ON POSITIVE KNOTS OF GENUS TWO

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Abstract. We show that positive knots of genus two are positive-alternating or almost positive-alternating. We also show that positive knots of genus two are quasi-alternating. In addition, we show that every prime positive knot of genus two is obtained from one of certain fourteen positive diagrams by $\overline{t_2'}$ moves.

1. Introduction

A diagram is *positive* if the signs of all crossings are positive, and a link is *positive* if it has a positive diagram. A diagram is *alternating* if over-crossings and under-crossings appear alternately along every component of the diagram, and a link is *alternating* if it has an alternating diagram. A link is *positive-alternating* if it has a diagram which is positive and alternating.

Proposition 1.1 ([12]). *If a link is positive and alternating, then the link is positive-alternating.*

A diagram is *almost alternating* (resp. *almost positive-alternating*) if a single crossing change turns it into an alternating diagram (resp. a positive-alternating diagram) (cf. [2]), and a link is *almost alternating* (resp. *almost positive-alternating*) if it has an almost alternating (resp. an almost positive-alternating) diagram and no alternating (resp. positive alternating) diagram.

Note that every positive and almost alternating knot is almost positive-alternating with up to eleven crossings (cf. [4]). Then the following question comes to mind.

Question 1.2. *Let K be a positive knot. If K is almost alternating, then is K almost positive-alternating ?*

In this paper, we show that the answer to Question 1.2 is affirmative up to genus two. Precisely, we show the following theorem.

Theorem 1.3. *Positive knots up to genus two are positive-alternating or almost positive-alternating.*

On the other hand, an almost alternating knot is one of generalizations of an alternating knot. In terms of a dealternating number (see [1], [2]), the class of almost alternating knots is "nearest" to that of alternating knots. By Theorem 1.3, positive knots up to genus two is "near" to the class of alternating knots. Here we consider another generalization of an alternating knot, that is, a quasi-alternating knot [14] which is closely related to both of the Khovanov homology and the knot Floer homology [10].

Theorem 1.4. *Positive knots up to genus two are quasi-alternating.*

This paper is organized as follows: In Section 2, we prove Theorem 1.3. The proof is achieved by using the generators for canonical genus two knots, which are introduced by Stoimenow [16], and some properties of the Jones polynomial. In Section 3, we refine Stoimenow's generators for positive knots of genus two (Theorem 3.1). In Section 4, we prove Theorem 1.4.

2. Proof of Theorem 1.3.

First we review the generators for positive knots of genus two and some properties of the Jones polynomial, and then we prove Theorem 1.3.

2.1. Generators.

A diagram depicted in Figure 1 is called a *generator* (for positive knots of genus two). We denote by $\mathcal{G}_2^+$ the set of generators. We say that a positive knot (of genus two) is *generated* by a generator $G \in \mathcal{G}_2^+$ if the knot has a diagram obtained by applying $\overline{t_2'}$ moves on the generator G. Here a $\overline{t_2'}$ *move* is a local move on diagrams applied at a crossing as shown in Figure 2.

Lemma 2.1 ([16]). *Every prime positive knot of genus two is generated by one of the twenty four generators in $\mathcal{G}_2^+$.*

Remark 2.2. Throughout this paper, unless otherwise specified, we use the notation of KnotScape [5] for knots.

Remark 2.3. The generators 5_1^+, 7_5^+, 8_{15}^+, 9_{23}^+, 9_{38}^+, 10_{101}^+, 10_{120}^+, 11_{123}^+, 11_{329}^+, 12_{1097}^+, and 13_{4233}^+ are positive-alternating diagrams. The generators 6_2^+, 6_3^+, 7_6^+, 7_7^+, and 8_{12}^+ represent the alternating knot 5_1. The generator 8_{14}^+ represents the alternating knot 7_5. The generators 9_{25}^+ and 10_{58}^+ represent the alternating knot 8_{15}. The generator 10_{97}^+ represents the alternating knot 9_{38}. The generator 11_{148}^+ represents the alternating knot 10_{101}. The generators 9_{39}^+ and 9_{41}^+ represent the non-alternating knot 9_{49}. The generator 12_{1202}^+ represents the non-alternating knot 12_{n881}.

2.2. Jones polynomial.

The *Jones polynomial* [6] $V_L(t)$ of a link L is the $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-valued invariant of a link, which satisfies the skein relationship

$$t^{-1}V_{\nearrow\!\!\!\!\nwarrow}(t) - tV_{\nwarrow\!\!\!\!\nearrow}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{)(}(t).$$

We define the Jones polynomial of a diagram D as that of the link L represented by D: $V_D(t) = V_L(t)$. For a non-zero polynomial $f(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$, we denote by $\operatorname{maxdeg} f$ (resp. $\operatorname{mindeg} f$) the maximal degree (resp. the minimal degree) of $f(t)$, and by $\operatorname{maxdegcf} f$ the leading coefficient of $f(t)$. Let $\operatorname{span} f = \operatorname{maxdeg} f - \operatorname{mindeg} f$. We give two lemmas needed to prove Theorem 1.3.

Lemma 2.4 ([9], [11], [17]). *For a link L with the crossing number $c(L)$, we have*

$$\operatorname{span} V_L \leq c(L)\,.$$

Lemma 2.5 ([15, Theorem 3.1]). *Let L be a positive link. Then we have*

$$\operatorname{mindeg} V_L = (1 - \chi(L))/2\,.$$

Here $\chi(L)$ is the Euler characteristic of L.

