

PRESERVATION OF AD VIA FORCINGS

DAISUKE IKEGAMI AND NAM TRANG

ABSTRACT. We show that assuming $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$, any poset which increases Θ does not preserve the truth of AD. We also show that in $\text{ZF} + \text{AD}$, any non-trivial poset on \mathbb{R} does not preserve the truth of AD. This answers the question of Chan and Jackson [2, Question 5.7]. Furthermore, we show that under the assumptions $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))” + “\Theta$ is regular”, there is a poset on Θ which adds a new subset of Θ while preserving the truth of AD. This answers the question of Cunningham [3, Section 5].

1. INTRODUCTION

In this paper, we discuss the relationship between forcing and the Axiom of Determinacy (AD), especially on the question what kind of forcings preserve the truth of AD.

Forcing was introduced by Cohen to prove the independence of the Continuum Hypothesis from Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). Using forcing, he also proved that the Axiom of Choice is independent of Zermelo-Fraenkel set theory (ZF). Since then, forcing has been a basic tool of constructing models of set theory, and it has been used to obtain various results on independence or unprovability of some mathematical statements from set theory as well as to analyze various kinds of models of set theory.

The Axiom of Determinacy (AD) was introduced by Mycielski and Steinhaus to consider a situation where every set of reals has good properties that simple sets (such as Borel sets and analytic sets) enjoy, and examples of those good properties are Lebesgue measurability and the Baire property. While AD contradicts the Axiom of Choice in ZF, it has many beautiful consequences on sets of reals. Furthermore, it has been shown that there are deep connections between models of $\text{ZF} + \text{AD}$ (or models of $\text{ZF} + \text{AD}^+$) and models of ZFC with Woodin cardinals, and AD has been playing an important role not only in descriptive set theory to analyze the properties of sets of reals, but also in the theory of large cardinals and inner model theory.

Let us mention how we got interested in the relationship between forcing and AD, especially on the question what kind of forcings preserve the truth of AD. By the result of Kunen, there

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is no non-trivial elementary embedding $j: V \rightarrow V$ such that (V, \in, j) is a model of ZFC.¹ Furthermore, Hamkins, Kirmayer, and Perlmutter proved that for any set generic G over V , there is no non-trivial elementary embedding $j: V \rightarrow V[G]$ such that $(V[G], \in, j)$ is a model of ZFC.

One can then ask questions such as what if the structure (V, \in, j) or $(V[G], \in, j)$ is a model of ZF or ZF + AD instead of ZFC. Using the method of symmetric models, Woodin proved that there are a set generic G over V and a non-trivial elementary embedding $j: V \rightarrow V[G]$ such that $(V[G], \in, j)$ is a model of ZF + AD. However, in his example, $j \upharpoonright \text{Ord}$ is the identity map, so there is no critical point of j .

As far as we know, it is still open whether there are a set generic G over V and an elementary embedding $j: V \rightarrow V[G]$ such that $(V[G], \in, j)$ is a model of ZF + AD and $j \upharpoonright \text{Ord}$ is not the identity map. We are especially interested in the case when the critical point of j is ω_1 in V because if the critical point of j is ω_1 in V , then the forcing to obtain $V[G]$ must add new reals to V and AD has influence on reals and sets of reals. To obtain such a j , one needs to have a poset \mathbb{P} to produce such a model $V[G]$, and the poset \mathbb{P} must add new reals while preserve the truth of AD from V to $V[G]$. Hence we have a test question: Is there any poset which adds a new real while preserving the truth of AD? This is how we got interested in the relationship between forcing and AD.

We still do not know if there is any poset which adds a new real while preserving the truth of AD. Considering this question, we have observed that many forcings adding a new real do not preserve the truth of AD. A typical example is Cohen forcing. It is well-known that if G is V -generic for Cohen forcing, then in $V[G]$, the set of reals in V does not have the Baire property. In particular, AD must fail in $V[G]$. On the other hand, there are posets which add a new set while preserving the truth of AD. By the result of Woodin [6, Section 3], if we assume ZF + AD + “ $V = L(\mathbb{R})$ ” and let κ be a sufficiently big cardinal and \mathbb{P} be the poset for adding a Cohen subset of κ in HOD, the class of hereditarily ordinal definable sets, then the poset \mathbb{P} adds a new set while preserving the truth of AD. Actually, the poset \mathbb{P} does not add any set of reals to V in this case.

We have been wondering what kind of forcings preserve the truth of AD. Our intuition was such a poset \mathbb{P} would not be able to change the structure of sets of reals drastically. The intuition was partially justified using the ordinal Θ , the supremum of ordinals which are surjective images of \mathbb{R} , by the following theorem:

Theorem 3.1. Assume ZF + AD⁺ + “ $V = L(\wp(\mathbb{R}))$ ”. Suppose that a poset \mathbb{P} increases Θ , i.e., $\Theta^V < \Theta^{V[G]}$ for any \mathbb{P} -generic filter G over V . Then AD fails in $V[G]$ for any \mathbb{P} -generic filter G over V .

However, the assumption “ $V = L(\wp(\mathbb{R}))$ ” is essential in Theorem 3.1:

¹Here V is the class of all sets, j in the structure (V, \in, j) is considered as the interpretation of a binary predicate on the universe, and the structure (V, \in, j) satisfies Comprehension and Replacement for first-order formulas with the binary predicates for \in and j .

Theorem 3.2. It is consistent relative to $\text{ZF} + \text{AD}_{\mathbb{R}}$ that $\text{ZF} + \text{AD}$ holds and there is a poset \mathbb{P} increasing Θ while preserving AD, i.e., for any \mathbb{P} -generic filter G over V , we have $\Theta^V < \Theta^{V[G]}$ and that AD holds in $V[G]$.²

In particular, there is a poset which adds a new set of reals (but does not add a new real) while preserving the truth of AD.

After we announced Theorem 3.1 and Theorem 3.2, Chan and Jackson [2] worked on the question what kind of forcings do not preserve the truth of AD. They proved that assuming $\text{ZF} + \text{AD}$, if a non-trivial poset \mathbb{P} is a wellorderable forcing of cardinality less than Θ , then \mathbb{P} does not preserve the truth of AD ([2, Theorem 3.2]). They also proved that assuming $\text{ZF} + \text{AD} + “\Theta$ is regular”, if a non-trivial poset \mathbb{P} is a surjective image of \mathbb{R} , then \mathbb{P} does not preserve the truth of AD ([2, Theorem 5.5]). Then they asked whether $\text{ZF} + \text{AD}$ only (i.e., without assuming the regularity of Θ) implies that if a non-trivial poset \mathbb{P} is a surjective image of \mathbb{R} , then \mathbb{P} does not preserve the truth of AD ([2, Question 5.7]). We give a positive answer to their question:

Theorem 4.1. Assume $\text{ZF} + \text{AD}$. Let \mathbb{P} be any non-trivial poset which is a surjective image of \mathbb{R} and G be any \mathbb{P} -generic filter over V . Then AD fails in $V[G]$.

We now turn to positive results on the question what kind of forcings preserve the truth of AD. As was mentioned in a previous paragraph, By the result of Woodin [6, Section 3], if we assume $\text{ZF} + \text{AD} + “V = \text{L}(\mathbb{R})”$ and let κ be a sufficiently big cardinal and \mathbb{P} be the poset for adding a Cohen subset of κ in HOD, the class of hereditarily ordinal definable sets, then the poset \mathbb{P} adds a new set while preserving the truth of AD. A natural question would be how small one can take κ for this result. Cunningham [3] worked on this question. He proved that κ can be taken as any regular cardinal larger than Θ^+ ([3, Subsection 4.1]). Then he asked whether κ can be taken as Θ ([3, Section 5]). We answer his question positively. In fact, we prove a more general theorem as follows:

Theorem 5.1. Assume $\text{ZF} + \text{AD}^+ + “V = \text{L}(\wp(\mathbb{R}))”$. Suppose that Θ is regular. Then there is a poset \mathbb{P} on Θ which adds a subset of Θ while preserving AD, i.e., for any \mathbb{P} -generic filter G over V , there is a subset of Θ^V which belongs to $V[G] \setminus V$ and AD holds in $V[G]$.³

Notice that $\text{ZF} + \text{AD} + “V = \text{L}(\mathbb{R})”$ implies the assumptions of Theorem 5.1 including the regularity of Θ . Also, in case of $“V = \text{L}(\mathbb{R})”$, the poset \mathbb{P} is the one for adding a Cohen subset of Θ in HOD as in Case 1 in the proof of Theorem 5.1. Therefore, the arguments for Theorem 5.1 answer the question of Cunningham [3, Section 5].

We also note that Theorem 5.1 is optimal in the following two senses: In one sense, the size of the poset \mathbb{P} cannot be smaller than Θ . As was mentioned in a previous paragraph, by the result of Chan and Jackson [2, Theorem 3.2], any wellorderable forcing of cardinality less than Θ cannot preserve the truth of AD. Also, by Theorem 4.1 (or by the result of Chan and Jackson [2, Theorem 5.5] in case Θ is regular), any poset on \mathbb{R} (or a surjective image of \mathbb{R}) cannot preserve the truth of AD. In the other sense, unless the poset \mathbb{P} adds a new real,

²For an expert on determinacy, the assumption $\text{ZF} + \text{AD}_{\mathbb{R}}$ is an overkill. The proof of Theorem 3.2 shows that the assumption $\text{ZF} + \text{AD}^+ + “\Theta > \Theta_0”$ is enough.

³The proof of Theorem 5.1 shows that in both Case 1 and Case 2, the poset \mathbb{P} does not add any new set of reals to V . In particular, the poset \mathbb{P} preserves the truth of AD^+ as well.

the poset \mathbb{P} cannot add a new bounded subset of Θ while preserving the truth of AD. This is because if \mathbb{P} does not add any real and both V and $V[G]$ are models of AD, then by the Moschovakis Coding Lemma, V and $V[G]$ have the same bounded subsets of Θ , leading to the situation that the poset \mathbb{P} cannot add a bounded subset of Θ .

