Chebyshev’s Bias for Algebraic Curves

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What is Chebyshev's Bias?

The phenomenon that most of the time, there are more primes of the form $4k+3$ than of the form $4k+1$, up to the same bound. (Chebyshev (1853))

Put $\pi(x;\, q,\, a) := \#\{p \leq x \mid p \equiv a \pmod{q}\}$.

- For all $x < 26861$, it holds $\pi(x;\, 4,\, 3) \geq \pi(x;\, 4,\, 1)$. Team 3 is in the lead.
- For $x = 26861$, the inequality "$<$" holds. Team 1 leads for an instant.
- For $x = 26863$, the equation "$=$" holds. Team 3 catches up.
- For $x = 26879$, the inequality "$>$" holds again. Team 3 gets ahead.
- For $26879 \leq x < 616841$, the inequality "$>$" holds. Team 3 maintains a lead.

It seems that $\pi(x;\, 4,\, 3) \geq \pi(x;\, 4,\, 1)$ more often than not.

Apparently there exists
- a bias towards primes of the form $4k+3$.
- a bias against primes of the form $4k+1$.

Chebyshev's Bias

x	$\pi(x;\,4,\,3)$	$\pi(x;\,4,\,1)$	
100	13	11	Team 3 leads by 2 points.
$1,000$	87	80	Team 3 leads by 7 points.
$10,000$	619	609	Team 3 leads by 10 points.
$100,000$	4808	4783	Team 3 leads by 25 points.
$1,000,000$	39322	39175	Team 3 leads by 147 points.
$2,000,000$	74516	74416	Team 3 leads by 100 points.
$3,000,000$	108532	108283	Team 3 leads by 249 points.

History on Chebyshev's Bias

Littlewood (1914)

$\pi(x;\,4,\,3) - \pi(x;\,4,\,1)$ changes its sign infinitely many times.

Knapowski-Turan Conjecture (1962)

The natural density of the set $A(X) = \{x < X \mid \pi(x;\,4,\,3) - \pi(x;\,4,\,1) > 0\}$ is 1.

$$\lim_{X\to\infty} \frac{\mathrm{vol}(A(X))}{X} = 1.$$

$\rightarrow$ This is false under GRH. The limit does not exist. (Kaczorowski 1995)

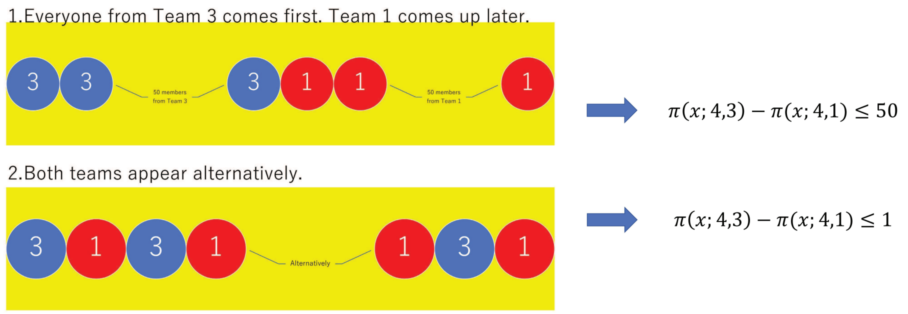
Rubinstein-Sarnak's Theorem (1994)

The logarithmic density of the set $A(X)$ exists under "GRH+LI".

$$\lim_{X\to\infty} \frac{1}{\log X} \int_{t\in A(X)} \frac{dt}{t} = 0.9959\ldots.$$

$A(X) = \{x < X \mid \pi(x;\,4,\,3) - \pi(x;\,4,\,1) > 0\}$ is insufficient.

Two exterme cases where $\pi(x;\,4,\,3) \geq \pi(x;\,4,\,1)$ holds ($\forall x \leq p_{100} = (100^{\text{th}}$ prime)).



Even if we know $\{x \mid 0 < x \leq p_{100}\} \subset A(X)$, we cannot distinguish Cases 1 and 2.

The length of the interval $A(X)$ does not lead us to the truth.

We need a method of estimating the "difference".

The key idea for attacking Chebyshev's bias

Prime number theorem in arithmetic progressions

Team 3 is as big as Team 1.

Reinterpretation of Chebyshev's bias

We regard Chebyshev's bias as follows:

There seem to be more members in Team 3 = Members in Team 3 appear earlier

The main idea (zeta parametrization)

In order to regard smaller primes as heavier elements,

we adopt a weighted counting function $$\pi_s(x;\, q,\, a) = \sum_{\substack{p<x:\,\text{prime} \\ p\equiv a \pmod q}} \frac{1}{p^s} \quad (s \geq 0),$$

which is a generalization of the counting function $\pi(x;\, q,\, a) = \pi_0(x;\, q,\, a)$.