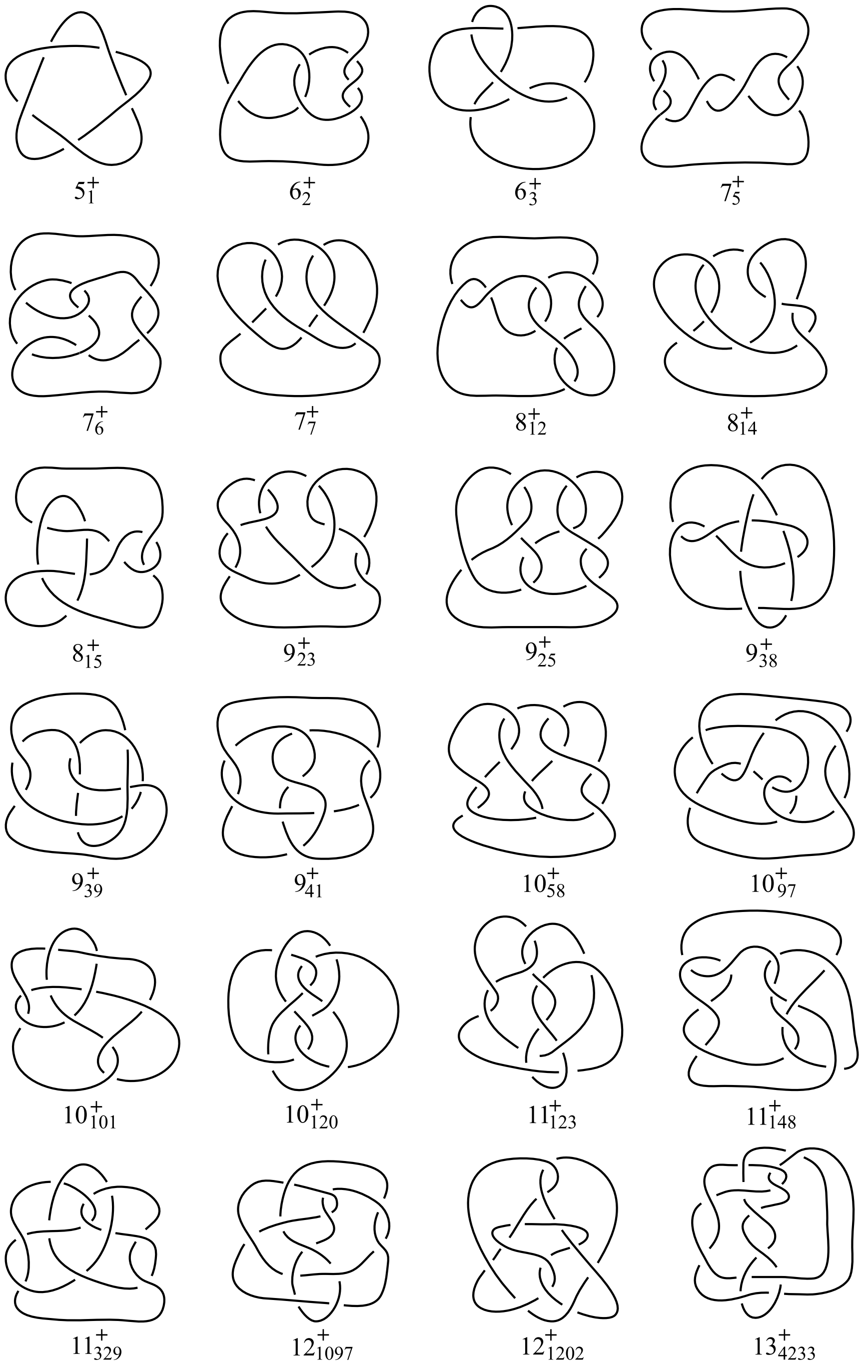

FIGURE 1. The generators of positive knots of genus two

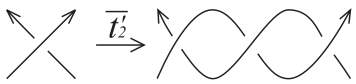

FIGURE 2. $\overline{t_2'}$ move

2.3. **Proof of Theorem 1.3.** Let G be a generator and $c_1, \dots, c_n$ the crossings of G. We denote by $G(a_1, \dots, a_n)$ the diagram obtained from G by applying $\overline{t_2'}$ moves a_i times at c_i for $i = 1, \dots, n$. Here $a_1, \dots, a_n$ are non-negative integers. Note that the crossing number of the diagram $G(a_1, \dots, a_n)$ is equal to $n + 2(a_1 + \cdots + a_n)$. We represent continuous twists by a white or a shaded box as shown in Figure 3. A white (resp. a shaded) box contains odd (resp. even) number of crossings.

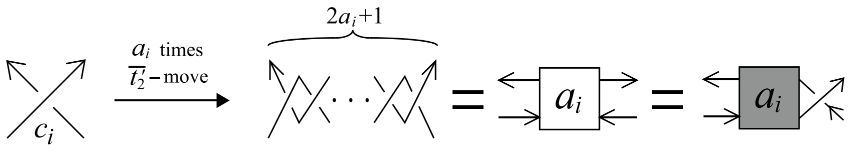

FIGURE 3

Proof of Theorem 1.3. A positive knot of genus one is a pretzel knot of type $(-2p-1, -2q-1, -2r-1)$ for some non-negative integers p, q, and r, and then, it is positive-alternating. Since a positive diagram of a non-prime positive knot is non-prime [13], every non-prime positive knot of genus two is the connected sum of two positive pretzel knots of genus one. Thus, every non-prime positive knot of genus two is positive-alternating.

By Lemma 2.1, it remains to prove that every positive knot generated by each generator in $\mathcal{G}_2^+$ is positive-alternating or almost positive-alternating. It is clear that positive knots generated by alternating generators (5_1^+, 7_5^+, 8_{15}^+, 9_{23}^+, 9_{38}^+, 10_{101}^+, 10_{120}^+, 11_{123}^+, 11_{329}^+, 12_{1097}^+, and 13_{4233}^+) are positive-alternating. As shown in Figures 4 and 5, positive knots generated by each one of the generators 6_2^+, 6_3^+, 7_6^+, 7_7^+, 8_{12}^+, 8_{14}^+, 9_{25}^+, 10_{58}^+, 10_{97}^+, and 11_{148}^+ have positive-alternating diagrams, and positive knots generated by each one of the generators 9_{39}^+, 9_{41}^+, and 12_{1202}^+ have almost positive-alternating diagrams. Thus, it suffices to show that positive knots generated by each one of the generators 9_{39}^+, 9_{41}^+, and 12_{1202}^+ are non-alternating. Since the Jones polynomial of an alternating link is monic [17], the following three claims (Claim 2.6, 2.7, and 2.8) guarantee that positive knots generated by each one of the generators 9_{39}^+, 9_{41}^+, and 12_{1202}^+ are non-alternating.