After looking at Theorem 5.1, it is natural to ask whether the assumption “ Θ is regular” is essential there. We do not know the answer to this question. However, in case Θ is singular, we have $\text{AD}_{\mathbb{R}}$ under the assumptions of Theorem 5.1. In case $\text{AD}_{\mathbb{R}}$ holds, which is Case 2 in the proof of Theorem 5.1, the poset \mathbb{P} in Theorem 5.1 is for adding a Cohen subset of Θ in HOD. We show that this particular poset cannot preserve the truth of AD if Θ is singular:

Theorem 5.2. Assume $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$. Suppose that Θ is singular and let \mathbb{P} be $\text{Add}(\Theta, 1)$ in HOD, where $\text{Add}(\Theta, 1) = \{p \mid p: \gamma \rightarrow 2 \text{ for some } \gamma < \Theta\}$. Then AD fails in $V[G]$ for any \mathbb{P} -generic filter G over V .

2. BASIC DEFINITIONS, THEOREMS, AND LEMMAS

In this section, we introduce basic definitions, theorems, and lemmas we will use in later sections of the paper. We assume that readers are familiar with the basics of forcing and descriptive set theory. For basic definitions not given in this paper, see Jech [4] and Moschovakis [9]. When we say “reals”, we mean elements of the Baire space ω^ω or of the Cantor space 2^ω .

We start with some basic definitions which will be used throughout the paper:

Definition 2.1.

- (1) The ordinal Θ is the supremum of ordinals which are surjective images of \mathbb{R} .
- (2) A set x is *OD from sets* y_1, \dots, y_n if x is definable by a first-order formula with an ordinal and y_1, \dots, y_n as parameters.
- (3) Let Y be a set. We say a set x is *hereditarily* OD_Y if any element of the transitive closure of $\{x\}$ is OD from some elements of Y .
- (4) For a set Y , we write HOD_Y for the collection of sets which are hereditarily OD_Y . When Y is the empty set, we simply write HOD for HOD_Y .

Definition 2.2. Let A and B be sets of reals (or subsets of the Baire space ω^ω). We say A is *Wadge reducible to* B if there is a continuous function $f: \omega^\omega \rightarrow \omega^\omega$ such that $A = f^{-1}(B)$. When A is Wadge reducible to B , we write $A \leq_W B$. The order \leq_W is called the *Wadge order on sets of reals*.

Lemma 2.3 (Wadge’s Lemma). Assume $\text{ZF} + \text{AD}$. Then for any sets A, B of reals, we have either $A \leq_W B$ or $B \leq_W \omega^\omega \setminus A$.

Proof. See e.g., [14, Lemma 2.1]. □

The following theorems will be used in Section 3.

Theorem 2.4 (Woodin). Assume $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$. Then the model HOD is of the form $L[Z]$ for some subset Z of Θ , and there are a poset \mathbb{Q} in HOD and a \mathbb{Q} -generic filter H over HOD such that $\text{HOD} \subseteq V \subseteq \text{HOD}[H]$.

Proof. See e.g., [13, Theorem 3.1.9]. □

Theorem 2.5 (Moschovakis). Assume $\text{ZF} + \text{AD}$. Then Θ is a limit of measurable cardinals.

Proof. For a proof without assuming $\text{DC}_{\mathbb{R}}$, one could first prove that Θ is a limit of strong partition cardinals under $\text{ZF} + \text{AD}$ as in [7] and then verify that every strong partition cardinal is measurable under ZF as in [5, 28.10 Theorem]. \square

Theorem 2.6 (Solovay). Assume $\text{ZF} + \text{AD}_{\mathbb{R}}$. Then for any set A of reals, there is a set B of reals which is not OD from A and any real.

Proof. See [10, Lemma 2.2]. \square

The following theorem will be used in Section 4:

Theorem 2.7 (Chan and Jackson). Assume $\text{ZF} + \text{AD}$ and Θ is regular. Then for any non-trivial poset \mathbb{P} on \mathbb{R} and any \mathbb{P} -generic filter G over V , the axiom AD fails in $V[G]$.

Proof. See [2, Theorem 5.5]. \square

The following theorems will be used in Section 5:

Theorem 2.8 (Moschovakis). Assume $\text{ZF} + \text{AD}$. Then for any non-zero ordinal $\gamma < \Theta$, there is a set A of reals such that there is a surjection from \mathbb{R} to $\wp(\gamma)$ which is OD from A .

Proof. For any surjection $\rho: \mathbb{R} \rightarrow \gamma$, the arguments in [5, 28.15 Theorem] give us a surjection from \mathbb{R} to $\wp(\gamma)$ which is OD from ρ . If A is a prewellordering on \mathbb{R} of length γ , then the surjection $\rho: \mathbb{R} \rightarrow \gamma$ induced from A is clearly OD from A . Hence there is a surjection from \mathbb{R} to $\wp(\gamma)$ which is OD from A , as desired. \square

Theorem 2.9 (Woodin). Assume $\text{ZF} + \text{AD}^+ + "V = \text{L}(\wp(\mathbb{R}))"$. Suppose also that $\text{AD}_{\mathbb{R}}$ fails. Then there is a set T of ordinals such that $V = \text{L}(T, \mathbb{R})$.

Proof. By the results of Woodin [7], the axiom AD^+ and the failure of $\text{AD}_{\mathbb{R}}$ imply that the set of Suslin cardinals is closed below Θ while not cofinal in Θ . Hence there is a largest Suslin cardinal in Θ . By the result of Woodin [12, Corollary 6], the assumptions AD^+ and " $V = \text{L}(\wp(\mathbb{R}))$ " imply that the ultrapower $V^{\mathcal{D}}/\mu$ is well-founded where \mathcal{D} is the set of Turing degrees and μ is the Martin measure on \mathcal{D} . Using the result of Woodin [7], it follows that there is a set T of ordinals such that $\wp(\mathbb{R}) \subseteq \text{L}(T, \mathbb{R})$. Since we assume $V = \text{L}(\wp(\mathbb{R}))$, we have $V = \text{L}(T, \mathbb{R})$, as desired. \square

Theorem 2.10 (Woodin). Assume $\text{ZF} + \text{AD}^+ + "V = \text{L}(T, \mathbb{R})"$ for some set T of ordinals. Then

- (1) for some subset Z of Θ , we have $\text{HOD}_{\{T\}} = \text{L}[T, Z]$, and
- (2) for any real x , we have $\text{HOD}_{\{T, x\}} = \text{HOD}_{\{T\}}[x]$.

Proof. For (1), one can argue in the same way as in [1, Corollary 7.21].

For (2), see [7]. \square

We next introduce Vopěnka algebras and their variants we will use in this paper:

Definition 2.11. Let γ be a non-zero ordinal and T be a set of ordinals.

- (1) Let n be a natural number with $n \geq 1$ and \mathcal{O}_n be the collection of all nonempty subsets of $(\gamma^\omega)^n$ which are OD from T . Fix a bijection $\pi_n: \eta \rightarrow \mathcal{O}_n$ which is OD from T , where η is some ordinal. Let \mathbb{Q}_n be the poset on η such that for each p, q in \mathbb{Q}_n , we have $p \leq q$ if $\pi_n(p) \subseteq \pi_n(q)$. We call \mathbb{Q}_n the *Vopěnka algebra for adding an element of $(\gamma^\omega)^n$ in $\text{HOD}_{\{T\}}$* .

- (2) For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $i_{\ell,m}: \mathbb{Q}_\ell \rightarrow \mathbb{Q}_m$ be the inclusion map induced from π_ℓ and π_m , i.e., for all $p \in \mathbb{Q}_\ell$, $\pi_m(i_{\ell,m}(p)) = \{x \in (\gamma^\omega)^m \mid x \upharpoonright \ell \in \pi_\ell(p)\}$. Then each $i_{\ell,m}$ is a complete embedding between posets. Let $(\mathbb{Q}_\omega, (i_n: \mathbb{Q}_n \rightarrow \mathbb{Q}_\omega \mid n < \omega))$ be the direct limit of the system $(i_{\ell,m}: \mathbb{Q}_\ell \rightarrow \mathbb{Q}_m \mid 1 \leq \ell \leq m < \omega)$. We call \mathbb{Q}_ω the *finite support direct limit of Vopěnka algebras for adding an element of γ^ω in $\text{HOD}_{\{T\}}$* .

The following lemmas will be useful in Section 5:

Lemma 2.12. Assume $\text{ZF} + \text{AD}^+ + "V = L(T, \mathbb{R})"$ for some set T of ordinals.

- (1) Let \mathbb{Q}_1 be the Vopěnka algebra for adding an element of 2^ω in $\text{HOD}_{\{T\}}$. Then the poset \mathbb{Q}_1 is of size at most Θ and \mathbb{Q}_1 has the Θ -c.c. in $\text{HOD}_{\{T\}}$.
- (2) Let \mathbb{Q}_ω be the finite support limit of the Vopěnka algebras for adding an element of 2^ω in $\text{HOD}_{\{T\}}$. Then \mathbb{Q}_ω has the Θ -c.c. in $\text{HOD}_{\{T\}}$.
- (3) (Woodin) There is a \mathbb{Q}_ω -generic filter H over $\text{HOD}_{\{T\}}$ such that $V = L(T, \mathbb{R}) \subseteq \text{HOD}_{\{T\}}[H]$ and the set \mathbb{R}^V is countable in $\text{HOD}_{\{T\}}[H]$.

Proof. For (1), we first show that the poset \mathbb{Q}_1 is of size at most Θ in $\text{HOD}_{\{T\}}$. Recall from Definition 2.11 that \mathbb{Q}_1 is a poset on some ordinal η and π_1 is a bijection from η to \mathcal{O}_1 which is OD from T , where \mathcal{O}_1 is the collection of all subsets of 2^ω which are OD from T . We will argue that the ordinal η is at most Θ . For each $\alpha < \Theta$, let W_α be the collection of sets of reals in \mathcal{O}_1 of Wadge rank α . Then we have $\mathcal{O}_1 = \bigcup_{\alpha < \Theta} W_\alpha$ and each W_α is a surjective image of \mathbb{R} . Since the set \mathcal{O}_1 is well-ordered, so is each W_α and there is a surjection from Θ to W_α which is OD from T . Hence there is a surjection from $\Theta \times \Theta$ to \mathcal{O}_1 which is OD from T , and therefore the set \mathbb{Q}_1 is of size at most Θ in $\text{HOD}_{\{T\}}$, as desired.

