H}^{(\aleph_\omega)}$ preserves cardinals, $\mathbb{H}^{(\aleph_\omega)}$ forces $\mathbf{b} \neq \mathbf{d}$ by Theorem 4.19, which contradicts $\mathsf{A}_{\mathbb{H}^{(*)}}$. □

4.4. **Gitik's model.** In this section, we show that the principles $\mathsf{A}_{\mathbb{C}^*}$, $\mathsf{A}_{\mathbb{R}_*}$ and $\mathsf{A}_{\mathbb{H}^{(*)}}$ fail in Gitik's model from [7] where all infinite cardinals have countable cofinality.

4.4.1. *Review of Gitik's construction.* All results of this subsection are due to Gitik [7], while some notation is taken from [1].[27] We assume that in the ground model V, ZFC holds and there is a proper class of strongly compact cardinals. We further assume that there is no regular limit of strongly compact cardinals.[28]. We further assume there is a predicate for a global wellorder on V with order type Ord. This can be added by the pretame class forcing $\operatorname{Add}(\mathsf{Ord}, 1)$ without adding new sets.[29] Let $\vec{\kappa} = \langle \kappa_\xi \mid \xi \in \mathsf{Ord}\rangle$ be the increasing enumeration of all strongly compact cardinals. In the resulting model, the closure of $\vec{\kappa}$ will equal the class of all uncountable cardinals.

For each inaccessible cardinal α and each regular cardinal $\kappa < \alpha$, fix a bijection $\mathcal{P}_\kappa(\alpha) \to \alpha$. Let $\iota_{\kappa,\alpha}$ denote the induced bijection between their power sets.

Notation 4.21. We distinguish the following types of $\alpha \in \operatorname{Reg}$:

(0) If $\alpha \in [\omega, \kappa_0)$, let $\operatorname{cof}'(\alpha) := \alpha$.
(1) If $\alpha \geq \kappa_0$ and there is a largest $\kappa_\xi \leq \alpha$,[30] let $\operatorname{cof}'(\alpha) := \alpha$.

[27] The version using [1] yields the same model as Gitik's. The two forcings have isomorphic dense subclasses.

[28] This additional assumption do not increase the consistency strength of the axiom system.

[29] This produces a model of Bernays-Gödel class theory BG. More formally, the following proofs can be translated to statements in V about $\operatorname{Add}(\mathsf{Ord}, 1)$-names forced by $\mathbf{1}_{\operatorname{Add}(\mathsf{Ord},1)}$.

[30] I.e., $\alpha \in [\kappa_\xi, \kappa_{\xi+1})$

(a) If $\alpha > \kappa_\xi$ is inaccessible, let Ψ_α be any fine ultrafilter on $\mathcal{P}_{\kappa_\xi}(\alpha)$ and define $\Phi_\alpha := \iota_{\kappa_\xi,\alpha}[\Psi_\alpha]$.
(b) If $\alpha > \kappa_\xi$ is accessible, let Φ_α be any κ_ξ-complete uniform ultrafilter on α.

(2) If $\alpha \geq \kappa_0$ and there is no largest strongly compact cardinal $\leq\alpha$, let α' denote the largest (singular) limit of strongly compacts $<\alpha$ and $\mathrm{cof}'(\alpha) := \mathrm{cof}(\alpha')$. Fix a strictly increasing sequence $\langle \kappa_\nu^\alpha \mid \nu < \mathrm{cof}'(\alpha)\rangle$ of strongly compact cardinals $>\mathrm{cof}'(\alpha)$ with supremum α'.
(a) If α is inaccessible, let $\Psi_{\alpha,\nu}$ be a fine ultrafilter on $\mathcal{P}_{\kappa_\nu^\alpha}(\alpha)$ for each $\nu < \mathrm{cof}'(\alpha)$ and $\Phi_{\alpha,\nu} := \iota_{\kappa_\nu^\alpha,\alpha}[\Psi_{\alpha,\nu}]$.
(b) If α is accessible, let $\Phi_{\alpha,\nu}$ be any κ_ν^α-complete uniform ultrafilter on α for each $\nu < \mathrm{cof}'(\alpha)$.