Claim 2.6. *For the diagram $D = 9_{39}^+(a_1, \dots, a_9)$, we have*

$$\operatorname{maxdegcf} V_D = -2,$$
$$\operatorname{maxdeg} V_D = c(D).$$

Proof. We prove the claim by induction on $\alpha = a_1 + \cdots + a_9$. If $\alpha = 0$, that is, $a_1 = \cdots = a_9 = 0$, then $D = 9_{39}^+$ is a positive diagram of the knot 9_{49}. Calculating the Jones polynomial, we have $V_{9_{39}^+}(t) = V_{9_{49}}(t) = -2t^9 + 3t^8 - 4t^7 + 5t^6 - 4t^5 + 4t^4 - 2t^3 + t^2$, and thus, the claim is true. Assuming that the claim is true for $\alpha = k \geq 0$, we prove it for $\alpha = k+1$. Then there exists $m \in \{1, \dots, 9\}$ such that $a_m \geq 1$. We denote by D' and D_0 the diagrams which differ from $D = 9_{39}^+(a_1, \dots, a_9)$ only in a small neighborhood of c_m as shown in Figure 6. By the skein relationship of the Jones polynomial, we have

$$V_D(t) = t^2 V_{D'}(t) + (t^{\frac{3}{2}} - t^{\frac{1}{2}}) V_{D_0}(t).$$

The diagram D' is obtained from 9_{39}^+ by applying $\overline{t_2'}$ moves $(\alpha - 1)$ times. By the assumption of induction, we have $\operatorname{maxdeg} V_{D'} = c(D') = c(D) - 2$ and $\operatorname{maxdegcf} V_{D'} = -2$. Therefore it suffices to show that $\operatorname{maxdeg} V_{D_0} \leq c(D) - 5/2$ holds. Now we show that this inequality holds for each $m = 1, \dots, 9$.

Case 1. $m = 1, \dots, 6$**:** We show only the case where $m = 1$, since other cases ($m = 2, \dots, 6$) are proved in the same way. Let D_0' be the diagram obtained from D_0 by

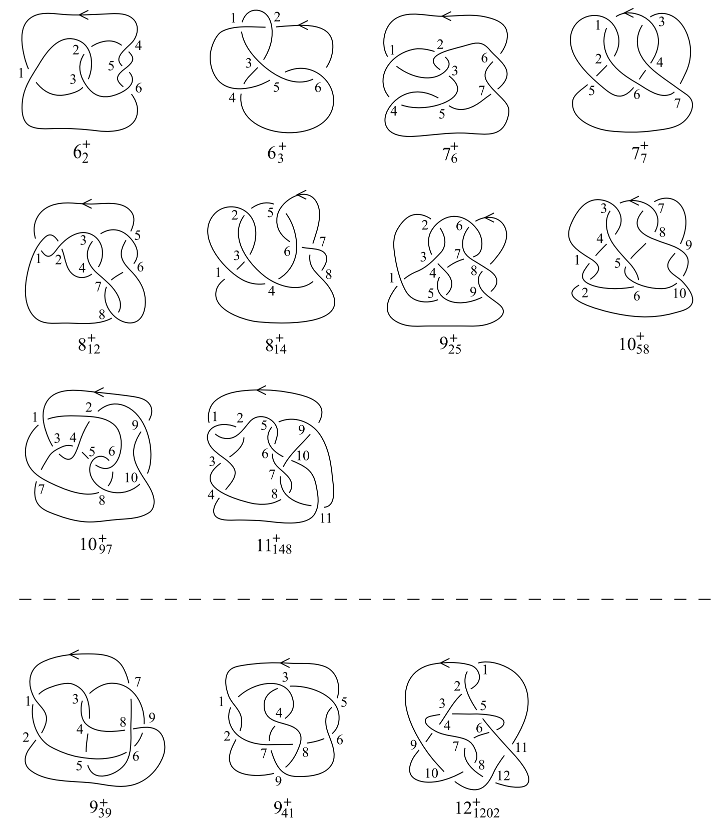

FIGURE 4. We label crossings as shown in the figure.

reducing nugatory crossings as shown in Figure 7. Note that D_0 and D_0' represent the same positive 2-component link. Since $a_1 \geq 1$, we have

$$\begin{aligned} c(D_0') &= c(D) - ((2a_1 + 1) + (2a_2 + 1)) \\ &\leq c(D) - 4. \end{aligned}$$

By Lemma 2.4, we have

$$\operatorname{span} V_{D_0} = \operatorname{span} V_{D_0'} \leq c(D) - 4.$$

On the other hand, the diagram D_0' represents a positive link. Notice that the Euler characteristic of the link represented by D_0' is equal to -2 (see [4]). Then, by Lemma 2.5, we have $\operatorname{mindeg} V_{D_0} = 3/2$. Therefore we have

$$\begin{aligned} \operatorname{maxdeg} V_{D_0} &= \operatorname{mindeg} V_{D_0} + \operatorname{span} V_{D_0} \\ &\leq 3/2 + (c(D) - 4) \\ &= c(D) - 5/2. \end{aligned}$$

Case 2. $m = 7$**:** As shown in Figure 8, the diagram D_0 is equivalent to the diagram D_0''. Since $a_7 \geq 1$, we have

$$c(D_0'') = c(D) - (2a_7 + 1) - 1 \leq c(D) - 4.$$

Then we can show that $\operatorname{maxdeg} V_{D_0} \leq c(D) - 5/2$ by the same argument applied in Case 1.