We next show that the poset \mathbb{Q}_1 has the Θ -c.c. in $\text{HOD}_{\{T\}}$. To derive a contradiction, suppose that there is an antichain $(p_\alpha \mid \alpha < \Theta)$ in \mathbb{Q}_1 in $\text{HOD}_{\{T\}}$. Then the family $\{\pi_1(p_\alpha) \mid \alpha < \Theta\}$ is a pairwise disjoint family of nonempty subsets of 2^ω , which would easily induce a surjection from \mathbb{R} to Θ , contradicting the definition of Θ . Therefore, the poset \mathbb{Q}_1 has the Θ -c.c. in $\text{HOD}_{\{T\}}$, as desired.

For (2), we first note that for all natural numbers n with $n \geq 1$, the poset \mathbb{Q}_n has the Θ -c.c. in $\text{HOD}_{\{T\}}$ by the same argument as in (1). Then using the facts that Θ is regular in $V = L(T, \mathbb{R})$ and that \mathbb{Q}_ω is the direct limit of \mathbb{Q}_n s, it follows that the poset \mathbb{Q}_ω has the Θ -c.c. in $\text{HOD}_{\{T\}}$ as well.

For (3), one can argue in the same way as in [11, Lemma 3.4 and Lemma 3.5] by replacing \mathcal{M} with V , and \mathcal{H} with $\text{HOD}_{\{T\}}$. \square

Lemma 2.13. Assume ZFC. Let λ be a regular uncountable cardinal, \mathbb{P} be a $<\lambda$ -closed poset, and \mathbb{Q} be a λ -c.c. poset. Then for any \mathbb{P} -generic filter G over V , the poset \mathbb{Q} still has the λ -c.c. in $V[G]$. Furthermore, if H is a \mathbb{Q} -generic filter over V , then H is \mathbb{Q} -generic over $V[G]$ as well.

Proof. Let G be a \mathbb{P} -generic filter and A be an antichain in \mathbb{Q} in $V[G]$. We will show that A is of size less than λ in $V[G]$.

Towards a contradiction, we assume that A is of size at least λ in $V[G]$.

Let \dot{A} be a \mathbb{P} -name with $\dot{A}^G = A$. Let p be a condition in G with $p \Vdash_{\mathbb{P}} \text{“}\dot{A} \text{ is an antichain in } \check{\mathbb{Q}} \text{ of size at least } \check{\lambda}\text{”}$. Using the $<\lambda$ -closure of \mathbb{P} in V , one can construct a decreasing sequence $(p_\alpha \mid \alpha < \lambda)$ in \mathbb{P} and a sequence $(a_\alpha \mid \alpha < \lambda)$ in \mathbb{P} with the following properties:

- (1) $p_0 = p$,
- (2) for all $\alpha, \beta < \lambda$ with $\alpha \neq \beta$, we have $a_\alpha \neq a_\beta$, and
- (3) for all $\alpha < \lambda$, $p_\alpha \Vdash_{\mathbb{P}} \text{“}\check{a}_\alpha \in \dot{A}\text{”}$.

Since $p \Vdash_{\mathbb{P}} \text{“}\dot{A} \text{ is an antichain in } \check{\mathbb{Q}}\text{”}$, by (1), (2), and (3) above, for all $\alpha, \beta < \lambda$ with $\alpha < \beta$, the condition p_β forces that a_α and a_β are incompatible in \mathbb{P} . Therefore, the set $B = \{a_\alpha \mid \alpha < \lambda\}$ is an antichain in \mathbb{Q} of size λ in V . This contradicts the assumption that \mathbb{Q} has the λ -c.c. in V . Therefore, the antichain A is of size less than λ in $V[G]$, as desired.

Let H be a \mathbb{Q} -generic filter over V . We will verify that H is \mathbb{Q} -generic over $V[G]$ as well. Let A be a maximal antichain in \mathbb{Q} in $V[G]$. We will see that $H \cap A \neq \emptyset$. By the arguments in the previous paragraphs, A is of size less than λ in $V[G]$. Since G is \mathbb{P} -generic over V and \mathbb{P} is $<\lambda$ -closed in V while \mathbb{Q} is in V , there is no subset of \mathbb{Q} of size less than λ in $V[G] \setminus V$. Hence the antichain A is in V as well. By the genericity of H over V , we have that $H \cap A \neq \emptyset$, as desired. \square

Lemma 2.14. Assume $\text{ZF} + \text{AD}^+ + \text{AD}_{\mathbb{R}}$. Then for any set C of reals, there is an $s \in \Theta^\omega$ such that C is OD from s and that C is in $\text{HOD}_{\{s\}}(\mathbb{R})$.

Proof. Let C be any set of reals. By the result of Woodin [7], under $\text{ZF} + \text{AD}^+ + \text{AD}_{\mathbb{R}}$, every set of reals is Suslin. By the result of Martin and Steel [8], every Suslin and co-Suslin set of reals is homogeneously Suslin. In particular, the complement $2^\omega \setminus C$ is homogeneously Suslin witnessed by the sequence $(\mu_u \mid u \in 2^{<\omega})$ of measures on $\kappa^{<\omega}$ for some $\kappa < \Theta$. By the result of Kunen [5, 28.21 Corollary], each measure μ_u is OD. Using the Moschovakis Coding Lemma and $\text{AD}_{\mathbb{R}}$, one can show that each measure μ_u is definable from an ordinal below Θ . Hence there is an $s \in \Theta^\omega$ such that the sequence $(\mu_u \mid u \in 2^{<\omega})$ is definable from s . Now from the sequence $(\mu_u \mid u \in 2^{<\omega})$, one can construct a Martin-Solovay tree T such that $C = p[T]$. By the construction of T , it follows that T is OD from $(\mu_u \mid u \in 2^{<\omega})$. Hence T is OD from s , which easily implies that the set C is OD from s and C is in $\text{HOD}_{\{s\}}(\mathbb{R})$, as desired. \square

Lemma 2.15. Assume $\text{ZF} + \text{AD}_{\mathbb{R}}$. Let $\gamma < \Theta$ and \mathbb{Q}_1 be the Vopěnka algebra for adding an element of γ^ω in HOD. Also let \mathbb{Q}_ω be the finite support limit of the Vopěnka algebras for adding an element of γ^ω in HOD.

- (1) The posets \mathbb{Q}_1 and \mathbb{Q}_ω are of size less than Θ in HOD.
- (2) Let $s \in \gamma^\omega$ and $h_s = \{p \in \mathbb{Q}_1 \mid s \in \pi_1(p)\}$, where $\pi_1: \mathbb{Q}_1 \rightarrow \mathcal{O}_1$ is as in Definition 2.11. Then the set h_s is a \mathbb{Q}_1 -generic filter over HOD such that $\text{HOD}[h_s] = \text{HOD}_{\{s\}}$.
- (3) (Woodin) There is a \mathbb{Q}_ω -generic filter H over HOD such that the set $(\gamma^\omega)^V$ is countable in $\text{HOD}[H]$.

Proof. For (1), we first show that the poset \mathbb{Q}_1 is of size less than Θ in HOD. Recall from Definition 2.11 that $\pi_1: \mathbb{Q}_1 \rightarrow \mathcal{O}_1$ is a surjection which is OD, where \mathcal{O}_1 is the collection of all subsets of γ^ω which are OD. Since $\gamma < \Theta$, by Theorem 2.8, there is a set A of reals such that there is a surjection from \mathbb{R} to $\wp(\gamma)$ which is OD from A . In particular, there is a surjection $\sigma: \mathbb{R} \rightarrow \gamma^\omega$ which is OD from A . Hence for each $b \in \mathcal{O}_1$, the set $\sigma^{-1}(b)$ of reals is OD from A . Since we assume $\text{AD}_{\mathbb{R}}$, by Theorem 2.6, there is a set B of reals which is not

OD from A . By Wadge's Lemma under $\text{ZF} + \text{AD}$, for each $b \in \mathcal{O}_1$, the set $\sigma^{-1}(b)$ is Wadge reducible to B . In particular, there is a surjection from \mathbb{R} to the family $\{\sigma^{-1}(b) \mid b \in \mathcal{O}_1\}$. Hence the family \mathcal{O}_1 is also a surjective image of \mathbb{R} and the poset \mathbb{Q}_1 is of size less than Θ in V . Since Θ is a cardinal in V , it follows that the poset \mathbb{Q}_1 is of size less than Θ in HOD as well.

We next show that the poset \mathbb{Q}_ω is of size less than Θ in HOD . Let $C = A \oplus B = \{x * y \mid x \in A \text{ and } y \in B\}$, where $x * y(2\ell) = x(\ell)$ and $x * y(2\ell + 1) = y(\ell)$ for all $\ell < \omega$. Then the argument in the last paragraph shows that there is a surjection from \mathbb{R} to \mathbb{Q}_1 which is OD from C . Similarly, one can argue that for each natural numbers n with $n \geq 1$, there is a surjection from \mathbb{R} to \mathbb{Q}_n which is OD from C . Since all such surjections are OD from C , one can pick a sequence $(\rho_n: \mathbb{R} \rightarrow \mathcal{O}_n \mid n \geq 1)$ of surjections, which would readily give us a surjection from \mathbb{R} to \mathbb{Q}_ω . Therefore, the poset \mathbb{Q}_ω is of size less than Θ in V . Since Θ is a cardinal in V , it follows that the poset \mathbb{Q}_ω is of size less than Θ in HOD as well.

For (2), for the \mathbb{Q}_1 -genericity of h_s over HOD , see e.g., [4, Theorem 15.46].