The Deep Riemann Hypothesis (DRH)

K: a (1-dimensional) global field

ρ : an n-dimensional irreducible representation of $\mathrm{Gal}(\overline{K}/K)$ $\qquad (\rho \neq \mathbf{1})$

$$L_K(s,\rho) := \prod_{v\text{: finite place}} \det\left(1 - \rho(\mathrm{Frob}_v)N(v)^{-s}\right)^{-1} \qquad (\Re(s) > 1)$$

($N(v)$: the norm, $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{K}/K)$: the Frobenius element)

Deep Riemann Hypothesis (DRH)

Put $m = m_\rho = \mathrm{ord}_{s=\frac{1}{2}} L_K(s,\rho)$ and let γ be the Euler constant. If $\rho \neq \mathbf{1}$, then

$$\lim_{x\to\infty}\left((\log x)^m \prod_{N(v)\le x} \det\left(1 - \rho(\mathrm{Frob}_v)N(v)^{-\frac{1}{2}}\right)^{-1}\right) = \frac{\sqrt{2}^{\nu(\rho)} L_K^{(m)}(\frac{1}{2},\rho)}{e^{m\gamma} m!},$$

where $\nu(\rho) = \mathrm{mult}(\mathbf{1}, \mathrm{sym}^2\rho) - \mathrm{mult}(\mathbf{1}, \wedge^2\rho)$.

DRH $\implies$ CC $=$ [EP of $L_K(s,\rho)$ converges at $s = \frac{1}{2}$]. (The limit may be 0.)

Euler's L-function $\quad L(s,\chi)=\prod_{p:\,\text{odd}}\left(1-\chi(p)p^{-s}\right)^{-1} \quad \left(\chi(p)=(-1)^{\frac{p-1}{2}}\right)$

Since $L(\frac{1}{2},\chi)\neq 0$ (i.e. $m=0$), DRH is equivalent to

$$\sum_{p\le x:\,\text{odd}} \log\left(1-\chi(p)p^{-\frac{1}{2}}\right)^{-1} = L+o(1) \quad (x\to\infty) \quad \text{with } L=\log\left(\sqrt{2}L(\tfrac{1}{2},\chi)\right).$$

Taylor expansion $\displaystyle \sum_{p\le x}\log\left(1-\frac{\chi(p)}{p^{\frac{1}{2}}}\right)^{-1} = \sum_{p\le x}\frac{\chi(p)}{\sqrt{p}}+\sum_{p\le x}\frac{\chi(p)^2}{2p}+\sum_{p\le x}\sum_{k=3}^{\infty}\frac{\chi(p)^k}{kp^{\frac{k}{2}}}$

- 1st term $=\pi_{1/2}(x;\,4,\,1)-\pi_{1/2}(x;\,4,\,3)$
- 2nd term $\displaystyle =\sum_{p\le x}\frac{\chi(p)^2}{2p}=\sum_{p\le x}\frac{1}{2p}=\frac{1}{2}\log\log x+c+o(1) \quad (x\to\infty) \quad (\exists c\in\mathbb{R})$
- 3rd term is absolutely convergent from $\displaystyle \sum_{p\le x}\sum_{k=3}^{\infty}\left|\frac{\chi(p)^k}{kp^{\frac{k}{2}}}\right| \le \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}=\zeta(\tfrac{3}{2})$.

Under DRH, we have $\pi_{1/2}(x;\,4,\,3)-\pi_{1/2}(x;\,4,\,1)=\frac{1}{2}\log\log x+C+o(1)\ (x\to\infty)$.

Formulations of Chebyshev Biases (in a global field K)

Definition 1

$a(\mathfrak{p}) \in \mathbb{R}$: a sequence over prime ideals $\mathfrak{p}$ of K s.t. $\lim_{x\to\infty} \frac{\#\{\mathfrak{p} \mid a(\mathfrak{p})>0,\ N(\mathfrak{p})\le x\}}{\#\{\mathfrak{p} \mid a(\mathfrak{p})<0,\ N(\mathfrak{p})\le x\}} = 1$.
We say $a(\mathfrak{p})$ has a *Chebyshev bias to being positive*, if there exists $C > 0$ such that

$$\sum_{N(\mathfrak{p})\le x} \frac{a(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}} \sim C \log\log x \quad (x \to \infty).$$