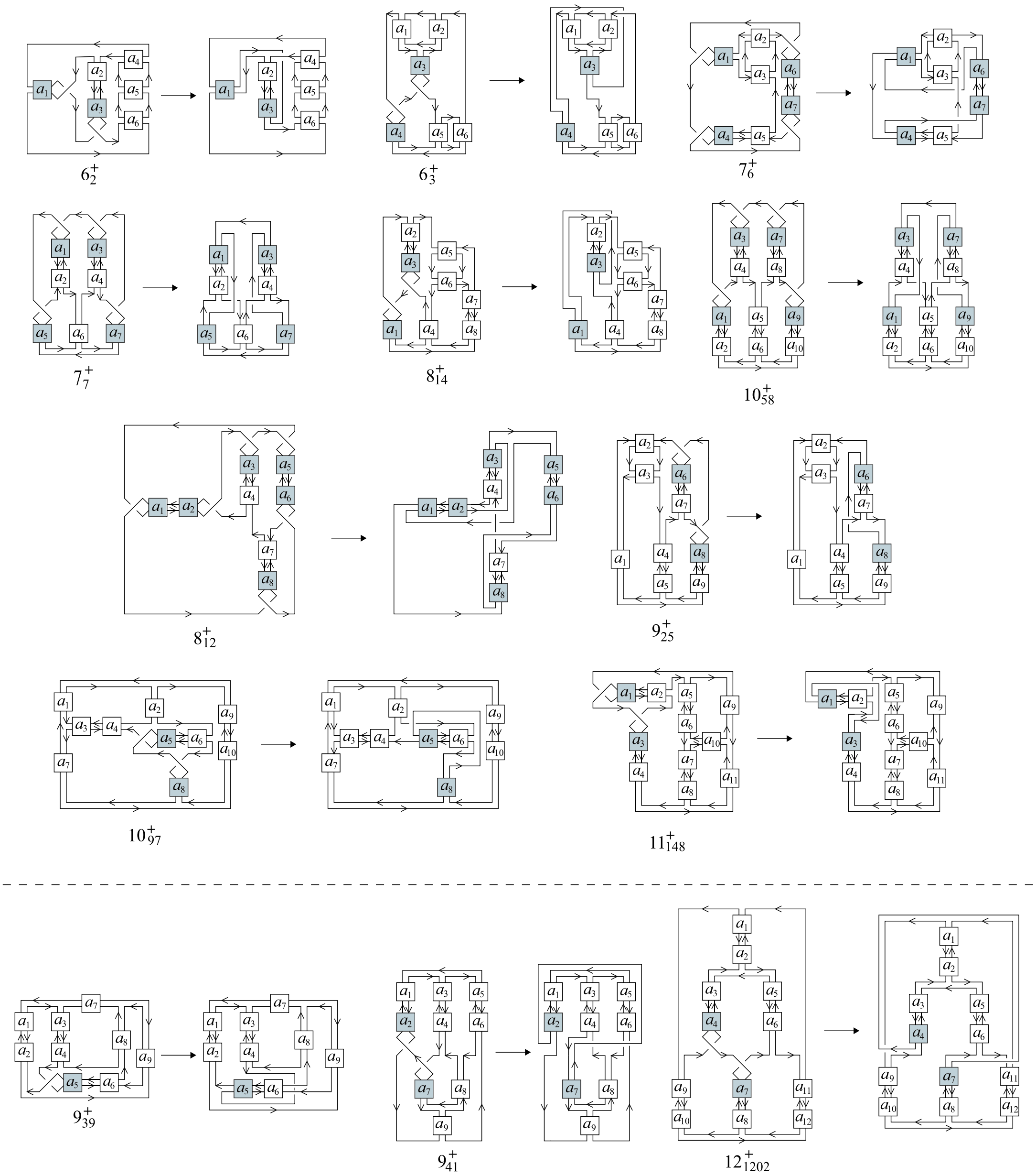

FIGURE 5

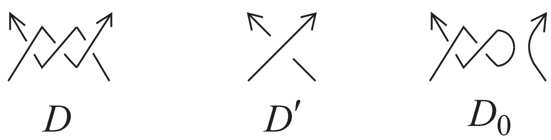

FIGURE 6

Case 3. $m = 8$**:** As shown in Figure 9, the diagram D_0 is equivalent to the diagram D_0''. Since $a_8 \geq 1$, we have

$$c(D_0'') = c(D) - (2a_8 + 1) - 1 \leq c(D) - 4.$$

Then we see that $\operatorname{maxdeg} V_{D_0} \leq c(D) - 5/2$ by the same argument applied in Case 1.

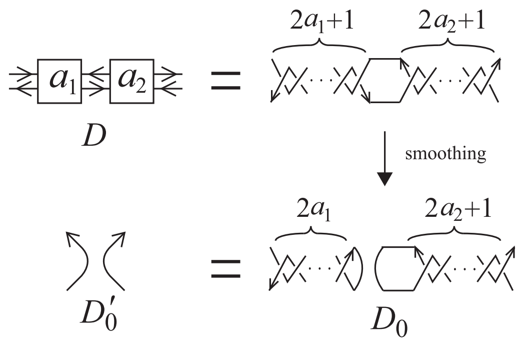

FIGURE 7

Case 4. $m = 9$**:** As shown in Figure 10, the diagram D_0 is equivalent to the diagram D_0''. Since $a_9 \geq 1$, we have

$$c(D_0'') = c(D) - (2a_9 + 1) - 1 \leq c(D) - 4.$$

Then we see that $\operatorname{maxdeg} V_{D_0} \leq c(D) - 5/2$ by the same argument applied in Case 1.