We will show the equality $\text{HOD}[h_s] = \text{HOD}_{\{s\}}$. The inclusion $\text{HOD}[h_s] \subseteq \text{HOD}_{\{s\}}$ is easy because h_s is OD from s and h_s is a set of ordinals. We will argue that $\text{HOD}_{\{s\}} \subseteq \text{HOD}[h_s]$. Since $\text{HOD}_{\{s\}}$ is a model of ZFC , it is enough to see that every set of ordinals in $\text{HOD}_{\{s\}}$ is also in $\text{HOD}[h_s]$. Let X be any set of ordinals in $\text{HOD}_{\{s\}}$. We will verify that X is also in $\text{HOD}[h_s]$. Let δ be an ordinal such that $X \subseteq \delta$. Since X is in $\text{HOD}_{\{s\}}$, the set X is OD from s . So there is a formula ϕ such that for all $\alpha < \delta$, we have that $\alpha \in X$ if and only if $\phi[\alpha, s]$ holds. For each $\alpha < \delta$, let $b_\alpha = \{x \in \gamma^\omega \mid \phi[\alpha, x]\}$. Then each set b_α is a subset of γ^ω which is OD. So each b_α is in \mathcal{O}_1 . Now we have the following equivalences: For all $\alpha < \delta$,

$$\alpha \in X \iff \phi[\alpha, s] \iff s \in b_\alpha \iff \pi_1^{-1}(b_\alpha) \in h_s.$$

Hence $X = \{\alpha < \delta \mid \pi_1^{-1}(b_\alpha) \in h_s\}$. Since the sequence $(\pi_1^{-1}(b_\alpha) \in \mathbb{Q}_1 \mid \alpha < \delta)$ is OD and \mathbb{Q}_1 is in HOD , the sequence $(\pi_1^{-1}(b_\alpha) \in \mathbb{Q}_1 \mid \alpha < \delta)$ belongs to HOD . Hence the set $\{\alpha < \delta \mid \pi_1^{-1}(b_\alpha) \in h_s\}$ is in $\text{HOD}[h_s]$. Therefore, the set X is in $\text{HOD}[h_s]$, as desired.

For (3), one can argue in the same way as in [11, Lemma 3.4 and Lemma 3.5] by replacing \mathbb{R} with γ^ω , \mathcal{M} with V , and \mathcal{H} with HOD . \square

3. ON FORCINGS INCREASING Θ

In this section, we prove the following theorems:

Theorem 3.1. Assume $\text{ZF} + \text{AD}^+ + "V = \text{L}(\wp(\mathbb{R}))"$. Suppose that a poset \mathbb{P} increases Θ , i.e., $\Theta^V < \Theta^{V[G]}$ for any \mathbb{P} -generic filter G over V . Then AD fails in $V[G]$ for any \mathbb{P} -generic filter G over V .

Theorem 3.2. It is consistent relative to $\text{ZF} + \text{AD}_\mathbb{R}$ that $\text{ZF} + \text{AD}$ holds and there is a poset \mathbb{P} increasing Θ while preserving AD , i.e., for any \mathbb{P} -generic filter G over V , we have $\Theta^V < \Theta^{V[G]}$ and that AD holds in $V[G]$.

Proof of Theorem 3.1. Let G be a \mathbb{P} -generic filter over V . We will show that AD fails in $V[G]$. Towards a contradiction, we assume that AD holds in $V[G]$.

Since we have AD^+ and $V = \text{L}(\wp(\mathbb{R}))$, by Theorem 2.4, the model HOD is of the form $\text{L}[Z]$ for some subset Z of Θ , and there are a poset \mathbb{Q} in HOD and a \mathbb{Q} -generic filter H over

HOD such that $\text{HOD} \subseteq V \subseteq \text{HOD}[H]$. In particular, $Z^\#$ does not exist in HOD. Since any poset does not add $Z^\#$, it follows that $Z^\#$ does not exist in V either.

We will argue that $Z^\#$ exists in $V[G]$, which would contradict the fact that $Z^\#$ does not exist in V . Since \mathbb{P} increases Θ , we have $\Theta^V < \Theta^{V[G]}$. By assumption, we have AD in $V[G]$, so by Theorem 2.5, it follows that $\Theta^{V[G]}$ is a limit of measurable cardinals in $V[G]$. In particular, there is a measurable cardinal κ in $V[G]$ such that $\Theta^V < \kappa$. Let U be a $<\kappa$ -complete nonprincipal ultrafilter on κ in $V[G]$. Then letting $M = L[U, Z]$, the cardinal κ is measurable also in M witnessed by $U \cap M$. Since M is a model of ZFC and Z is a bounded subset of κ in M , it follows that $Z^\#$ exists in M . By absoluteness of $Z^\#$, we have $Z^\#$ in $V[G]$, contradicting the fact that $Z^\#$ does not exist in V .

Therefore, the assumption that AD holds in $V[G]$ was wrong, and AD fails in $V[G]$.

This completes the proof of Theorem 3.1. \square

Proof of Theorem 3.2. We assume $\text{ZF} + \text{AD}_{\mathbb{R}}$ and will show that there is an inner model M of $\text{ZF} + \text{AD}$ satisfying that there is a poset \mathbb{P} increasing Θ while preserving AD.

Let $M = \text{HOD}_{\mathbb{R}}$, the class of all sets hereditarily ordinal definable from some real. We will show that M is the desired inner model.

First notice that M is a model of ZF. Also since M contains all the reals and V satisfies AD, we have that M is a model of AD as well. Since we have $\text{AD}_{\mathbb{R}}$ in V , by Theorem 2.6, there is a set B of reals which is not definable from any ordinal and any real. Hence the set B is not in M .

We will show that M satisfies that there is a poset \mathbb{P} increasing Θ while preserving AD. The idea is to consider a variant of Vopěnka algebra in M adding the set B to M . Let $\mathcal{O} = \{b \subseteq \wp(\mathbb{R}) \mid b \text{ is nonempty and OD from some real}\}$ ordered by inclusion. Then \mathcal{O} is a poset which is OD. Let η be a sufficiently large ordinal and let $\pi: \eta \times \mathbb{R} \rightarrow \mathcal{O}$ be a surjection which is OD such that if a set b is in \mathcal{O} and OD from a real x , then there is some $\alpha < \eta$ such that $\pi(\alpha, x) = b$. Let $\mathbb{P} = \pi^{-1}(\mathcal{O})$ and for $p_1, p_2 \in \mathbb{P}$, we set $p_1 \leq p_2$ if $\pi(p_1) \subseteq \pi(p_2)$. Then since π is OD, the poset \mathbb{P} is in M . For an r in \mathbb{P} , let $\mathbb{P} \upharpoonright r = \{p \in \mathbb{P} \mid p \leq r\}$.

We will show that there is some \mathbb{P} -generic filter G over M such that $\Theta^M < \Theta^{M[G]}$ and $M[G]$ is a model of AD. This is enough to end the arguments for the theorem because then there is some $r \in \mathbb{P}$ forcing the desired two statements for $M[G]$ over M , and the poset $\mathbb{P} \upharpoonright r$ is the desired poset in M .

Let $H = \{b \in \mathcal{O} \mid B \in b\}$ and $G = \pi^{-1}(H)$. We will see that G is the desired filter.

We first verify that G is \mathbb{P} -generic over M . Let D be a dense subset of \mathbb{P} in M . We will argue that $G \cap D \neq \emptyset$. Let $E = \pi[D]$ and $b_E = \bigcup E$. By the definition of \mathbb{P} , the set E is dense in \mathcal{O} . We claim that $b_E = \wp(\mathbb{R})$. Suppose not. Then since D is in M and π is OD, the set b_E is OD from some real. So b_E is in \mathcal{O} . But then $\wp(\mathbb{R}) \setminus b_E$ is a nonempty set which is in \mathcal{O} incompatible with any element of E , contradicting that E is dense in \mathcal{O} . Hence $b_E = \wp(\mathbb{R})$. Since B is in $\wp(\mathbb{R})$, we have that B is in b_E , so there is a b' in E such that B is in b' . By the definition of H , the condition b' is also in H . Hence $H \cap E \neq \emptyset$. Since $G = \pi^{-1}(H)$ and $E = \pi[D]$, it follows that $G \cap D \neq \emptyset$, as desired. Therefore, G is \mathbb{P} -generic over M .

We next verify that $M[G]$ is a model of AD. Since B is in V , $H = \{b \in \mathcal{O} \mid B \in b\}$, and $G = \pi^{-1}(H)$, it follows that G is in V and $M[G]$ is a submodel of V . Since M contains all the reals, so does $M[G]$. Finally, since V is a model of AD, it follows that $M[G]$ is also a model of AD, as desired.

Finally, we verify that $\Theta^M < \Theta^{M[G]}$. Since both M and $M[G]$ are models of AD containing all the reals, by the Wadge lemma under ZF + AD, it is enough to see that there is a set of reals in $M[G] \setminus M$. Since B is not in M , it suffices to argue that B is in $M[G]$. For each real x , let $b_x = \{A \in \wp(\mathbb{R}) \mid x \in A\}$. Then b_x is OD from x , so b_x is in \mathcal{O} . By the choice of π , for each real x , there is an ordinal α such that $\pi(\alpha, x) = b_x$. For each real x , let α_x be the least ordinal with $\pi(\alpha_x, x) = b_x$. Then since π and \mathcal{O} are OD, the sequence $(\alpha_x \mid x \in \mathbb{R})$ is OD and is in $M = \text{HOD}_{\mathbb{R}}$. From the sequence $(\alpha_x \mid x \in \mathbb{R})$ and G , one can compute the set B as follows: for any real x ,

$$x \in B \iff B \in b_x \iff b_x \in H \iff (\alpha_x, x) \in G.$$

Therefore, the set B is in $M[G]$, as desired.

We have verified that G is the desired filter, and this completes the proof of Theorem 3.2. \square

4. ON FORCINGS ON THE REALS

In this section, we prove the following theorem which answers a question by Chan and Jackson [2, Question 5.7]:

Theorem 4.1. Assume ZF + AD. Let \mathbb{P} be any non-trivial poset which is a surjective image of \mathbb{R} and G be any \mathbb{P} -generic filter over V . Then AD fails in $V[G]$.