We will define the forcing $\mathbb{P}$ successively via the next definitions. For $t \subseteq \mathrm{Reg} \times \omega \times \mathrm{Ord}$, write $\mathrm{dom}(t) := \{\alpha \in \mathrm{Reg} \mid \exists m\ \exists\gamma\, (\alpha, m, \gamma) \in t\}$. First, let P_1 denote the set of all finite subsets t of $\mathrm{Reg} \times \omega \times \mathrm{Ord}$ such that for every $\alpha \in \mathrm{dom}(t)$, $t(\alpha) := \{(m, \beta) \mid (\alpha, m, \beta) \in t\}$ is an injective function from a finite subset of ω to α.

Definition 4.22. Let P_2 denote the set of all $t \in P_1$ such that:
(a) For every $\alpha \in \mathrm{dom}(t)$, $\mathrm{cof}'\alpha \in \mathrm{dom}(t)$ and $\mathrm{dom}\big(t(\mathrm{cof}'\alpha)\big) \supseteq \mathrm{dom}\big(t(\alpha)\big)$.
(b) If $(\alpha_0, \ldots, \alpha_{n-1})$ is the increasing sequence enumeration of $\mathrm{dom}(t) \setminus \kappa_0$, then there exist $m \geq 1$ and $j \leq n-1$ such that:
(i) For each $k < j$, $\mathrm{dom}\big(t(\alpha_k)\big) = m + 1$.
(ii) For each k with $j \leq k < n$, $\mathrm{dom}\big(t(\alpha_k)\big) = m$.

The values m and α_j in (b) are unique for t and can thus be denoted $m(t) := m$ and $\alpha(t) := \alpha_j$. Note that $\big(\alpha(t), m(t)\big)$ is the point that needs to be filled in next in order to extend t. For any $t \in T \subseteq P_2$, let
$$\mathrm{Suc}_T(t) := \{\beta \mid t \cup \{\big(\alpha(t), m(t), \beta\big)\} \in T\}.$$

Definition 4.23. Let P_3 denote the set of all pairs (s, T) such that $s \in T \subseteq P_2$ and:
(a) For every $t \in T$:
(i) Either $t \supseteq s$ or $t \subseteq s$.
(ii) $\mathrm{dom}(t) = \mathrm{dom}(s)$.
(iii) *(tree-like)* If $t = r \cup \{\big(\alpha(r), m(r), \beta\big)\}$, then $r \in T$.
(b) For every $t \in T$ with $s \subseteq t$:
(i) If $\alpha(t)$ is of type 1, then $\mathrm{Suc}_T(t) \in \Phi_{\alpha(t)}$.
(ii) If $\alpha(t)$ is of type 2 and $m(t) \in \mathrm{dom}\big(t(\mathrm{cof}'(\alpha(t)))\big)$, then
$$\mathrm{Suc}_T(t) \in \Phi_{\alpha(t), t\left(\mathrm{cof}'(\alpha(t))\right)(m(t))}.$$

s is called the *stem* of (s, T).

Let $\mathbb{P}$ denote P_3 with the following partial order. For $(r, R), (t, T) \in P_3$, let $(r, R) \leq (t, T)$ if $r{\restriction}\kappa_0 \supseteq t{\restriction}\kappa_0$, $R{\restriction}\big(\mathrm{dom}(t) \setminus \kappa_0\big) \subseteq T$ and $\mathrm{dom}(r) \supseteq \mathrm{dom}(t)$. Let I denote the class of finite subsets of Reg closed under cof'. For any $s \in I$, let
$$\mathbb{P}_s := \{(t, T) \in P_3 \mid\ \mathrm{dom}(t) \subseteq s\}.$$
We have $(t, T){\restriction}\mathbb{P}_s := \big(t{\restriction}s, \{u{\restriction}s \mid u \in T\}\big) \in P_3$ for all $(t, T) \in P_3$ and $s \in I$ [7, Lemma 2.4]. The next lemma shows how $\mathbb{P}_s$ can be factored.