Now we complete the proof of Claim 2.6. □

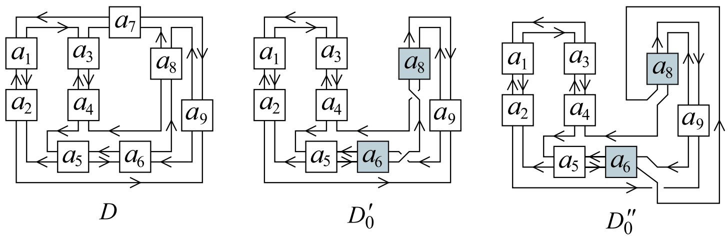

FIGURE 8

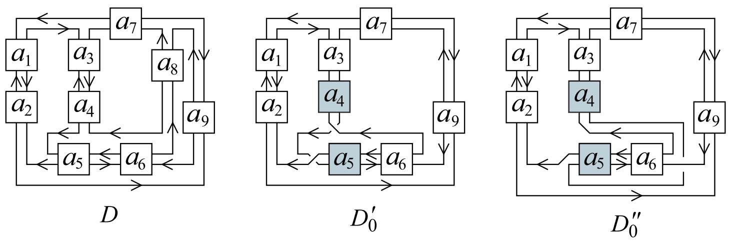

FIGURE 9

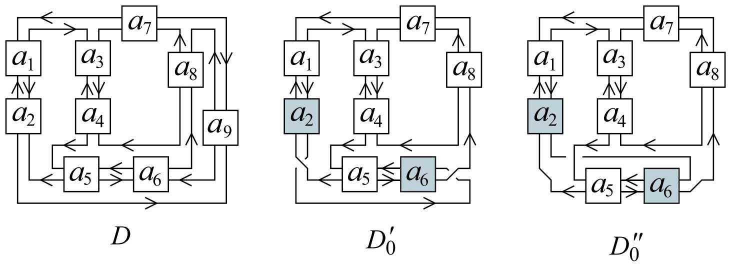

FIGURE 10

Claim 2.7. *For the diagram $D = 9^+_{41}(a_1, \dots, a_9)$, we have*

$$\operatorname{maxdegcf} V_D = -2,$$

$$\operatorname{maxdeg} V_D = c(D).$$

Proof. We prove the claim by induction on $\alpha = a_1 + \cdots + a_9$. If $\alpha = 0$, then $D = 9^+_{41}$ is a positive diagram of the knot 9_{49}. Calculating the Jones polynomial, we have $V_{9_{49}}(t) = -2t^9 + 3t^8 - 4t^7 + 5t^6 - 4t^5 + 4t^4 - 2t^3 + t^2$, and thus the claim is true. Assuming that the claim is true for $\alpha = k \geq 0$, we prove it for $\alpha = k + 1$. Then there exists some $m \in \{1, \dots, 9\}$ such that $a_m \geq 1$.

Case 1. $m = 1, \dots, 6$**:** We omit the proof since it is similar to that of Case 1 in the proof of Claim 2.6.

Case 2. $m = 7, 8$**:** We show only the case where $m = 7$, since the case where $m = 8$ is proved by the same argument. As shown in Figure 11, the diagram D_0 is equivalent to the diagram D''_0. The rest of the proof is similar to that of Case 1 in the proof of Claim 2.6.

Case 3. $m = 9$**:** As shown in Figure 12, the diagram D_0 is equivalent to the diagram D''_0. The rest of the proof is similar to that of Case 1 in the proof of Claim 2.6.

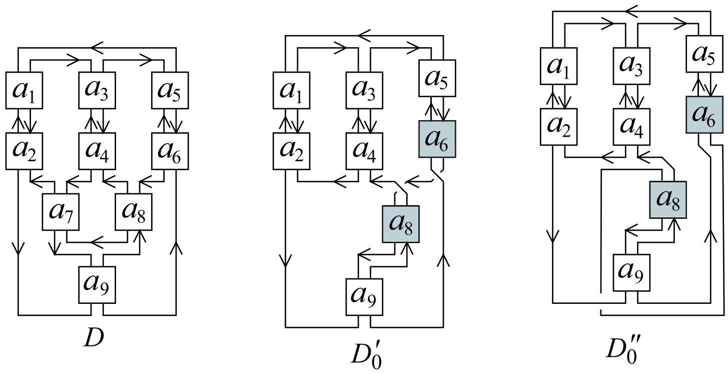

FIGURE 11

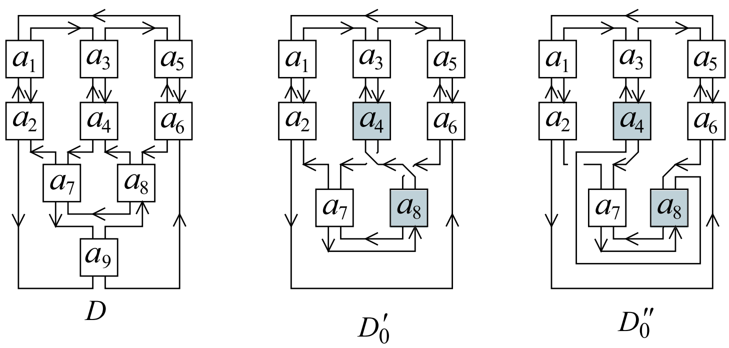

FIGURE 12

Now we complete the proof of Claim 2.7. □

Claim 2.8. *For the diagram $D = 12^+_{1202}(a_1, \dots, a_{12})$, we have*

$$\operatorname{maxdegcf} V_D = 2,$$

$$\operatorname{maxdeg} V_D = c(D).$$

Proof. We prove Claim 2.8 by induction on $\alpha = a_1 + \cdots + a_{12}$. If $\alpha = 0$, then $D = 12^+_{1202}$ is a positive diagram of the knot 12_{n881}. Calculating the Jones polynomial, we have $V_{12_{n881}}(t) = 2t^{12} - 6t^{11} + 10t^{10} - 16t^9 + 19t^8 - 20t^7 + 19t^6 - 14t^5 + 10t^4 - 4t^3 + t^2$, and then the claim is true. Assuming that the claim is true for $\alpha = k \geq 0$, we prove it for $\alpha = k+1$. The rest of the proof is similar to that of Case 1 in the proof of Claim 2.6. □

Now we complete the proof of Theorem 1.3. □

3. Minimal set of generators for positive knots of genus two

In Section 2, we use Stoimenow's twenty four generators to study positive knots of genus two. In this section, we show that just fourteen generators are needed to study positive knots of genus two.

