AN INTEGRAL REGION CHOICE PROBLEM WITH TWO PROHIBITED REGIONS

YAMATO FUKUSHIMA AND IN DAE JONG

Abstract. We consider an integral region choice problem on a knot projection with two prohibited regions, where the use of these regions is restricted. We show that solvability of the problem is closely related to the difference between Alexander numbers of the two prohibited regions. In particular, if the difference of Alexander numbers is equal to one, then there exists a unique solution for the problem. We also provide a necessary and sufficient condition for solvability when the difference is not equal to one.

1. Introduction

We consider *an integral region choice problem* on a knot projection proposed by Ahara and Suzuki [1], which is a generalization of *a region choice game* originally proposed by Kawauchi, Kishimoto, and Shimizu based on a study of a region crossing change operation [6].

The rules of the integral region choice problem are as follows. Let P be a knot projection with n crossings $C_1, \ldots, C_n$. Throughout this paper, we assume that a knot projection contains at least one crossing. Each crossing C_i is equipped with an integer c_i for $i = 1, \ldots, n$, referred to as its *color* [3]. Colors can be changed by assigning an integer r to a region R of P as follows:

$$\begin{cases} c_i \mapsto c_i + 2r & (\text{if } \partial R \text{ touches } C_i \text{ twice}), \\ c_i \mapsto c_i + r & (\text{if } \partial R \text{ touches } C_i \text{ once}), \\ c_i \mapsto c_i & (\text{if } C_i \notin \partial R). \end{cases}$$

Here ∂R denotes the boundary of a region R. This rule for changing colors is called the *double count rule* in [1]. Another rule, the *single count rule*, is defined by:

$$\begin{cases} c_i \mapsto c_i + r & (\text{if } \partial R \text{ touches } C_i), \\ c_i \mapsto c_i & (\text{if } C_i \notin \partial R). \end{cases}$$

Date: April 21, 2024.
2020 *Mathematics Subject Classification.* Primary 57K10, Secondary 15A03, 15A06.
Key words and phrases. integral region choice problem, Alexander numbering, knot projection.
Jong is partially supported by JSPS KAKENHI Grant Number 22K03324.

Recently, Batal and Gügümcü [3] introduced more general rules defined by:

$$\begin{cases} c_i \mapsto c_i + a_i r & \text{(if } \partial R \text{ touches } C_i \text{ twice)}, \\ c_i \mapsto c_i + r & \text{(if } \partial R \text{ touches } C_i \text{ once)}, \\ c_i \mapsto c_i & \text{(if } C_i \notin \partial R). \end{cases}$$

Here a_i is an integer called an *increment number* of C_i [3]. Note that setting each a_i equal to two (resp. one) results in the double (resp. single) count rule. If a knot projection P is reduced, that is, P contains no nugatory crossings, then there are no difference among these rules. In this paper, we employ the double count rule for a technical reason (cf. Lemma 2.2). For studies based on other rules, refer to Section 4. Let $R_1, \ldots, R_{n+2}$ be regions of P. Note that there are two more regions than crossings. The goal of the integral region choice problem is to find suitable integers $r_1, \cdots, r_{n+2}$ assigned to the regions, which transform all colors to zeros. See Figure 1 for an example.

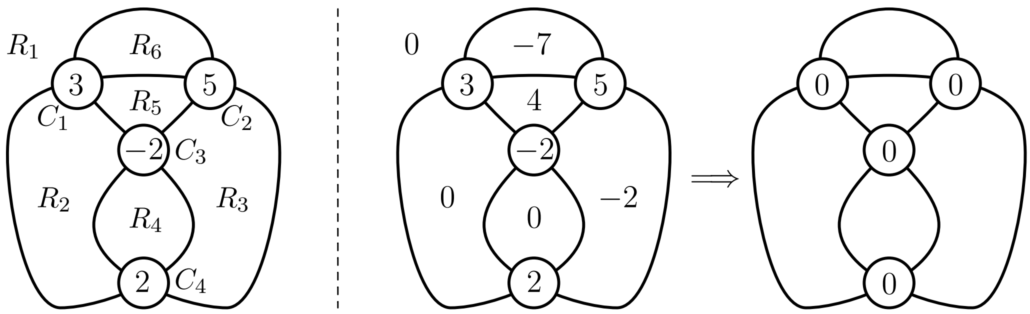

FIGURE 1. Given colors are $c_1 = 3$, $c_2 = 5$, $c_3 = -2$, $c_4 = 2$ respectively. Assigning integers to regions as $r_1 = 0$, $r_2 = 0$, $r_3 = -2$, $r_4 = 0$, $r_5 = 4$, $r_6 = -7$, all colors are changed to zeros.