Proof of Theorem 4.1. Let \mathbb{P} be any non-trivial poset which is a surjective image of \mathbb{R} and G be any \mathbb{P} -generic filter over V . We will show that AD fails in $V[G]$. Since \mathbb{P} is a surjective image of \mathbb{R} , there is a poset on \mathbb{R} which is forcing equivalent to \mathbb{P} . Hence we may assume \mathbb{P} is a poset on \mathbb{R} .

Towards a contraction, we assume that AD holds in $V[G]$.

Case 1. When the set \mathbb{R}^V is uncountable in $V[G]$.

Here is the key point:

Claim 1. There is a real r_0 in $V[G]$ such that $\mathbb{R}^{V[G]} \subseteq L(\mathbb{R}^V, r_0)$.

Proof of Claim 1. Since $V[G]$ satisfies AD, the set \mathbb{R}^V has the perfect set property in $V[G]$. Since \mathbb{R}^V is uncountable in $V[G]$, the set \mathbb{R}^V contains a perfect set C in $V[G]$. Let r_0 code a perfect tree T on $2 = \{0, 1\}$ with $[T] = C$ in $V[G]$.

We will show that $\mathbb{R}^{V[G]} \subseteq L(\mathbb{R}^V, r_0)$. Let x be any element of 2^ω in $V[G]$. We will see that x is in $L(\mathbb{R}^V, r_0)$.

We say a node $t \in T$ is *splitting in T* if both $t \frown \langle 0 \rangle$ and $t \frown \langle 1 \rangle$ are in T . Let $\{t_s \in T \mid s \in 2^{<\omega}\}$ be the set of all splitting nodes in T such that if s_1 is a subsequence of s_2 in $2^{<\omega}$, then t_{s_1} is a subsequence of t_{s_2} in T . Let $y = \bigcup \{t_{x \upharpoonright n} \mid n < \omega\}$. Then y is in $[T]$. Since $[T] = C \subseteq \mathbb{R}^V$, the real y is in \mathbb{R}^V . However, for all $n < \omega$ and $k \in 2 = \{0, 1\}$,

$$x(n) = k \iff t_{(x \upharpoonright n) \frown \langle k \rangle} \in y.$$

Hence x can be simply computed from y and T . So $x \in L[y, T]$. Since $L[y, T] \subseteq L[y, r_0] \subseteq L(\mathbb{R}^V, r_0)$, the real x is in $L(\mathbb{R}^V, r_0)$, as desired.

This completes the proof of Claim 1. \square

Continuing to argue in Case 1, let r_0 be a real in $V[G]$ such that $\mathbb{R}^{V[G]} \subseteq L(\mathbb{R}^V, r_0)$ as in Claim 1. Since the poset \mathbb{P} is on \mathbb{R}^V , there is a \mathbb{P} -name \dot{x} such that $\dot{x}^G = r_0$ and \dot{x} is coded by some set A of reals in V . Then setting $M = L(\mathbb{R}^V, \mathbb{P}, A)$, we have that M is an inner model of V satisfying AD and the statement “ Θ is regular”. However, since $\mathbb{R}^{V[G]} \subseteq L(\mathbb{R}^V, r_0) \subseteq M[G] \subset V[G]$ and we assumed that $V[G]$ satisfies AD, the model $M[G]$ also satisfies AD, contradicting Theorem 2.7. Therefore, the assumption that $V[G]$ satisfies AD was wrong and AD must fail in $V[G]$, as desired.

This finishes the arguments for Theorem 4.1 in Case 1.

Case 2. When the set \mathbb{R}^V is countable in $V[G]$.

Since \mathbb{R}^V is countable in $V[G]$, any ordinal α below Θ^V is countable in $V[G]$ as well. Hence $\Theta^V \leq \omega_1^{V[G]}$.

We will show that $\Theta^{V[G]} \leq (\Theta^+)^V$, which would contradict the assumption that AD holds in $V[G]$, because AD in $V[G]$ would imply that $\Theta^{V[G]} > \omega_2^{V[G]} \geq (\Theta^+)^V$ since $\Theta^V \leq \omega_1^{V[G]}$.

To see that $\Theta^{V[G]} \leq (\Theta^+)^V$, let $f: \mathbb{R}^{V[G]} \rightarrow (\Theta^+)^V$ be any function in $V[G]$. We will show that f is not surjective. As in the arguments in Case 1, since \mathbb{P} is on \mathbb{R} , any real in $V[G]$ can be coded by a set of reals in V . Hence we may assume that $f: \wp(\mathbb{R})^V \rightarrow (\Theta^+)^V$. Also, since \mathbb{P} is on \mathbb{R}^V , there is a function $g: \wp(\mathbb{R})^V \times \mathbb{R}^V \rightarrow (\Theta^+)^V$ in V such that $\text{rng}(f) \subseteq \text{rng}(g)$. Therefore, it is enough to see that g is not surjective in V .

We now work in V . To see that g is not surjective, for each $\alpha < \Theta$, let $W_\alpha = \{B \in \wp(\mathbb{R}) \mid |B|_W = \alpha\}$, where $|B|_W$ is the Wadge ordinal of B . Then each W_α is a surjective image of \mathbb{R} and so is the set $R_\alpha = \{g(B, x) \mid B \in W_\alpha, x \in \mathbb{R}\}$. Hence, for every $\alpha < \Theta$, the order type of R_α is less than Θ , and $\text{rng}(g) = \bigcup_{\alpha < \Theta} R_\alpha$ is a surjective image of $\Theta \times \Theta$. Therefore, $\text{rng}(g)$ is of cardinality at most Θ which is smaller than Θ^+ . Hence g is not surjective in V , as desired.

This finishes the arguments for Theorem 4.1 in Case 2.

This completes the proof of Theorem 4.1. □

5. ON FORCINGS ADDING A SUBSET OF Θ

In this section, we prove the following theorems:

Theorem 5.1. Assume $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$. Suppose that Θ is regular. Then there is a poset \mathbb{P} on Θ which adds a subset of Θ while preserving AD, i.e., for any \mathbb{P} -generic filter G over V , there is a subset of Θ^V which belongs to $V[G] \setminus V$ and AD holds in $V[G]$.

Theorem 5.2. Assume $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$. Suppose that Θ is singular and let \mathbb{P} be $\text{Add}(\Theta, 1)$ in HOD, where $\text{Add}(\Theta, 1) = \{p \mid p: \gamma \rightarrow 2 \text{ for some } \gamma < \Theta\}$. Then AD fails in $V[G]$ for any \mathbb{P} -generic filter G over V .

Proof of Theorem 5.1. Throughout the proof of the theorem, we write λ for Θ^V .

We prove the theorem by considering the two cases whether $\text{AD}_{\mathbb{R}}$ holds or not.

Case 1. When $\text{AD}_{\mathbb{R}}$ fails.

Since $\text{AD}_{\mathbb{R}}$ fails while we assume AD^+ and $V = L(\wp(\mathbb{R}))$, by Theorem 2.9, there is a set T of ordinals such that $V = L(T, \mathbb{R})$. We fix such a T throughout the arguments for Case 1.

Let \mathbb{P} be $\text{Add}(\lambda, 1)$ in $\text{HOD}_{\{T\}}$, where $\text{Add}(\lambda, 1) = \{p \mid p: \gamma \rightarrow 2 = \{0, 1\} \text{ for some } \gamma < \lambda\}$. Since \mathbb{P} is computed in $\text{HOD}_{\{T\}}$ and $\lambda = \Theta^V$ is inaccessible in $\text{HOD}_{\{T\}}$, the poset \mathbb{P} can be considered as a poset on λ .

We will show that \mathbb{P} is the desired poset in Case 1, i.e., \mathbb{P} adds a subset of $\lambda = \Theta^V$ while preserving AD in Case 1.

Let G be any \mathbb{P} -generic filter over V . Then the function $\bigcup G: \lambda \rightarrow 2$ can be considered as a subset of λ and by the genericity of G over V , the subset is not in V . Hence the poset \mathbb{P} adds a new subset of λ to V .

We will show that AD holds in $V[G]$. We start with showing that the poset \mathbb{P} does not add any bounded subset of λ :

Claim 1. For any $\gamma < \lambda$, we have $\wp(\gamma)^V = \wp(\gamma)^{V[G]}$. In particular, $\mathbb{R}^V = \mathbb{R}^{V[G]}$.

Proof of Claim 1. Let γ be an ordinal less than λ and $f: \gamma \rightarrow 2$ in $V[G]$. We will show that f is in V .

Let \dot{f} be a \mathbb{P} -name with $\dot{f}^G = f$. Since \mathbb{P} can be seen as a poset on λ and $f: \gamma \rightarrow 2$, we may assume that \dot{f} is a subset of $\lambda \times \gamma \times 2$. Since $V = L(T, \mathbb{R})$ and \dot{f} is in V , we have that \dot{f} is $\text{OD}_{\{T, x\}}$ for some real x . Then since \dot{f} is essentially a set of ordinals, \dot{f} is in $\text{HOD}_{\{T, x\}}$. By Theorem 2.10, we have that $\text{HOD}_{\{T, x\}} = \text{HOD}_{\{T\}}[x]$ and $\text{HOD}_{\{T\}} = L[T, Z]$ for some subset Z of λ . Since $f = \dot{f}^G$, it follows that f is in $\text{HOD}_{\{T, x\}}[G] = \text{HOD}_{\{T\}}[x][G]$.

Let \mathbb{Q}_1 be the Vopěnka algebra for adding an element of 2^ω in $\text{HOD}_{\{T\}}$. Then the real x induces a \mathbb{Q}_1 -generic filter h_x over $\text{HOD}_{\{T\}}$ such that $x \in \text{HOD}_{\{T\}}[h_x]$. Since G was chosen to be \mathbb{P} -generic over V , it is also \mathbb{P} -generic over $\text{HOD}_{\{T\}}[h_x]$. Hence the filter $G \times h_x$ is $\mathbb{P} \times \mathbb{Q}_1$ -generic over $\text{HOD}_{\{T\}}$ and $\text{HOD}_{\{T\}}[x][G] \subseteq \text{HOD}_{\{T\}}[h_x][G] = \text{HOD}_{\{T\}}[G][h_x]$.