Theorem 3.1. *Every prime positive knot of genus two is generated by one of the following fourteen generators:* 5^+_1, 7^+_5, 8^+_{15}, 9^+_{23}, 9^+_{38}, 9^+_{39}, 9^+_{41}, 10^+_{101}, 10^+_{120}, 11^+_{123}, 11^+_{329}, 12^+_{1097}, 12^+_{1202}, 13^+_{4233}. *Furthermore, the set of the fourteen generators is minimal to obtain all prime positive knots of genus two.*

For two generators $G_1, G_2 \in \mathcal{G}^+_2$, we say that G_1 and G_2 are *independent* if one of two sets of knots generated by G_1 and G_2 is not a subset of the other.

Proof of Theorem 3.1. As shown in the proof of Theorem 1.3, prime positive knots generated by the generators in $\mathcal{G}^+_2$ except for 9^+_{39}, 9^+_{41}, and 12^+_{1202} are positive-alternating. Therefore we need only the following eleven generators for prime positive-alternating knots of genus two: 5^+_1, 7^+_5, 8^+_{15}, 9^+_{23}, 9^+_{38}, 10^+_{101}, 10^+_{120}, 11^+_{123}, 11^+_{329}, 12^+_{1097}, and 13^+_{4233}. Note that these eleven generators are mutually independent (see [7, 8] or [16]).

Next we show that the generators 9^+_{39}, 9^+_{41}, and 12^+_{1202} are also mutually independent. For a generator $G \in \mathcal{G}^+_2$, we denote by $\mathcal{K}_n(G)$ the set of all knots obtained from G by applying $\overline{t'_2}$ moves at most n times. Observing the case where $n = 3$, we see that

$$\begin{aligned}\mathcal{K}_3(9^+_{39}) = \{\, & 9_{49}, 11_{n171}, 11_{n181}, 13_{n4365}, 13_{n4795}, 13_{n4879}, 13_{n4929}, 13_{n4975}, 13_{n4996}, 13_{n5105},\\ & 15_{n135929}, 15_{n150753}, 15_{n152806}, 15_{n154713}, 15_{n158454}, 15_{n159290}, 15_{n159866}, 15_{n160477},\\ & 15_{n160544}, 15_{n162263}, 15_{n162490}, 15_{n162543}, 15_{n167671}, 15_{n167688}, 15_{n167734}, 15_{n167843},\\ & 15_{n168014}\,\},\\ \mathcal{K}_3(9^+_{41}) = \{\, & 9_{49}, 11_{n171}, 11_{n181}, 13_{n4365}, 13_{n4795}, 13_{n4879}, 13_{n4975}, 13_{n4996}, 13_{n5105}, 15_{n135929},\\ & 15_{n150753}, 15_{n152806}, 15_{n158454}, 15_{n159290}, 15_{n159866}, 15_{n160415}, 15_{n162263}, 15_{n162490},\\ & 15_{n162543}, 15_{n167671}, 15_{n167688}, 15_{n167843}, 15_{n168014}\}.\end{aligned}$$

As in the proofs of Claims 2.6 and 2.7, a $\overline{t'_2}$ move on the generator 9^+_{39} or 9^+_{41} increases the span degree of the Jones polynomial. Thus, by Lemma 2.4, $\mathcal{K}_n(9_{39})$ and $\mathcal{K}_n(9_{41})$ do not contain the knot with crossing number twelve. Then the generator 12^+_{1202} and each of 9^+_{39} and 9^+_{41} are independent. Further, we see that the generators 9^+_{39} and 9^+_{41} are also independent. In fact, the knot $15_{n160415}$ cannot be generated by the generator 9^+_{39} and that the knots 13_{n4929}, $15_{n154713}$, $15_{n160477}$, $15_{n160544}$, and $15_{n167734}$ cannot be generated by the generator 9^+_{41}. □

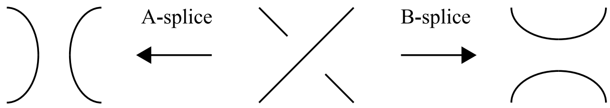

FIGURE 13. A-splice and B-splice

4. Positive knots of genus two are quasi-alternating

An A-*splice* (or a B-*splice*) is a local move on diagrams as shown in Figure 13. For a diagram D and a crossing c of D, let D_c^A (resp. D_c^B) be the diagram obtained from D by applying A-splice (resp. a B-splice) at c. For a diagram D, we denote by $L(D)$ the link represented by the diagram D. The set $\mathcal{Q}$ of *quasi-alternating links* is the smallest set of links satisfying the following properties:

- The unknot is in $\mathcal{Q}$.
- If the link L has a diagram D with a crossing c such that
  (1) $L(D_c^A)$ and $L(D_c^B)$ are in $\mathcal{Q}$, and
  (2) $\det(L) = \det(L(D_c^A)) + \det(L(D_c^B))$,
  then L is in $\mathcal{Q}$, where $\det(L)$ is the determinant of the link L.

Then we say that D is *quasi-alternating* at c.

Remark 4.1. A non-split alternating link is quasi-alternating.

We consider a crossing c of a diagram as a 2-tangle diagram. According to whether the overstrand has positive or negative slope, we set $\varepsilon(c) = \pm 1$. We say that a rational tangle diagram with the Conway notation $R = C(a_1, \dots, a_m)$ *extends* a crossing c if R contains c, and $\varepsilon(c) \cdot a_i \geq 1$ for $i = 1, \dots, m$. Then R is an alternating rational tangle diagram.

Lemma 4.2 ([3]). *Let D be quasi-alternating at a crossing c, and let D' be obtained by replacing c of D with a rational tangle diagram that extends c. Then the link $L(D')$ is quasi-alternating.*

Proof of Theorem 1.4. Since an alternating knot is quasi-alternating, it suffices to prove that non-alternating positive knots of genus two are quasi-alternating. Actually, we show that the diagrams $G(a_1, \dots, a_n)$ for $G = 9_{39}^+, 9_{41}^+$, and 12_{1202}^+ are quasi-alternating.