The *region choice matrix* (of the double count rule) of P is the matrix $A(P) = \big(a_{ij}\big)$ of size $n \times (n+2)$ defined by

$$a_{ij} = \begin{cases} 2 & \text{(if } \partial R_j \text{ touches } C_i \text{ twice)}, \\ 1 & \text{(if } \partial R_j \text{ touches } C_i \text{ once)}, \\ 0 & \text{(if } C_i \notin \partial R). \end{cases}$$

Then the goal of the integral region choice problem on P with colors $c_1, \ldots, c_n$ is equivalent to finding a vector $\boldsymbol{r} = \big(r_i\big) \in \mathbb{Z}^{n+2}$ which satisfies

$$A(P)\boldsymbol{r} + \boldsymbol{c} = \boldsymbol{0} \tag{1}$$

for $\boldsymbol{c} = \big(c_i\big) \in \mathbb{Z}^n$. A vector $\boldsymbol{r} \in \mathbb{Z}^{n+2}$ in Equation (1) is called a *solution* for an integral region choice problem on a knot projection P with colors $\boldsymbol{c}$. On the other hand, a vector $\boldsymbol{u} \in \mathbb{Z}^{n+2}$ satisfying

$$A(P)\boldsymbol{u} = \boldsymbol{0} \tag{2}$$

is called a *kernel solution*[1] for a knot projection P in [1]. Note that a kernel solution does not depend on colors $\boldsymbol{c}$. We call the set of all kernel solutions for P the *kernel solution space of* P, which is a free $\mathbb{Z}$-module.

Ahara and Suzuki proved the following.

Proposition 1.1 ([1, Theorem 3.1])**.** *There exists a solution for any knot projection and colors.*

That is to say, an integral region choice problem is always solvable.

Proposition 1.2 ([1, Theorem A.2])**.** *For any knot projection P with n crossings,* $\operatorname{rank} A(P) = n$*.*

Example 1.3. For an integral region choice problem of Figure 1, $\boldsymbol{c} = \begin{pmatrix} 3 \\ 5 \\ -2 \\ 2 \end{pmatrix}$ and $A(P) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$. Then $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 4 \\ -7 \end{pmatrix}$ is a solution, that is,

$$A(P)\boldsymbol{r} + \boldsymbol{c} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 4 \\ -7 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the size of $A(P)$ is $n \times (n+2)$, considering Proposition 1.2, it is natural to ask whether a solution exists or not by setting $(n+2) - n = 2$ *prohibited regions*, namely, the regions where we cannot assign integers to change colors. Here, we call such a problem an *integral region choice problem with two prohibited regions*. We denote by $\boldsymbol{r}_{jk}$ a solution for the problem when R_j and R_k are prohibited regions. In other words, $\boldsymbol{r}_{jk}$ represents a solution for an integral region choice problem, which satisfies $r_j = r_k = 0$. As a preceding result on this problem, Batal and Gügümcü proved the following in their modified rules [3, Theorem 3.7].

Proposition 1.4. *If two prohibited regions R_j and R_k are adjacent, then there exists a unique solution $\boldsymbol{r}_{jk}$.*

We note that Shimizu [7] showed the same result with modulo 2 reduction.

[1] It is called a *null pattern* for P in [3].

In this paper, we provide a necessary and sufficient condition for the solvability of an integral region choice problem with two prohibited regions. One of key ingredients is an Alexander numbering, which is originally introduced in [2] to give a combinatorial definition of a now well-known polynomial invariant. For the definition of Alexander numbers, see Section 2.

Let R_1 be the unbounded region, and R_2 a region adjacent to R_1. As shown in Section 2, there exists a 'canonical' solution $\boldsymbol{b} = \big(b_i\big) \in \mathbb{Z}^{n+2}$ with prohibited regions R_1 and R_2 (for details, see Lemma 2.6). The following is the main theorem of this paper.