Lemma 4.24. [1, Theorem 2.5] *For any $s \in I$ and any strongly compact $\kappa_\xi \in s$, $\mathbb{P}_s$ is forcing equivalent*[31] *to a forcing of the form $\mathbb{P}_{s\cap\kappa_\xi} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s\cap\kappa_\xi}$ forces that $\dot{\mathbb{Q}}$ does not add any bounded subset of κ_ξ.*

Let $D := \{(\alpha, n, \beta) \in \mathrm{Reg} \times \omega \times \mathsf{Ord} \mid \beta < \alpha\}$. In the next definition, we work with proper class functions to simplify the notation. One can obtain a formally correct definition by restricting each function to its support.

Definition 4.25.

[31] I.e., the Boolean completions are isomorphic.

(1) Let $\mathcal{G}$ be the group of permutations $g\colon D \to D$[32] such that there exists a sequence of permutations g_α of $\alpha \in \mathsf{Reg}$ with:
   (a) $g(\alpha, n, \beta) = \big(\alpha, n, g_\alpha(\beta)\big)$ for all $\alpha \in \mathsf{Reg}$, $\beta < \alpha$ and $n \in \omega$.
   (b) $\mathrm{supp}(g_\alpha) := \{\beta < \alpha \mid g_\alpha(\beta) \neq \beta\}$ is finite for all $\alpha \in \mathsf{Reg}$.
   (c) $\mathrm{supp}(g) := \{\alpha \in \mathsf{Reg} \mid g_\alpha \neq \mathrm{id}\}$ is finite.
(2) For each $g \in \mathcal{G}$, let $\mathbb{P}^g \subseteq P_3$ be the set of all $(t, T) \in P_3$ such that $\mathrm{dom}(t) \supseteq \mathrm{supp}(g)$ and for all $\alpha \in \mathrm{dom}(t)$:
   (a) $\mathrm{dom}\big(t(\alpha)\big) = \mathrm{dom}\big(t(\mathrm{cof}'(\alpha))\big)$.
   (b) If $\alpha \geq \kappa_0$, then $\mathrm{ran}\big(t(\alpha)\big) \supseteq \{\beta \in \mathrm{supp}(g_\alpha) \mid \exists r \in T\ \big(\beta \in \mathrm{ran}(r(\alpha))\big)\}$.
(3) For each $g \in \mathcal{G}$ and $(t, T) \in P_3$, let $g'(t, T) := (g[t], \{g[t'] \mid t' \in T\})$.

Lemma 4.26. [7, Lemma 3.2] *For each $g \in \mathcal{G}$, $\mathbb{P}^g$ is a dense subclass of $\mathbb{P}$ and $g'\colon \mathbb{P}^g \to \mathbb{P}^g$ is an automorphism of $\mathbb{P}^g$.*

Since the restriction of $\mathbb{P}$ to any ordinal is a complete subforcing, the Boolean algebra $\mathbb{B} := \mathbb{B}(\mathbb{P})$ of regular open subsets of $\mathbb{P}$ is complete.[33] Let $\iota\colon \mathbb{P} \to \mathbb{B}$ denote the canonical $\leq$- and $\perp$-homomorphism. It is easy to see that every $g \in \mathcal{G}$ induces an automorphism $\bar{g}$ of $\mathbb{B}$ defined by $\bar{g}(U) = g'[U \cap \mathbb{P}^g]_{\mathrm{down}}$, where $U \in \mathbb{B}$ is a regular open subset of $\mathbb{P}$ and U'_{down} denotes the downward closure of any $U' \subseteq \mathbb{P}$. Using density of $\mathbb{P}^g$, it can be show that one can recover g from $\bar{g}$ and $\overline{g \circ h} = \bar{g} \circ \bar{h}$ for all $g, h \in \mathcal{G}$. Thus $\bar{\mathcal{G}} := \{\bar{g} \mid g \in \mathcal{G}\} \cong \mathcal{G}$.