Since \dot{f} is in $\text{HOD}_{\{T\}}[x] \subseteq \text{HOD}_{\{T\}}[h_x]$, there is a \mathbb{Q}_1 -name τ in $\text{HOD}_{\{T\}}$ such that $\tau^{h_x} = \dot{f}$. Let ν be a sufficiently big cardinal in $\text{HOD}_{\{T\}}[G]$ and let N be V_ν in $\text{HOD}_{\{T\}}[G]$. By the $< \lambda$ -closure of \mathbb{P} in $\text{HOD}_{\{T\}}$, the ordinal λ is regular in $\text{HOD}_{\{T\}}[G]$. Since $\text{HOD}_{\{T\}}[G]$ is a model of ZFC, there is an elementary substructure X of N in $\text{HOD}_{\{T\}}[G]$ such that $\gamma + 1 \subseteq X$, $X \cap \lambda \in \lambda$, X is of size less than λ , and $T, Z, G, \mathbb{P}, \mathbb{Q}_1, \tau \in X$. Let M be the transitive collapse of X and let $\pi: M \rightarrow X$ be the inverse of the collapsing map. Then letting $\kappa = X \cap \lambda$, we have that κ is the critical point of π and $\pi(\kappa) = \lambda$. For any $a \in X$, we write \bar{a} for $\pi^{-1}(a)$, i.e., $\pi(\bar{a}) = a$.

We claim that M is in $\text{HOD}_{\{T\}}$. Let $g = \bigcup G$. Then by the genericity of G , we have $g: \lambda \rightarrow 2$. Since G is simply definable from g , we have $\text{HOD}_{\{T\}}[G] = \text{HOD}_{\{T\}}[g]$. Recall that $\text{HOD}_{\{T\}} = L[T, Z]$, so $\text{HOD}_{\{T\}}[G] = L[T, Z][G] = L[T, Z][g]$. Hence the model M is of the form $L_\mu[\bar{T}, \bar{Z}][\bar{g}]$ for some μ . Since Z is a subset of λ , we have $\bar{Z} = Z \cap \kappa$ and hence $\bar{Z} \in \text{HOD}_{\{T\}}$. Since \bar{T} is a set of ordinals of size less than λ in $\text{HOD}_{\{T\}}[G]$, by the $< \lambda$ -closure of \mathbb{P} in $\text{HOD}_{\{T\}}$, the set \bar{T} is in $\text{HOD}_{\{T\}}$. Since $g: \lambda \rightarrow 2$, we have $\bar{g} = g \upharpoonright \mu$, which is in \mathbb{P} . So \bar{g} is in $\text{HOD}_{\{T\}}$. Since $M = L_\mu[\bar{T}, \bar{Z}, \bar{g}]$, the model M is in $\text{HOD}_{\{T\}}$, as desired.

Let $\bar{h}_x = \{\bar{q} \mid q \in h_x \cap X\}$. We claim that \bar{h}_x is $\bar{\mathbb{Q}}_1$ -generic over M . Recall that \mathbb{Q}_1 is the Vopěnka algebra for adding an element of 2^ω in $\text{HOD}_{\{T\}}$. By Lemma 2.12, we may assume that \mathbb{Q}_1 is on $\Theta^V = \lambda$ and \mathbb{Q}_1 has the λ -c.c. in $\text{HOD}_{\{T\}}$. Let A be a maximal antichain in $\bar{\mathbb{Q}}_1$ such that A is in M . We will verify that $A \cap \bar{h}_x \neq \emptyset$. Since \mathbb{P} is $< \lambda$ -closed and \mathbb{Q}_1 has the λ -c.c. in $\text{HOD}_{\{T\}}$, by Lemma 2.13, the poset \mathbb{Q}_1 still has the λ -c.c. in $\text{HOD}_{\{T\}}[G]$. By elementarity of π , the poset $\bar{\mathbb{Q}}_1$ has the κ -c.c. in M . In particular, the antichain A is of size

less than κ in M . Since \mathbb{Q}_1 is on λ , the poset $\bar{\mathbb{Q}}_1$ is on κ in M . So the antichain A is a bounded subset of κ . Since κ is the critical point of π , we have that $\pi(A) = A$.

By elementarity of π , the antichain $\pi(A) = A$ is maximal in \mathbb{Q}_1 in $\text{HOD}_{\{T\}}[G]$. Since M is in $\text{HOD}_{\{T\}}$ and A is in M , the antichain A is maximal in \mathbb{Q}_1 in $\text{HOD}_{\{T\}}$ as well. By the genericity of h_x over $\text{HOD}_{\{T\}}$, the set $A \cap h_x$ is nonempty. Let q be an element of $A \cap h_x$. Since A is in M and M is transitive, the condition q is in M . Since \mathbb{Q}_1 is on λ , $\pi(\kappa) = \lambda$, and $q \in h_x \cap M$, we have that $\pi(q) = q$ and hence $q \in \bar{h}_x$. Therefore, $q \in A \cap \bar{h}_x$ and the set $A \cap \bar{h}_x$ is nonempty, as desired.

Since the poset \mathbb{Q}_1 has the κ -c.c. in M , by a standard argument, one can lift the embedding $\pi: M \rightarrow N$ to an elementary embedding $\hat{\pi}: M[\bar{h}_x] \rightarrow N[h_x]$ such that $\hat{\pi}(\bar{h}_x) = h_x$.

We now argue that the function f is in V . It is enough to verify that f is in $M[\bar{h}_x]$ because M is in $\text{HOD}_{\{T\}}$, $\text{HOD}_{\{T\}} \subseteq V$, and $\bar{h}_x = \{\bar{q} \mid q \in h_x \cap X\} = \{q \mid q \in h_x \cap X\} = \bar{\mathbb{Q}}_1 \cap h_x$. Recall that τ is a \mathbb{Q}_1 -name in $\text{HOD}_{\{T\}}$ such that $\tau^{h_x} = \dot{f}$ and that \dot{f} is a \mathbb{P} -name in $\text{HOD}_{\{T\}}[h_x]$ such that $\dot{f}^G = f$. Since τ is in X , letting $\dot{g} = \bar{\tau}^{\bar{h}_x}$ and $g = \dot{g}^{\bar{G}}$, we have that $\hat{\pi}(g) = f$. By elementarity of $\hat{\pi}$, the set g is a function from $\pi^{-1}(\gamma)$ to 2. We now verify that $f = g$, which would imply that f is in $M[\bar{h}_x]$ because g is in $M[\bar{h}_x]$. Since $\gamma + 1 \subseteq X$, we have that $\pi \upharpoonright (\gamma + 1) = \text{id}$. Hence $\pi^{-1}(\gamma) = \gamma$ and $g: \gamma \rightarrow 2$. Also, since $\hat{\pi}(g) = f$, by elementarity of $\hat{\pi}$, for any $\alpha < \gamma$ and $i < 2$, we have that $g(\alpha) = i$ if and only if $f(\alpha) = i$. Therefore, $f = g$, as desired.

This completes the proof of Claim 1. \square

By Claim 1, we know that $\mathbb{R}^V = \mathbb{R}^{V[G]}$. So we simply write \mathbb{R} for \mathbb{R}^V or $\mathbb{R}^{V[G]}$. Recall that we write λ for Θ^V .

We now show that \mathbb{P} does not add any set of reals to V :

Claim 2. The equality $\wp(\mathbb{R})^V = \wp(\mathbb{R})^{V[G]}$ holds.

Proof of Claim 2. Let A be any set of reals in $V[G]$. We will show that A is in V as well.

We first claim that λ is regular in $V[G]$ and $\lambda = \Theta^{V[G]}$. Let \mathbb{Q}_ω be the finite support direct limit of Vopěnka algebras for adding an element of 2^ω in $\text{HOD}_{\{T\}}$. Then by Lemma 2.12, the poset \mathbb{Q}_ω has the λ -c.c. in $\text{HOD}_{\{T\}}$ and there is a \mathbb{Q}_ω -generic filter H over $\text{HOD}_{\{T\}}$ such that $V = L(T, \mathbb{R}) \subseteq \text{HOD}_{\{T\}}[H]$ and the set \mathbb{R} is countable in $\text{HOD}_{\{T\}}[H]$. Since \mathbb{P} is $<\lambda$ -closed in $\text{HOD}_{\{T\}}$ and \mathbb{Q}_ω has the λ -c.c. in $\text{HOD}_{\{T\}}$, by Lemma 2.13, the poset \mathbb{Q}_ω still has the λ -c.c. in $\text{HOD}_{\{T\}}[G]$ and the filter H is \mathbb{Q}_ω -generic over $\text{HOD}_{\{T\}}[G]$ as well. Hence λ is still regular uncountable in $\text{HOD}_{\{T\}}[G][H]$, the filter $G \times H$ is $\mathbb{P} \times \mathbb{Q}_\omega$ -generic over $\text{HOD}_{\{T\}}$, and $\text{HOD}_{\{T\}}[G][H] = \text{HOD}_{\{T\}}[H][G]$. Therefore, λ is still regular uncountable in $\text{HOD}_{\{T\}}[H][G]$. Since $V[G] \subseteq \text{HOD}_{\{T\}}[H][G]$, the ordinal λ is regular in $V[G]$ as well. Also, since \mathbb{R} is countable in $\text{HOD}_{\{T\}}[H]$ and $\mathbb{R}^V = \mathbb{R}^{V[G]}$ while $V[G] \subseteq \text{HOD}_{\{T\}}[H][G]$, the ordinal $\Theta^{V[G]}$ is at most ω_1 in $\text{HOD}_{\{T\}}[H][G]$. Since λ is regular uncountable in $\text{HOD}_{\{T\}}[H][G]$, we have that $\Theta^{V[G]} \leq \lambda$. Since $V \subseteq V[G]$ and $\lambda = \Theta^V$, the inequality $\lambda \leq \Theta^{V[G]}$ also holds. Hence $\lambda = \Theta^{V[G]}$, as desired.