Theorem 1.5. *Consider an integral region choice problem on a knot projection P with prohibited regions R_j and R_k. Let d be the absolute value of the difference between Alexander numbers of R_j and R_k. Let b_j (resp. b_k) be the j-th (resp. k-th) element of $\boldsymbol{b} = \big(b_i\big) \in \mathbb{Z}^{n+2}$, as defined in Lemma 2.6.*

(i) Assume that $d = 1$. Then there exists a unique solution $\boldsymbol{r}_{jk}$.
(ii) Assume that $d \geq 2$. Then there exists a solution $\boldsymbol{r}_{jk}$ if and only if $b_j \equiv (-1)^d b_k \pmod{d}$. In this case, the solution is unique if it exists.
(iii) Assume that $d = 0$. Then there exists a solution $\boldsymbol{r}_{jk}$ if and only if $b_j = b_k$. In this case, we have infinitely many solutions if it exists.

Remark 1.6. Theorem 1.5 (i) is a generalization of Proposition 1.4 since if R_j and R_k are adjacent, then $d = 1$.

Organization of the paper. In Section 2, we recall the definition of Alexander numbers and construct a basis of the kernel solution space using Alexander numbers. In Section 3, we prove Theorem 1.5. In Section 4, we provide concrete examples of Theorem 1.5 and propose some problems for future studies.

2. A Basis of the kernel solution space

In this section, we construct a basis of the kernel solution space of a given knot projection using Alexander numbers.

Let P be an oriented knot projection with $n+2$ regions $R_1, \ldots, R_{n+2}$. As introduced in [2], for each region R_i, one can assign an integer $a(R_i)$ called *an Alexander number* (or *an Alexander index*) satisfying the following condition: When R_i and R_j share an oriented edge of P and R_i (resp. R_j) is on the left side (resp. right side) of the edge as depicted in Figure 2, $a(R_j) = a(R_i) + 1$ holds. Assigning such numbers on all regions is called an *Alexander numbering* (or an *Alexander indexing*). Note that an Alexander numbering may begin with an arbitrary region assigned an arbitrary integer, called the *initial number*, for the given oriented knot projection.

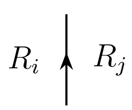

Figure 2. $a(R_j) = a(R_i) + 1$

Example 2.1. Let P be the oriented knot projection depicted in Figure 3. Let $-P$ be the same knot projection as P but with the opposite orientation. Assigning the initial number 0 to R_1, we obtain an Alexander numbering of P as depicted in the center of Figure 3. On the other hand, assigning to the initial number 1 to R_1, we obtain an Alexander numbering of $-P$ as depicted in the right side of Figure 3.

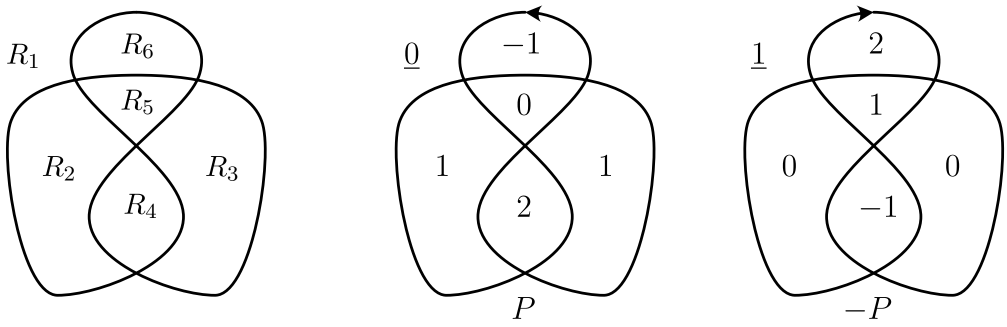

FIGURE 3. Example of Alexander numbering

Lemma 2.2. *Let P be an oriented knot projection with $n+2$ regions $R_1, \dots, R_{n+2}$, where R_1 is unbounded and R_2 is adjacent to R_1. Take an Alexander numbering so that $a(R_1) = 0$ and $a(R_2) = 1$ by reversing the orientation of P if necessary. Let $a_j = a(R_j)$, and set $u_j = (-1)^{a_j - 1} a_j$, $u'_j = u_j + (-1)^{a_j}$. Then two vectors $\boldsymbol{u} = \big(u_i\big)$, $\boldsymbol{u}' = \big(u'_i\big) \in \mathbb{Z}^{n+2}$ form a basis of the kernel solution space of P.*