Definition 4.27.
(1) Let $\mathrm{fix}(s) := \{\bar{g} \in \bar{\mathcal{G}} \mid g \in \mathcal{G},\ \forall \alpha \in s\ g_\alpha = \mathrm{id}_\alpha\}$ for $s \in I$.
(2) Let $\bar{\mathcal{F}}$ be the normal filter of subgroups of $\bar{\mathcal{G}}$ generated by $\mathrm{fix}(s)$ for all $s \in I$.

As usual, for any $\mathbb{B}$-name $\dot{x}$, let
$$\mathrm{sym}(\dot{x}) := \{\bar{g} \in \bar{\mathcal{G}} \mid \bar{g}(\dot{x}) = \dot{x}\}.$$
A $\mathbb{B}$-name $\dot{x}$ is called *symmetric* if $\mathrm{sym}(\dot{x})$ is in $\bar{\mathcal{F}}$. Let $\mathrm{HS}_{\bar{\mathcal{F}}}$ be the class of all hereditarily symmetric $\mathbb{B}$-names. We say that $s \in I$ *supports* a name $\dot{x} \in \mathrm{HS}_{\bar{\mathcal{F}}}$ if $\mathrm{fix}(s) \subseteq \mathrm{sym}(\dot{x})$. The next lemma is important in the proof.

Lemma 4.28. [7, Lemma 3.3] *Suppose that φ is a formula with n free variables and* $\mathrm{fix}(s)$ *supports $\dot{x}_0, \ldots, \dot{x}_{n-1} \in \mathrm{HS}_{\bar{\mathcal{F}}}$, where $s \in I$. Then for every $(t, T) \in P_3$:*
$$(t, T)_\iota \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}) \iff ((t, T){\restriction}\mathbb{P}_s)_\iota \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}).$$

Fix a $\mathbb{B}$-generic filter G over V. The symmetric model is defined as $V(G) = \{\dot{x}^G \mid \dot{x} \in \mathrm{HS}_{\bar{\mathcal{F}}}\}$. Note that for any $s \in I$, $G{\restriction}\mathbb{P}_s := \{(t, T){\restriction}\mathbb{P}_s \mid (t, T)_\iota \in G\}$ is a $\mathbb{P}_s$-generic filter over V.

Theorem 4.29.
(1) [1, Lemma 2.4] *For any set of ordinals $X \in V(G)$, there exists some $s \in I$ with $X \in V[G \restriction \mathbb{P}_s]$.*
(2) [1, Corollary 2.10] *$V(G)$ is a model of* ZF *where every infinite cardinal has cofinality ω. Moreover, the closure of $\vec{\kappa}$ equals the class of all uncountable cardinals of $V(G)$.*

4.4.2. *Absoluteness fails over Gitik's model.* The following variants of $\mathbf{c}$ will play a key role. For any cardinal κ, let
$$\mathbf{c}_\kappa = \sup\{\lambda \in \mathsf{Card} \mid \lambda \leq_i \kappa^\omega\}.$$
We will see that each of $\mathsf{A}_{\mathbb{C}^*}$ and $\mathsf{A}_{\mathbb{R}_*}$ implies that $\mathbf{c}_\kappa > \kappa$ for some infinite cardinal κ, while in Gitik's model, $\mathbf{c}_\kappa = \kappa$ for all infinite cardinals κ. Recall the notation $\mathbf{c}^{\mathbb{C}^\kappa}$ from Lemma 4.12.

Lemma 4.30. $\mathbf{c}_\kappa = \mathbf{c}_\omega^{\mathbb{C}^\kappa}$ *for all infinite cardinals κ.*

Proof. To show $\mathbf{c}_\omega^{\mathbb{C}^\kappa} \leq \mathbf{c}_\kappa$, suppose that $\dot{f}$ is a $\mathbb{C}^\kappa$-name and $p \in \mathbb{C}^\kappa$ forces that $\dot{f}\colon \gamma \to 2^\omega$ is injective. Since $\mathbb{C}^\kappa$ is nice, we have $\gamma \leq_i \kappa^\omega$ and thus $\gamma \leq \mathbf{c}_\kappa$, as desired. To show $\mathbf{c}_\omega^{\mathbb{C}^\kappa} \leq \mathbf{c}_\kappa$, it suffices to construct an injective function from $\kappa^\omega \cap V$ into the set of Cohen reals over V in