Let ν be a sufficiently large cardinal in $V[G]$ and let N be V_ν in $V[G]$. Since $V = L(T, \mathbb{R})$, the model N is of the form $L_\nu(T, \mathbb{R})[G]$. Since every element of N is definable from T, G , an ordinal, and some real while λ is regular in $V[G]$ and $\lambda = \Theta^{V[G]}$, one can find an elementary substructure X of N in $V[G]$ such that $\mathbb{R} \subseteq X$, $\lambda \cap X \in \lambda$, the structure X is a surjective image of \mathbb{R} , and $T, \mathbb{P}, G, A \in X$. Let M be the transitive collapse of X and let $\pi: M \rightarrow X$

be the inverse of the collapsing map. Then letting $\kappa = \lambda \cap X$, the critical point of π is κ and $\pi(\kappa) = \lambda$. For any a in X , we write \bar{a} for $\pi^{-1}(a)$, i.e., $\pi(\bar{a}) = a$.

We will finish arguing that the set A is in V . Since \mathbb{R} is contained in M and $\pi(\bar{A}) = A$, we have $\bar{A} = A$ and the set A is in M . Hence it is enough to verify that the model M is in V . Recall that $g = \bigcup G$ and $g: \lambda \rightarrow 2$. Since G is simply definable from g , we have that $N = L_\nu(T, \mathbb{R})[G] = L_\nu(T, \mathbb{R})[g]$. Since N is of the form $L_\nu(T, \mathbb{R})[g]$, X is a surjective image of \mathbb{R} in $V[G]$, and $\lambda = \Theta^{V[G]}$, it follows that M is of the form $L_\mu(\bar{T}, \mathbb{R})[\bar{g}]$ for some ordinal $\mu < \lambda$. Since $\mu < \lambda$, the set \bar{T} is a bounded subset of λ in $V[G]$. By Claim 1, the set \bar{T} is in V as well. Since $g: \lambda \rightarrow 2$ and $\pi(\kappa) = \lambda$, we have that $\bar{g} = g \upharpoonright \kappa$ and \bar{g} is in \mathbb{P} . So \bar{g} is in V as well. Since $M = L_\mu(\bar{T}, \mathbb{R})[\bar{g}]$, the model M is in V , and the set A is in V , as desired.

This completes the proof of Claim 2. \square

By Claim 1 and Claim 2, we have that $\mathbb{R}^V = \mathbb{R}^{V[G]}$ and $\wp(\mathbb{R})^V = \wp(\mathbb{R})^{V[G]}$. Since we assume **AD** in V , the axiom **AD** holds in $V[G]$ as well.

This finishes the arguments for Theorem 5.1 in Case 1 when **AD** $_{\mathbb{R}}$ fails.

Case 2. When **AD** $_{\mathbb{R}}$ holds.

Recall that we write λ for Θ^V . Let \mathbb{P} be $\text{Add}(\lambda, 1)$ in **HOD**, where $\text{Add}(\lambda, 1) = \{p \mid p: \gamma \rightarrow 2 = \{0, 1\} \text{ for some } \gamma < \lambda\}$. Since \mathbb{P} is computed in **HOD** and $\lambda = \Theta^V$ is inaccessible in **HOD**, the set \mathbb{P} can be considered as a poset on λ . Let G be a \mathbb{P} -generic filter over V . We will show that **AD** holds in $V[G]$.

Claim 3. The forcing \mathbb{P} does not add any new set of reals, i.e., $\wp(\mathbb{R})^V = \wp(\mathbb{R})^{V[G]}$.

Proof of Claim 3. We will show that for any $f: \mathbb{R}^V \rightarrow 2$ in $V[G]$, the function f is also in V . Since any real in $V[G]$ can be simply coded as a subset of \mathbb{R}^V in $V[G]$, this will show that $\mathbb{R}^V = \mathbb{R}^{V[G]}$ and $\wp(\mathbb{R})^V = \wp(\mathbb{R})^{V[G]}$ as well.

From now on, we write \mathbb{R} for \mathbb{R}^V .

Subclaim 1. For some sequence $s \in \lambda^\omega$, the function f is in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$.

Proof of Subclaim 1. Let \dot{f} be a \mathbb{P} -name with $\dot{f}^G = f$. Since \mathbb{P} can be considered as a poset on λ , we may assume that \dot{f} can be considered as a subset of $\lambda \times \mathbb{R} \times 2$. To make it simpler, we regard \dot{f} as a subset of $\lambda \times \mathbb{R}$.

Since $V = L(\wp(\mathbb{R}))$, there is a set A of reals such that \dot{f} is OD from A . For each $\alpha < \lambda$, let $X_\alpha = \{x \in \mathbb{R} \mid (\alpha, x) \in \dot{f}\}$ and let ξ_α be the least ordinal $\xi < \lambda$ such that $X_\xi = X_\alpha$. For $\alpha, \beta < \lambda$, we write $\alpha \preceq \beta$ if $\xi_\alpha \leq \xi_\beta$. Then the structure (λ, \preceq) is a prewellordering. Let $\pi: (\lambda, \preceq) \rightarrow (\gamma, \leq)$ be the Mostowski collapsing map. For each $\delta < \gamma$, let $\eta_\delta = \min \pi^{-1}(\delta)$ and $Y_\delta = X_{\eta_\delta}$. Set $Y = (Y_\delta \mid \delta < \gamma)$.

Since \dot{f} is OD from A , so is π . Also π is essentially a set of ordinals, so π is in $\text{HOD}_{\{A\}}$.

We next verify that there is a set B of reals such that Y is in $L(B, \mathbb{R})$. Since we have **AD** $_{\mathbb{R}}$ in Case 2, by Theorem 2.6, there is a set B_0 of reals which is not OD from A and any real. Since \dot{f} is OD from A , so is Y . So each set Y_δ of reals is OD from A . By the Wadge Lemma under **ZF** + **AD**, each Y_δ is Wadge reducible to B_0 . In particular, there is a surjection from \mathbb{R} to $\{Y_\delta \mid \delta < \gamma\}$ in $L(B_0, \mathbb{R})$. Since the sequence $Y = (Y_\delta \mid \delta < \gamma)$ is injective, there is a surjection $\rho: \mathbb{R} \rightarrow \gamma$ in $L(B_0, \mathbb{R})$ as well. Let $B_1 = \{x * y \mid x \in Y_{\rho(y)}\}$, where

$x * y(2n) = x(n)$ and $x * y(2n + 1) = y(n)$ for all $n \in \omega$. Then Y is in $L(\rho, B_1, \mathbb{R})$. So letting $B = B_0 \oplus B_1 = \{x * y \mid x \in B_0 \text{ and } y \in B_1\}$, we have that Y is in $L(B, \mathbb{R})$, as desired.

We now argue that for some sequence $s \in \lambda^\omega$, the \mathbb{P} -name \dot{f} is in $\text{HOD}_{\{s\}}(\mathbb{R})$. Since π is in $\text{HOD}_{\{A\}}$ and Y is in $L(B, \mathbb{R})$, letting $C = A \oplus B$, we have that π is in $\text{HOD}_{\{C\}}$ and Y is in $L(C, \mathbb{R})$. Since we have $\text{AD}_{\mathbb{R}}$ in Case 2, by Lemma 2.14, there is an $s \in (\Theta^V)^\omega = \lambda^\omega$ such that C is OD from s and that C is in $\text{HOD}_{\{s\}}(\mathbb{R})$. Hence both π and Y are in $\text{HOD}_{\{s\}}(\mathbb{R})$. Since \dot{f} is simply definable from π and Y , we have that \dot{f} is in $\text{HOD}_{\{s\}}(\mathbb{R})$, as desired.

Since $f = \dot{f}^G$ and \dot{f} is in $\text{HOD}_{\{s\}}(\mathbb{R})$, we have that f is in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$.

This completes the proof of Subclaim 1. \square

Since we assume that $\lambda = \Theta^V$ is regular in V and $s \in \lambda^\omega \cap V$, we can pick an ordinal $\gamma < \lambda$ such that $s \in \gamma^\omega$. Let \mathbb{Q}_1 be the Vopěnka algebra for adding an element of γ^ω in HOD and let $h_s = \{p \in \mathbb{Q}_1 \mid s \in \pi_1(p)\}$, where $\pi_1: \mathbb{Q}_1 \rightarrow \mathcal{O}_1$ is as in Definition 2.11. Then by Lemma 2.15, we have that h_s is a \mathbb{Q}_1 -generic filter over HOD such that $\text{HOD}[h_s] = \text{HOD}_{\{s\}}$.

Subclaim 2. The ordinal λ is regular in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ and there is no surjection from \mathbb{R}^V to λ in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$.

Proof of Subclaim 2. Let \mathbb{Q}_ω be the finite support limit of the Vopěnka algebras for adding an element of γ^ω in HOD. Since \mathbb{Q}_1 is a complete suborder of \mathbb{Q}_ω , by Lemma 2.15, there is a \mathbb{Q}_ω -generic filter H over HOD such that $h_s \in \text{HOD}[H]$ and that the set $(\gamma^\omega)^V$ is countable in $\text{HOD}[H]$. In particular, $\text{HOD}_{\{s\}}(\mathbb{R}) \subseteq \text{HOD}[h_s](\mathbb{R}) \subseteq \text{HOD}[H]$ and \mathbb{R}^V is countable in $\text{HOD}[H]$. Since we have $\text{AD}_{\mathbb{R}}$ in Case 2, by Lemma 2.15, the poset \mathbb{Q}_ω is of size less than $\Theta^V = \lambda$. Since \mathbb{P} is $<\lambda$ -closed in HOD and G is \mathbb{P} -generic over HOD, we have that any subset of \mathbb{Q}_ω in $\text{HOD}[G]$ is also in HOD. Hence the filter H is \mathbb{Q}_ω -generic over $\text{HOD}[G]$ as well.