Proof. Four regions near a crossing C_i of P have Alexander numbers $a, a+1, a+1,$ and $a+2$ for some $a \in \mathbb{Z}$ as depicted in Figure 4. Thus, the i-th row of $A(P)\boldsymbol{u}$ is given by:

$$\begin{cases} -a + 2(a+1) - (a+2) = 0 & \text{if } a \text{ is even,} \\ a - 2(a+1) + (a+2) = 0 & \text{if } a \text{ is odd.} \end{cases}$$

This follows from the double count rule. Hence, $\boldsymbol{u}$ is a kernel solution for P. Similarly, the i-th row of $A(P)\boldsymbol{u}'$ is given by:

$$\begin{cases} -a + 1 + 2a - (a+1) = 0 & \text{if } a \text{ is even,} \\ a - 1 - 2a + (a+1) = 0 & \text{if } a \text{ is odd.} \end{cases}$$

Hence, $\boldsymbol{u}'$ is also a kernel solution for P. Notice that $\boldsymbol{u}$ and $\boldsymbol{u}'$ are linearly independent since $u_1 = 0, u_2 = 1$ and $u'_1 = 1, u'_2 = 0$ by construction. Since the rank-nullity theorem holds for $\mathbb{Z}$-modules, by Proposition 1.2, a basis of the kernel solution space consists of two integral vectors. Thus, the conditions $u_1 = 0, u_2 = 1$ and $u'_1 = 1, u'_2 = 0$ guarantee that $\boldsymbol{u}$ and $\boldsymbol{u}'$ span the kernel solution space of P. □

Remark 2.3. The construction of the kernel solution $\boldsymbol{u}$ in Lemma 2.2 was provided by Kawamura [5, Lemma 7.1]. A key factor to construct the basis $\boldsymbol{u}, \boldsymbol{u}'$ is that a knot projection admits just two orientations. By selecting an orientation of P such that $a(R_1) = 0$ and $a(R_2) = 1$, and reversing the sign of even Alexander numbers,

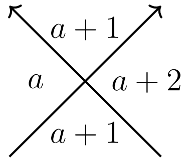

FIGURE 4. Alexander numbers of four regions around a crossing C_i

we obtain the kernel solution $\boldsymbol{u}$. Subsequently, by reversing the orientation of P and assigning Alexander numbers such that $a(R_1) = 1$ and $a(R_2) = 0$, and reversing the sign of even Alexander numbers, we obtain the kernel solution $\boldsymbol{u}'$.

Remark 2.4. The absolute value of the difference between Alexander numbers of two regions does not depend on the orientation of a knot projection and the initial number.

Example 2.5. For a knot projection in Figure 3, $\boldsymbol{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}$, $\boldsymbol{u}' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ -2 \end{pmatrix}$.

By Proposition 1.4 and Lemma 2.2, we have the following.

Lemma 2.6. *Let P be a knot projection with n crossings. For any colors $\boldsymbol{c} \in \mathbb{Z}^n$, there exists a unique solution $\boldsymbol{b} = \big(b_i\big) \in \mathbb{Z}^{n+2}$ satisfying $b_1 = b_2 = 0$ such that any solution $\boldsymbol{r} \in \mathbb{Z}^{n+2}$ for $A(P)\boldsymbol{r} + \boldsymbol{c} = \boldsymbol{0}$ is given by*

$$\boldsymbol{r} = \boldsymbol{b} + \alpha\boldsymbol{u} + \beta\boldsymbol{u}' \quad (\alpha, \beta \in \mathbb{Z}). \tag{3}$$

Here, $\boldsymbol{u}$ and $\boldsymbol{u}'$ are the kernel solutions constructed in Lemma 2.2.

The 'canonical' solution $\boldsymbol{b}$ coincides with the solution of the integral region choice problem with two prohibited regions R_1 and R_2, denoted as $\boldsymbol{r}_{12}$. Since R_1 is adjacent to R_2, $\boldsymbol{r}_{12}$ exists uniquely by Proposition 1.4.