Case 1. $G = 9_{39}^+$**:** Put $D' = 9_{39}^+(a_1, \dots, a_9)$. Let D and c be the diagram and the crossing of D as shown in Figure 14. Let $b_i = 2a_i + 1$ and $b_{i,j} = b_i + b_j$. Using a checkerboard surface and the Goeritz matrix, we calculate the determinants, and then, we have the following:

$$\begin{aligned}
\det(L(D)) = & - b_7 - b_9 - b_{1,2} - b_7 b_8 - b_7 b_9 - b_8 b_9 - b_{1,2} b_7 - b_{1,2} b_8 \\
& + b_{1,2} b_{5,6} b_8 + b_{1,2} b_{5,6} b_9 + b_{5,6} b_7 b_8 + b_{5,6} b_7 b_9 + b_{5,6} b_8 b_9 \\
& + b_{1,2} b_{5,6} b_7 b_8 + b_{1,2} b_{5,6} b_7 b_9 + b_{1,2} b_{5,6} b_8 b_9 \, . \\
\det(L(D_c^A)) = & - b_7 - b_9 - b_{1,2} + b_{1,2} b_{5,6} b_8 + b_{1,2} b_{5,6} b_9 + b_{5,6} b_7 b_8 \\
& + b_{5,6} b_7 b_9 + b_{5,6} b_8 b_9 \, . \\
\det(L(D_c^B)) = & - b_7 b_8 - b_7 b_9 - b_8 b_9 - b_{1,2} b_7 - b_{1,2} b_8 + b_{1,2} b_{5,6} b_7 b_8 \\
& + b_{1,2} b_{5,6} b_7 b_9 + b_{1,2} b_{5,6} b_8 b_9 \, .
\end{aligned}$$

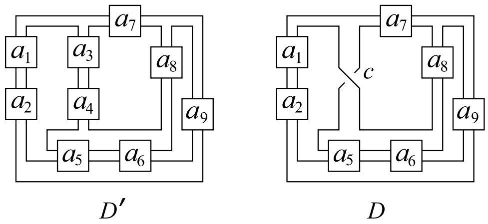

FIGURE 14

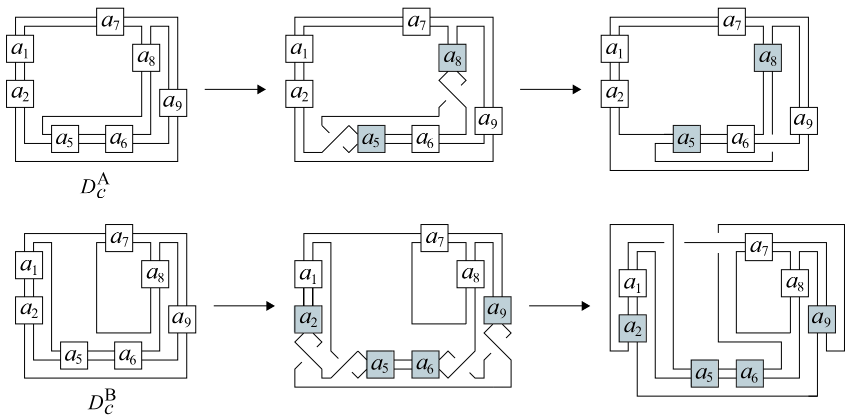

FIGURE 15

Then the equality $\det(L(D)) = \det(L(D_c^A)) + \det(L(D_c^B))$ holds. On the other hand, the diagrams D_c^A and D_c^B can be deformed into connected alternating diagrams as shown in Figure 15. Thus, by Remark 4.1, the links $L(D_c^A)$ and $L(D_c^B)$ are in $\mathcal{Q}$. Therefore D is quasi-alternating at c. By Lemma 4.2, the link $L(D')$ is quasi-alternating.

Case 2. $G = 9_{41}^+$**:** Put $D' = 9_{41}^+(a_1, \dots, a_9)$. Let D and c be the diagram and the crossing as shown in Figure 16. Calculating the determinants of the diagrams, we have the following:

$$\begin{aligned}\det(L(D)) =& 1 - b_{1,2} - b_{5,6} - b_{1,2}b_7 - b_{3,4}b_7 - b_{3,4}b_8 - b_{5,6}b_8 + b_{1,2}b_{3,4}b_7 \\ &+ b_{1,2}b_{3,4}b_8 + b_{1,2}b_{5,6}b_7 + b_{1,2}b_{5,6}b_8 + b_{3,4}b_{5,6}b_7 + b_{3,4}b_{5,6}b_8 \\ &+ b_{1,2}b_{3,4}b_7b_8 + b_{1,2}b_{5,6}b_7b_8 + b_{3,4}b_{5,6}b_7b_8\,. \\ \det(L(D_c^A)) =& -b_{1,2} - b_{5,6} + b_{1,2}b_{3,4}b_7 + b_{1,2}b_{3,4}b_8 + b_{1,2}b_{5,6}b_7 + b_{1,2}b_{5,6}b_8 \\ &+ b_{3,4}b_{5,6}b_7 + b_{3,4}b_{5,6}b_8\,. \\ \det(L(D_c^B)) =& 1 - b_{1,2}b_7 - b_{3,4}b_7 - b_{3,4}b_8 - b_{5,6}b_8 + b_{1,2}b_{3,4}b_7b_8 \\ &+ b_{1,2}b_{5,6}b_7b_8 + b_{3,4}b_{5,6}b_7b_8\,.\end{aligned}$$

Then the equality $\det(L(D)) = \det(L(D_c^A)) + \det(L(D_c^B))$ holds. On the other hand, the diagrams D_c^A and D_c^B can be deformed into connected alternating diagrams as shown in Figure 17. Therefore D is quasi-alternating at c. By Lemma 4.2, the link $L(D')$ is quasi-alternating.