We now argue that λ is regular in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$. Since \mathbb{P} is $<\lambda$ -closed in HOD and G is \mathbb{P} -generic over HOD, the ordinal λ is still regular in $\text{HOD}[G]$. Also, since \mathbb{Q}_ω is of size less than λ in HOD and H is \mathbb{Q}_ω -generic over $\text{HOD}[G]$, the ordinal λ is also regular in $\text{HOD}[G][H]$. Since $\text{HOD}_{\{s\}}(\mathbb{R})[G] \subseteq \text{HOD}[H][G] = \text{HOD}[G][H]$, the ordinal λ is regular in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$, as desired.

We next show that there is no surjection from \mathbb{R}^V to λ in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$. Since \mathbb{R}^V is countable in $\text{HOD}[H]$ while λ is regular uncountable in $\text{HOD}[H][G]$, there is no surjection from \mathbb{R}^V to λ in $\text{HOD}[H][G]$. Since $\text{HOD}_{\{s\}}(\mathbb{R})[G] \subseteq \text{HOD}[H][G]$, there is no surjection from \mathbb{R}^V to λ in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$, as desired.

This completes the proof of Subclaim 2. \square

Recall that \mathbb{Q}_1 is the Vopěnka algebra for adding an element of γ^ω and h_s is the \mathbb{Q}_1 -generic filter over HOD derived from s with $\text{HOD}[h_s] = \text{HOD}_{\{s\}}$. Since we have $\text{AD}_{\mathbb{R}}$ in Case 2, by Lemma 2.15, the poset \mathbb{Q}_1 is of size less than $\Theta^V = \lambda$ in HOD. So the filter h_s is essentially a bounded subset of λ . Since we assume $\text{ZF} + \text{AD}^+ + \text{“}V = L(\wp(\mathbb{R}))\text{”}$, by Theorem 2.4, there is a set $Z \subseteq \Theta^V = \lambda$ such that $\text{HOD} = L[Z]$. So the model $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ is of the form $L(Z, h_s, \mathbb{R})[G]$ where Z is a subset of λ and h_s is a bounded subset of λ . By Subclaim 1, the function f is in the model $\text{HOD}_{\{s\}}(\mathbb{R})[G]$.

We are now ready to finish the arguments for Claim 3, which are similar to those for Claim 2. Let ν be a sufficiently big cardinal in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ and let N be V_ν in $V[G]$. Let $g = \bigcup G$. Then by the genericity of G , we have $g: \lambda \rightarrow 2$. Since G is simply definable from

g , we also have $\text{HOD}_{\{s\}}(\mathbb{R})[G] = \text{HOD}_{\{s\}}(\mathbb{R})[g]$. Since $\text{HOD}_{\{s\}}(\mathbb{R}) = \text{L}(Z, h_s, \mathbb{R})$, the model N is of the form $\text{L}_\nu(Z, h_s, \mathbb{R})[G] = \text{L}_\nu(Z, h_s, \mathbb{R})[g]$. Since every element of N is definable from Z, h_s, g , an ordinal, and some real while λ is regular in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ and there is no surjection from \mathbb{R} to λ in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ by Subclaim 2, one can find an elementary substructure X of N in $\text{HOD}_{\{s\}}(\mathbb{R})[G]$ such that $\mathbb{R} \subseteq X$, $\lambda \cap X \in \lambda$, the structure X is a surjective image of \mathbb{R} , and $Z, \mathbb{P}, h_s, G, f \in X$. Let M be the transitive collapse of X and let $\pi: M \rightarrow X$ be the inverse of the collapsing map. Then letting $\kappa = \lambda \cap X$, the critical point of π is κ and $\pi(\kappa) = \lambda$. For any a in X , we write \bar{a} for $\pi^{-1}(a)$, i.e., $\pi(\bar{a}) = a$.

We will finish arguing that the function f is in V . Since \mathbb{R} is contained in M and $\pi(\bar{f}) = f$, we have $\bar{f} = f$ and the set f is in M . Hence it is enough to verify that the model M is in V . Since N is of the form $\text{L}_\nu(Z, h_s, \mathbb{R})[g]$, the set M is of the form $\text{L}_\mu(\bar{Z}, \bar{h}_s, \mathbb{R})[\bar{g}]$ for some ordinal μ . Since Z is a subset of λ , we have $\bar{Z} = Z \cap \kappa$, which is in V . The filter h_s is essentially bounded subset of λ and h_s is in X . So by elementarity of X , we have that $h_s \subseteq X$ and it follows that $\bar{h}_s = h_s$, which is also in V . Since $g: \lambda \rightarrow 2$, we have $\bar{g} = g \upharpoonright \kappa$ and so \bar{g} is in \mathbb{P} . Hence we have $\bar{g} \in V$. Since $M = \text{L}_\mu(\bar{Z}, \bar{h}_s, \mathbb{R})[\bar{g}]$, the model M is in V , and hence the function f is in V , as desired.

This completes the proof of Claim 3. \square

By Claim 3, we have $\wp(\mathbb{R})^{V[G]} = \wp(\mathbb{R})^V$. Since **AD** holds in V , so does in $V[G]$, as desired. This finishes the arguments for Theorem 5.1 in Case 2 when **AD** $_{\mathbb{R}}$ holds.

This completes the proof of Theorem 5.1. \square

Proof of Theorem 5.2. Let G be any \mathbb{P} -generic filter over V . We will show that **AD** fails in $V[G]$. To derive a contradiction, we assume **AD** in $V[G]$.

Since we have **ZF** + **AD** $^+$ + “ $V = \text{L}(\wp(\mathbb{R}))$ ” in V , by Theorem 3.1, it is enough to show that \mathbb{P} increases Θ , i.e., $\Theta^V < \Theta^{V[G]}$.

Let γ be the cofinality of Θ in V . Since Θ is singular in V , we have that $\gamma < \Theta$.

We will show that there is an injection from Θ^V to $\wp(\gamma)^{V[G]}$ in $V[G]$, which would imply $\Theta^V < \Theta^{V[G]}$ as follows: Since we assumed **AD** in $V[G]$, by Theorem 2.8, there is a surjection from $\mathbb{R}^{V[G]}$ to $\wp(\gamma)^{V[G]}$ in $V[G]$. By the existence of an injection from Θ^V to $\wp(\gamma)^{V[G]}$, there would be a surjection from $\mathbb{R}^{V[G]}$ to Θ^V in $V[G]$. By the definition of $\Theta^{V[G]}$, we would have that $\Theta^V < \Theta^{V[G]}$, as desired.

We will construct a function $\iota: \Theta^V \rightarrow \wp(\gamma)^{V[G]}$ in $V[G]$ which is verified to be injective. Since $\mathbb{P} = \text{Add}(\Theta, 1)$ in **HOD** and G is \mathbb{P} -generic over V , the set $g = \bigcup G$ is a function from Θ^V to $2 = \{0, 1\}$. Since γ is the cofinality of Θ in V , we can fix a cofinal increasing sequence $(\beta_\alpha: \alpha < \gamma)$ in Θ in V . For each $\delta < \Theta^V$, let a_δ be the sequence $(\beta_\alpha + \delta \mid \alpha < \gamma)$ in Θ in V . Now let $\iota(\delta) = \{\alpha < \gamma \mid g(a_\delta(\alpha)) = 1\}$. Then $\iota(\delta)$ is a subset of γ for each $\delta < \Theta^V$.

We will verify that the function $\iota: \Theta^V \rightarrow \wp(\gamma)^{V[G]}$ is injective. Let δ, ϵ be distinct ordinals less than Θ^V . We will see that $\iota(\delta) \neq \iota(\epsilon)$. First notice that the functions a_δ and a_ϵ are different everywhere: For all $\alpha < \gamma$, we have $a_\delta(\alpha) = \beta_\alpha + \delta \neq \beta_\alpha + \epsilon = a_\epsilon(\alpha)$. Now since a_δ and a_ϵ are different everywhere and both are in V , by the genericity of G , there is an $\alpha < \gamma$ such that $g(a_\delta(\alpha)) \neq g(a_\epsilon(\alpha))$, and hence $\alpha \in \iota(\delta) \triangle \iota(\epsilon)$. Therefore, we have $\iota(\delta) \neq \iota(\epsilon)$, as desired.

This completes the proof of Theorem 5.2. \square

6. QUESTIONS

We close this paper by raising two questions.

Question 6.1. Assume $\text{ZF} + \text{AD}$. Let \mathbb{P} be a poset which adds a new real and let G be \mathbb{P} -generic over V . Then must AD fail in $V[G]$?

To answer ‘No’ to Question 6.1, one would need to find a poset which changes the structure of cardinals below Θ drastically as follows: Woodin proved that if there is a poset which adds a new real while preserving the truth of AD, then the poset must collapse ω_1 . Also, by Theorem 3.1, if such a poset exists in a model of $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$, then the poset must preserve Θ . Furthermore, by the arguments for [2, Lemma 2.10] by Chan and Jackson, if a poset adds a new real while preserving the truth of AD, then any weak partition property of a cardinal in its generic extension cannot be witnessed by a club in the ground model. Hence, if such a poset preserves Θ as well, then for cofinally many cardinals κ below Θ , the poset must shoot a club in κ which does not contain any club in κ in the ground model.

There are many things we do not know on forcings over $\text{ZF} + \text{AD}$ especially when Θ is singular. One of them is whether the assumption “ Θ is regular” in Theorem 5.1 is essential or not:

Question 6.2. Assume $\text{ZF} + \text{AD}^+ + “V = L(\wp(\mathbb{R}))”$. Suppose that Θ is singular. Then is there any poset which adds a new subset of Θ while preserving AD?

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(D. Ikegami) COLLEGE OF ENGINEERING, SHIBAURA INSTITUTE OF TECHNOLOGY, 307 FUKASAKU,
MINUMA-WARD, SAITAMA CITY, 337-8570, SAITAMA JAPAN
Email address, D. Ikegami: ikegami@shibaura-it.ac.jp

(N. Trang) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE
311430, DENTON, TX 76203-5017, USA
Email address, N. Trang: nam.trang@unt.edu