Setting $\boldsymbol{r} = \big(r_i\big)$, Equation (3) is represented as:

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{n+2} \end{pmatrix} = \begin{pmatrix} b_1 = 0 \\ b_2 = 0 \\ b_3 \\ \vdots \\ b_{n+2} \end{pmatrix} + \alpha \begin{pmatrix} u_1 = 0 \\ u_2 = 1 \\ u_3 \\ \vdots \\ u_{n+2} \end{pmatrix} + \beta \begin{pmatrix} u'_1 = 1 \\ u'_2 = 0 \\ u'_3 \\ \vdots \\ u'_{n+2} \end{pmatrix} \quad (\alpha, \beta \in \mathbb{Z}). \tag{4}$$

Note that $\boldsymbol{u}$ and $\boldsymbol{u}'$ depend on P, but not on colors $\boldsymbol{c}$. Also note that the solution $\boldsymbol{b}$ depends on $\boldsymbol{u}, \boldsymbol{u}'$ (and P), and colors $\boldsymbol{c}$. Lemma 2.6 provides all solutions for an integral region choice problem. Although we can directly obtain the basis $\boldsymbol{u}, \boldsymbol{u}'$ of the kernel solution space from P without solving simultaneous linear equations, obtaining the solution $\boldsymbol{b}$ is not as straightforward. See Question 4.1 in Section 4.

Remark 2.7. Equation (4) ensures that one can solve an integral region choice problem by assigning an arbitrary pair of integers (α, β) to an adjacent region pair. Furthermore, for a given pair of integers, the solution is unique. This fact is also observed in [3, Proposition 3.10].

Example 2.8. Consider an integral region choice problem on Figure 1. Notice that the solution $\boldsymbol{r}$ in Example 1.3 coincides with $\boldsymbol{b}$ in Lemma 2.6, meaning that the first and second elements are zeros. Thus, by Examples 1.3 and 2.5, all solutions are given by:

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 4 \\ -7 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ -2 \end{pmatrix} \qquad (\alpha, \beta \in \mathbb{Z}).$$

3. Proof of Theorem 1.5

Let P be a knot projection with $n+2$ regions $R_1, \dots, R_{n+2}$, where R_1 is unbounded and R_2 is adjacent to R_1. From now on, by Lemma 2.6, we may assume that any solution for an integral region choice problem on P with color $\boldsymbol{c}$ is given by Equation (4).

Proof of Theorem 1.5. Recall that

$$\begin{aligned} u_j &= (-1)^{a_j - 1} a_j, \\ u'_j &= u_j + (-1)^{a_j} = (-1)^{a_j - 1}(a_j - 1), \end{aligned}$$

where a_j is an Alexander number of R_j such that $a_1 = 0$ and $a_2 = 1$. Without loss of generality, we may assume that $a_k = a_j + d$ with an integer $d \geq 0$. Extracting the j-th and k-th rows of Equation (4), we have

$$\begin{aligned} \begin{pmatrix} r_j \\ r_k \end{pmatrix} &= \begin{pmatrix} b_j \\ b_k \end{pmatrix} + \alpha \begin{pmatrix} u_j \\ u_k \end{pmatrix} + \beta \begin{pmatrix} u'_j \\ u'_k \end{pmatrix} \\ &= \begin{pmatrix} b_j \\ b_k \end{pmatrix} + \begin{pmatrix} (-1)^{a_j - 1} a_j & (-1)^{a_j - 1}(a_j - 1) \\ (-1)^{a_k - 1} a_k & (-1)^{a_k - 1}(a_k - 1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} b_j \\ b_k \end{pmatrix} - (-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ (-1)^d (a_j + d) & (-1)^d (a_j + d - 1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \end{aligned} \tag{5}$$

There is a one-to-one correspondence between a solution with two prohibited regions R_j and R_k, denoted as $\boldsymbol{r}_{jk}$ with $r_j = r_k = 0$, and a solution of the following equation with integral variables α, β, obtained by setting $r_j = r_k = 0$ in Equation (5):

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_j \\ b_k \end{pmatrix} - (-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ (-1)^d (a_j + d) & (-1)^d (a_j + d - 1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6}$$

We proceed with a case-by-case argument depending on the value of d.

(i) Assume that $d = 1$. Then Equation (6) becomes:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_j \\ b_k \end{pmatrix} - (-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ -a_j - 1 & -a_j \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{7}$$

Solving Equation (7), we obtain the unique integral solution:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (-1)^{a_j} \begin{pmatrix} -b_k + (b_j + b_k)a_j \\ -b_j - (b_j + b_k)a_j \end{pmatrix}.$$

(ii) Assume that $d \geq 2$. Note that $\begin{vmatrix} a_j & a_j - 1 \\ (-1)^d(a_j + d) & (-1)^d(a_j + d - 1) \end{vmatrix} = (-1)^d d.$

Since $d \geq 2$, the matrix $(-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ (-1)^d(a_j + d) & (-1)^d(a_j + d - 1) \end{pmatrix}$ is singular over $\mathbb{Z}$, but regular over $\mathbb{Q}$. Solving Equation (6) over $\mathbb{Q}$, we have