Case 3. $G = 12_{1202}^+$**:** Put $D' = 12_{1202}^+(a_1, \dots, a_{12})$. Let D and c be the diagram and the crossing as shown in Figure 18. Calculating the determinants of the diagrams,

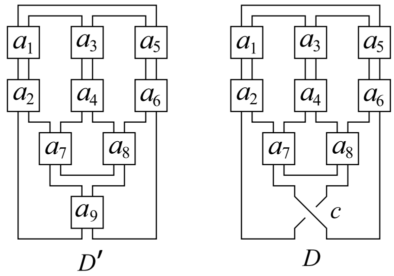

FIGURE 16

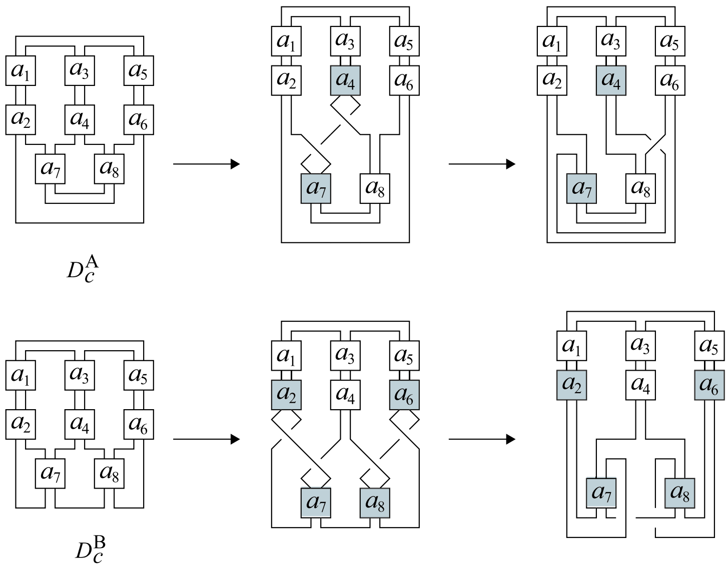

FIGURE 17

we have the following:

$$\begin{aligned}\det(L(D)) =& 1 - b_{9,10} - b_{11,12} - b_{3,4}b_{7,8} - b_{3,4}b_{9,10} - b_{5,6}b_{7,8} - b_{5,6}b_{11,12}\\ &+ b_{3,4}b_{7,8}b_{9,10} + b_{3,4}b_{7,8}b_{11,12} + b_{3,4}b_{9,10}b_{11,12} + b_{5,6}b_{7,8}b_{9,10}\\ &+ b_{5,6}b_{7,8}b_{11,12} + b_{5,6}b_{9,10}b_{11,12} + b_{3,4}b_{5,6}b_{7,8}b_{9,10}\\ &+ b_{3,4}b_{5,6}b_{7,8}b_{11,12} + b_{3,4}b_{5,6}b_{9,10}b_{11,12}\,.\\ \det(L(D_c^A)) =& - b_{9,10} - b_{11,12} + b_{3,4}b_{7,8}b_{9,10} + b_{3,4}b_{7,8}b_{11,12} + b_{3,4}b_{9,10}b_{11,12}\\ &+ b_{5,6}b_{7,8}b_{9,10} + b_{5,6}b_{7,8}b_{11,12} + b_{5,6}b_{9,10}b_{11,12}\,.\\ \det(L(D_c^B)) =& 1 - b_{3,4}b_{7,8} - b_{3,4}b_{9,10} - b_{5,6}b_{7,8} - b_{5,6}b_{11,12}\\ &+ b_{3,4}b_{5,6}b_{7,8}b_{9,10} + b_{3,4}b_{5,6}b_{7,8}b_{11,12} + b_{3,4}b_{5,6}b_{9,10}b_{11,12}\,.\end{aligned}$$

Then the equality $\det(L(D)) = \det(L(D_c^A)) + \det(L(D_c^B))$ holds. On the other hand, the diagrams D_c^A and D_c^B can be deformed into connected alternating diagrams as shown in Figure 19. Therefore D is quasi-alternating at c. By Lemma 4.2, the link $L(D')$ is quasi-alternating.

This completes the proof of Theorem 1.4. □

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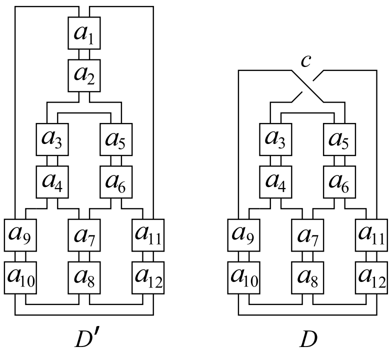

FIGURE 18

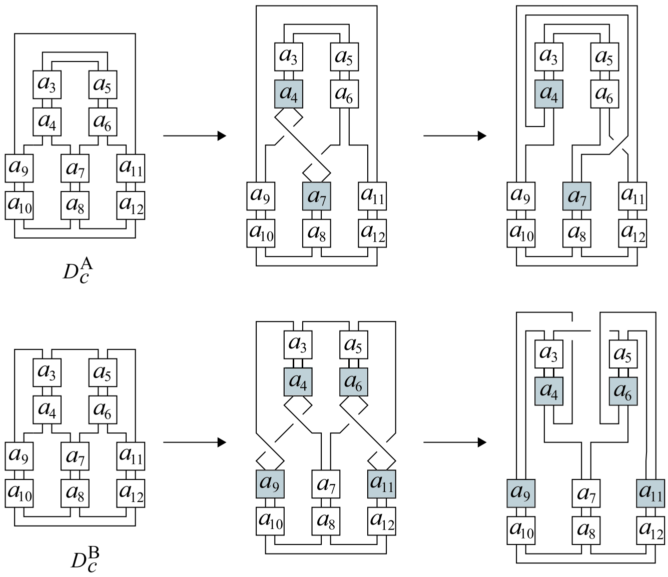

FIGURE 19

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