[32]We use the domain D instead of $\mathrm{Reg} \times \omega \times \mathsf{Ord}$ to ensure that g is supported on a set.
[33]A class Boolean algebra is called *complete* if every subset has a supremum.

a $\mathbb{C}^\kappa$-generic extension $V[G]$. Let $\langle i_\alpha \mid \alpha < \kappa\rangle$ denote the sequence of the first digits of the Cohen reals added by G. Let S denote a set of size κ^ω of injective functions in κ^ω such that for any $f, g \in S$ with $f \neq g$, there exist infinitely many $n \in \omega$ with $f(n) \neq g(n)$. For each $f \in S$, consider the Cohen real x_f over V defined by $x_f(n) = i_{f(n)}$. These reals are pairwise distinct by the choice of S. □

Lemma 4.31. *In Gitik's model, $\mathbf{c}_\kappa = \kappa$ for all infinite cardinals κ.*

Proof. Suppose that κ is an infinite cardinal in $V(G)$ and $f\colon \gamma \to {}^\omega\kappa$ is an injective function in $V(G)$. It suffices to show $\gamma < (\kappa^+)^{V(G)}$. By Theorem 4.29 (2), $(\kappa^+)^{V(G)} = \kappa_\xi$ for some ξ, where κ_ξ is a strongly compact cardinal in V. We will show $\gamma < \kappa_\xi$. By Theorem 4.29 (1), there is some $s \in I$ with $f \in V[G{\restriction}\mathbb{P}_s]$. We may assume $\kappa_\xi \in s$. By Lemma 4.24, $\mathbb{P}_s$ is forcing equivalent to a forcing of the form $\mathbb{P}_{s\cap\kappa_\xi} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s\cap\kappa_\xi}$ forces that $\dot{\mathbb{Q}}$ does not add new bounded subsets of κ_ξ. Let λ be an inaccessible cardinal in V with $\max(s\cap\kappa_\xi) < \lambda < \kappa_\xi$. Since $|\mathbb{P}_{s\cap\kappa_\xi}| < \lambda$ and $\mathbb{P}_{s\cap\kappa_\xi}$ forces that $\dot{\mathbb{Q}}$ does not add new bounded subsets of κ_ξ, λ remains inaccessible in $V[G{\restriction}\mathbb{P}_s]$. Since $f \in V[G{\restriction}\mathbb{P}_s]$, we have $\gamma < \lambda < \kappa_\xi = (\kappa^+)^{V(G)}$, as desired. □

Note that $\mathbf{c}_\omega = \omega$ implies that no cardinal characteristics of the reals exist. Using the previous lemma, we obtain the failure of the above absoluteness principles in Gitik's model.

Theorem 4.32. *$\mathsf{A}_{\mathbb{C}^*}$, $\mathsf{A}_{\mathbb{R}_*}$ and $\mathsf{A}_{\mathbb{H}_*}$ fail in Gitik's model.*

Proof. Each of $\mathsf{A}_{\mathbb{C}^*}$ and $\mathsf{A}_{\mathbb{R}_*}$ implies $\mathbf{c}_\kappa > \kappa$ for all ω-strong limit cardinals κ by Theorem 4.14 (1) and Lemma 4.30. But in Gitik's model, $\mathbf{c}_\kappa = \kappa$ for all such κ by Lemma 4.31. Moreover, $\mathsf{A}_{\mathbb{H}_*}$ implies that the bounding number equals ω_1 by Theorem 4.19. But in Gitik's model, the bounding number does not exist, since $\mathbf{c}_\omega = \omega$ by Lemma 4.31. □

5. Open problems

The absoluteness principle $\mathsf{A}_{\mathrm{Col}(\omega,*)}$ for the class of collapse forcings $\mathrm{Col}(\omega,\kappa)$ for arbitrary cardinals κ is consistent even with ZFC [9]. Our main open problem is:

Problem 5.1. *Are the principles $\mathsf{A}_{\mathbb{C}^*}$, $\mathsf{A}_{\mathbb{R}_*}$ and $\mathsf{A}_{\mathbb{H}^{(*)}}$ consistent?*

Our results in Sections 4.2-4.4 indicate some properties that a model of these principles must have. In particular, we argued that they fail in Gitik's model [7]. One could aim to show that they also fail in all set generic extensions of this model. For $\mathsf{A}_{\mathbb{C}^*}$ and $\mathsf{A}_{\mathbb{R}_*}$, it suffices that set forcing preserves $\mathbf{c}_\lambda = \lambda$ for sufficiently large cardinals λ, but the argument for Lemma 4.30 does not work for forcings which are not wellordered. Can the construction of Gitik's model [7] or Gitik's alternative construction of a model where all uncountable cardinals are singular from an almost huge cardinal [8] be adapted to obtain models of the above absoluteness principles? If these principles are consistent, we would like to understand the structure of their models. For instance, does the HOD of a model of $\mathsf{A}_{\mathbb{C}^*}$ contain large cardinals?

Section 4.1 shows that Cohen and random extensions have different theories. This suggests to study the same problem for other classical c.c.c. forcings. For instance:

Problem 5.2. *Do extensions by sufficiently large forcings in $\mathbb{C}^*$ and $\mathbb{H}^{(*)}$ necessarily have different theories?*

The problem of differentiating theories for different forcing extensions is interesting from the viewpoint of the modal logic of forcing [10]. Switches $\varphi_0, \dots, \varphi_n$ are called *independent* if any choice of truth values for these sentences can be realised in the relevant generic extensions.

Problem 5.3. *Is it provable in ZF for each natural number n that there exist n independent switches?*

Regarding preservation of cardinals in Section 3, we ask whether all $(\theta, 1)$-narrow forcings already preserve θ^+.[34] This would improve Lemma 3.2. Is every $(\theta, 1)$-narrow forcing θ-narrow? Is every θ-narrow forcing uniformly θ-narrow? Is every wellordered θ^+-c.c. forcing uniformly

[34] A recent argument of Karagila, Schilhan and the second-listed author shows that this is the case for $\theta = \omega$.

θ-narrow? Cardinal preservation is also a problem for many classical forcings. For example, we do not know whether Sacks forcing preserves ω_1 even if one assumes that ω_1 is inaccessible to reals. The problem is to show that every new real has a name in H_{ω_1}. If this is the case, then one can show that ω_1 is preserved using capturing as in [4, 21]. Regarding random algebras in Section 3.4, note that we would have a shortcut for some of our results if $\mathbb{R}_\kappa$ is $\mathbb{C}^\kappa$-linked. However, it is open whether even ZFC provides a negative answer to the next problem.

Problem 5.4. *Is $\mathbb{R}_{\omega_1}$ σ-linked?*

A further natural question regarding $\mathbb{R}_\kappa$ is whether it is provable in ZF that $\mathbb{R}_\kappa$ does not have uncountable antichains for any infinite cardinal κ. Regarding Section 3.5.2, one can ask for a finer understanding of the effect of $\mathrm{Add}(\kappa, 1)$. For instance, is it consistent that $\mathrm{Add}(\omega_3, 1)$ collapses ω_1 and ω_3 while preserving ω_2? It would further be interesting to extend the results about Hechler forcing in Section 4.3 to obtain a precise characterisation of the bounding number in finite support iterations of Hechler forcing or arbitrary ordinal length even in ZFC.

Gitik's model in Section 4.4 is a useful test case to study forcing over choiceless models. What are the answers to the following questions for Gitik's model: Does every nonatomic σ-closed forcing collapse ω_1? Does the dominating number exist in some generic extension? Is the bounding number ω_1 in any generic extension where it exists? Does Sacks forcing preserve ω_1? Can one increase HOD by forcing?

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