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \left((-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ (-1)^d(a_j + d) & (-1)^d(a_j + d - 1) \end{pmatrix} \right)^{-1} \begin{pmatrix} b_j \\ b_k \end{pmatrix} \\ &= \frac{(-1)^{a_j}}{d} \begin{pmatrix} a_j + d - 1 & (-1)^d(-a_j + 1) \\ -(a_j + d) & (-1)^d a_j \end{pmatrix} \begin{pmatrix} b_j \\ b_k \end{pmatrix} \\ &= (-1)^{a_j} \begin{pmatrix} b_j + \dfrac{(a_j - 1)(b_j - (-1)^d b_k)}{d} \\ -b_j - \dfrac{a_j(b_j - (-1)^d b_k)}{d} \end{pmatrix}. \end{aligned} \tag{8}$$

Assuming that $b_j \equiv (-1)^d b_k \pmod{d}$, both α and β become integers, leading to a unique solution $\boldsymbol{r}_{jk}$ with two prohibited regions R_j and R_k. Conversely, if a solution $\boldsymbol{r}_{jk}$ exists, making α and β in the LHS of Equation (8) integers, then $b_j \equiv (-1)^d b_k \pmod{d}$ must hold since $a_j - 1$ and a_j are coprime.

(iii) Assume that $d = 0$. Then Equation (6) becomes:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_j \\ b_k \end{pmatrix} - (-1)^{a_j} \begin{pmatrix} a_j & a_j - 1 \\ a_j & a_j - 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{9}$$

If $b_k = b_j$, then Equation (9) is equivalet to:

$$a_j\alpha + (a_j - 1)\beta = (-1)^{a_j} b_j\,.$$

Since a_j and $a_j - 1$ are coprime, this equation has infinitely many integral solutions $\{\alpha, \beta\}$ Conversely, if a solution exists for Equation (9), then $b_k = b_j$ must hold.

This completes the proof. In fact, we have shown that for $d = 1$, there exists a unique solution $\boldsymbol{r}_{jk}$. For $d \geq 2$, a solution $\boldsymbol{r}_{jk}$ exists if and only if $b_j \equiv (-1)^d b_k \pmod{d}$, and it is unique when it exists. Additionally, for $d = 0$, a solution $\boldsymbol{r}_{jk}$ exists if and only if $b_j = b_k$, and there are infinitely many solutions when it exists. □

4. Example and problems

In this section, we present concrete examples of Theorem 1.5 and propose some problems for future studies.

4.1. **Example.** Let P be the knot projection depicted in Figure 5. Assigning the initial number 0 to R_1, we obtain the Alexander numbering as depicted in the right side of Figure 5.

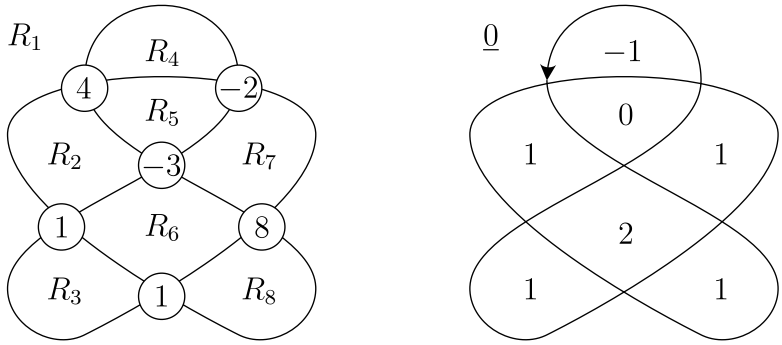

FIGURE 5

Consider the region choice problem with two prohibited regions, as depicted in the left side of Figure 5, where the initial colors are provided. By solving simultaneous linear equations, we find

$$\boldsymbol{r}_{12} = \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 13 \\ -15 \\ 11 \\ -14 \\ 6 \\ 0 \end{pmatrix}.$$

See the left side of Figure 6. Now we consider three cases (i) $d = 1$, (ii) $d = 2$, and (iii) $d = 0$ as stated in Theorem 1.5.

(i) Let R_3 and R_5 be the prohibited regions. They are not adjacent each other, but $d = |a(R_3) - a(R_5)| = 1$. By Theorem 1.5 (i), there is a unique solution $\boldsymbol{r}_{35}$. Actually, we can find $\boldsymbol{r}_{35} = \begin{pmatrix} -11 \\ -13 \\ 0 \\ 20 \\ 0 \\ 23 \\ -7 \\ -13 \end{pmatrix}$. See the right side of Figure 6.

(ii) Let R_1 and R_6 be the prohibited regions. We have $d = |a(R_1) - a(R_6)| = 2$, and $b_1 = 0$, $b_6 = -14$. Then $b_1 \equiv (-1)^2 b_6 \pmod 2$ holds. By Theorem 1.5 (ii),

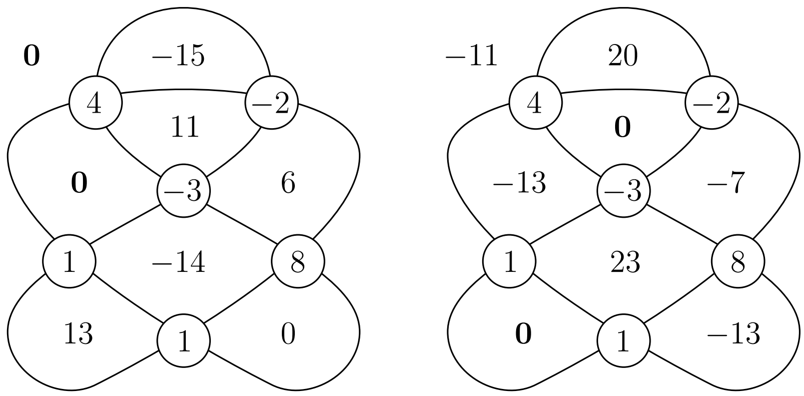

FIGURE 6. $\boldsymbol{b} = \boldsymbol{r}_{12}$ (left side) and $\boldsymbol{r}_{35}$ (right side)

there is a unique solution $\boldsymbol{r}_{16}$. Actually, we can find $\boldsymbol{r}_{16} = \begin{pmatrix} 0 \\ -7 \\ 6 \\ -8 \\ 11 \\ 0 \\ -1 \\ -7 \end{pmatrix}$. See the left side of Figure 7.

On the other hand, let R_5 and R_6 be the prohibited regions. We have $d = |a(R_5) - a(R_6)| = 2$, and $b_5 = 11$, $b_6 = -14$. Then $b_5 \not\equiv (-1)^2 b_6 \pmod 2$. By Theorem 1.5 (ii), there is no solution. In this case, we have the 'non-integral solution' $\begin{pmatrix} -11 \\ -3/2 \\ 23/2 \\ 17/2 \\ 0 \\ 0 \\ 9/2 \\ -3/2 \end{pmatrix}$. See the right side of Figure 7.

(iii) Let R_2 and R_8 be the prohibited regions. We have $d = |a(R_2) - a(R_8)| = 0$, and $b_2 = b_8 (= 0)$. By Theorem 1.5 (iii), there are infinitely many solutions. Actually, we can find $\boldsymbol{r}_{28} = \begin{pmatrix} \alpha \\ 0 \\ 13 \\ -2\alpha - 15 \\ \alpha + 11 \\ -\alpha - 14 \\ 6 \\ 0 \end{pmatrix}$ for any $\alpha \in \mathbb{Z}$. See Figure 8.

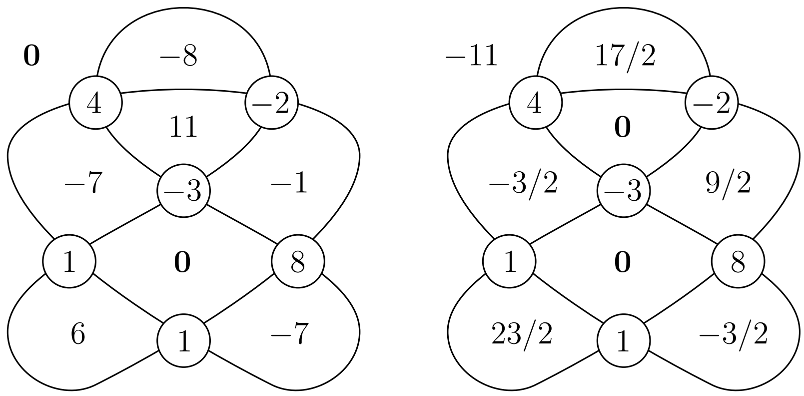

FIGURE 7. $\boldsymbol{r}_{16}$ (left side) and "non-integral solution" (right side)

On the other hand, let R_2 and R_3 be the prohibited regions. We have $d = |a(R_2) - a(R_3)| = 0$ and $b_2 \neq b_3$. Hence, there is no solution. In this case, there are no 'non-integral solutions'.

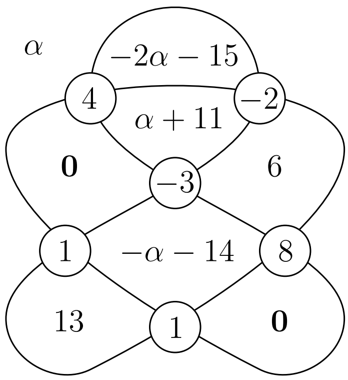

FIGURE 8. Infinitely many solutions $\boldsymbol{r}_{28}$

4.2. **Problems.** Finally, we propose some problems for future research. As previously mentioned, a basis $\{\boldsymbol{u}, \boldsymbol{u}'\}$ of the kernel solution space is directly obtained from a given knot projection P by using Alexander numberings without the need to solve any simultaneous linear equations. According to Lemma 2.6, if we can also obtain $\boldsymbol{b} = \boldsymbol{r}_{12}$ directly, we could solve every integral region choice problem without resorting to linear algebra. Of course, we can determine $\boldsymbol{b} = \boldsymbol{r}_{12}$ by utilizing a square matrix $A'(P)$, derived by removing the first and second columns from $A(P)$. More precisely, assume that R_1 and R_2 are adjacent, then $A'(P)$ is regular over $\mathbb{Z}$, and thus we have

$$\boldsymbol{b} = -A'(P)^{-1}\boldsymbol{c}.$$

Problem 4.1. *For a given knot projection P and colors $\boldsymbol{c}$, can we obtain the 'canonical' solution $\boldsymbol{b} = \boldsymbol{r}_{12}$ without linear algebra? In other words, can we derive $\boldsymbol{b}$ directly from P and $\boldsymbol{c}$?*

Throughout this paper, the double count rule has been applied. As mentioned in Section 1, there are other rules by utilizing increment numbers introduced in [3], which contain the single and the double count rule originally introduced in [1]. Changing the rules makes a difference when the knot projection is reducible. For a reducible knot projection, the following problem is one of interest.

Problem 4.2. *Let P be a reducible knot projection. Study an integral region choice problem on P with two prohibited regions using the modified rule that assigns increment numbers to reducible crossings.*

We can consider subspecies of a region choice problem, that is, a region freeze version [4], alternating version [5], and so on. The case where a projection has more than one components is also interest.

Problem 4.3. *Study an integral region choice problem with prohibited regions on a link projection.*

Acknowledgements

The authors express their gratitude to Tetsuya Abe for his valuable suggestions on our draft. Additionally, they extend their thanks to Megumi Hashizume, Ayumu Inoue, Yeonhee Jang, Kengo Kawamura, Ayaka Shimizu, and Masaaki Suzuki for their discussions. Special thanks are also due to the referees for their thorough review and helpful suggestions.

References

[1] K. Ahara and M. Suzuki, An integral region choice problem on knot projection, J. Knot Theory Ramifications **21** (2012), No. 11, 1250119, 20 pp.
[2] J. W. Alexander, Topological invariants of knots and links, Trans. Amer. Math. Soc. **30** (1928), 275–306.
[3] A. Batal and N. Gügümcü, k-color region select game, J. Math. Soc. Japan **75** (2023), No. 4, 1379–1399.
[4] A. Inoue and R. Shimizu, A subspecies of region crossing change, region freeze crossing change, J. Knot Theory Ramifications **25** (2016), no. 14, 1650075, 9 pp.
[5] T. Kawamura, Integral region choice problem on link diagrams, Osaka J. Math. **60** (2023), no. 4, 835–872.
[6] A. Shimizu, Region crossing change is an unknotting operation, J. Math. Soc. Japan **66** (2014), No. 3, 693–708.
[7] A. Shimizu, On region unknotting numbers, RIMS Kôkyûroku **1766** (2011), 15–22.

Kishiwada City Office, 7-1 Kishiki Cho, Kishiwada City, Osaka 596-0073, Japan
Email address: yamatake02031119@gmail.com

Department of Mathematics, Kindai University, 3-4-1 Kowakae, Higashiosaka City, Osaka 577-0818, Japan
Email address: jong@math.kindai.ac.jp