Definition 2

Assume that $\{\mathfrak{p} \mid N(\mathfrak{p}) \le x\} = P_1(x) \cup P_2(x)$ (disjoint) and that $\delta = \lim_{x\to\infty} \frac{|P_1(x)|}{|P_2(x)|}$.
We say there exists a *Chebyshev bias toward* P_1 (or *Chebyshev bias against* P_2), if

$$\sum_{p\in P_1(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{p\in P_2(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} \sim C \log\log x \quad (x \to \infty,\ \exists C > 0).$$

DRH over function fields

Theorem (DRH in char> 0) [Kaneko-Koyama-Kurokawa (2021)]

When K is an algebraic function field (of one variable) with char$(K) > 0$,
DRH holds for any automorphic L-function over GL_n.

In what follows, whenever we say "Under DRH", it means the following:

- In case of char$(K) = 0$, the theorem holds under the assumption of DRH.
- In case of char$(K) > 0$, the theorem holds unconditionally.

Overview of the preceding results

$$\pi_s(x;\,q,\,a) = \sum_{\substack{p<x:\,\text{prime}\\ p\equiv a \pmod q}} \frac{1}{p^s} \quad (s \geq 0), \qquad A_s = \{x > 0 \mid \pi_s(x;\,4,\,3) - \pi_s(x;\,4,\,1) > 0\}$$

The density of A_0 does not exist.

Preceding Results (under DRH for char 0, and unconditionally for char > 0)

- The density of $A_{1/2}$ is equal to 1. More precisely,

$$\pi_s(x;\,4,\,3) - \pi_s(x;\,4,\,1) = \begin{cases} \frac{1}{2}\log\log x + O(1) \quad (x \to \infty) & (s = \frac{1}{2}) \\ O(1) \quad (x \to \infty) & (s > \frac{1}{2}) \end{cases}$$

  $\rightarrow$ A formulation of the *Chebyshev bias towards Team 3 (against Team 1)*.

- The Chebyshev bias against splitting ideals in an extension of global fields.
- The Chebyshev bias against principal ideals in a global field of class number 2.
- Ramanujan's $\tau(p)$ has a Chebyshev bias to being positive.

Preceding results on abelian extensions

L/K: a finite abelian extension of (one-dimensional) global fields

$G := \mathrm{Gal}(L/K) \ni \sigma$, $\quad \mathfrak{p} \subset K$: a prime ideal

$$\pi_s(x;\sigma) := \sum_{\substack{\mathfrak{p}\in S_\sigma \\ N(\mathfrak{p})\le x}} \frac{1}{N(\mathfrak{p})^s} \quad \left(S_\sigma := \left\{ \mathfrak{p} \;\middle|\; \mathfrak{p} \nmid D_{L/K},\ \left(\frac{L/K}{\mathfrak{p}}\right) = \sigma \right\} \right)$$

Theorem 1 (Aoki-Koyama, JNT 2022)(Bias against squares in Gal(L/K))

Under DRH, it holds for any $\sigma \in G^2$ and $\tau \in G \setminus G^2$ that as $x \to \infty$

$$\pi_{\frac{1}{2}}(x;\tau) - \pi_{\frac{1}{2}}(x;\sigma) = \frac{1}{[L:K]} \left(\frac{|G/G^2|}{2} + m(\sigma) - m(\tau) \right) \log\log x + O(1),$$

where $m(\sigma) := \sum_{\chi \in \widehat{G}\setminus\{1\}} \Re(\chi(\sigma)^{-1}) \mathrm{ord}_{s=\frac{1}{2}} L_K(s,\chi).$

The case of cyclotomic fields ($K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_q)$ $(q \geq 3)$)

For $q \in \mathbb{Z}$, we have $G = \mathrm{Gal}(L/\mathbb{Q}) \ni \left(\frac{L/\mathbb{Q}}{(a)}\right) \xleftarrow{\sim} a \in (\mathbb{Z}/q\mathbb{Z})^\times$.

$\sigma \in G^2 \leftrightarrow R_q := \{a \in (\mathbb{Z}/q\mathbb{Z})^\times \,|\, a \in (\mathbb{Z}/q\mathbb{Z})^{\times 2}\}$: quadratic residues

$\tau \notin G^2 \leftrightarrow N_q := \{a \in (\mathbb{Z}/q\mathbb{Z})^\times \,|\, a \in (\mathbb{Z}/q\mathbb{Z})^\times \setminus (\mathbb{Z}/q\mathbb{Z})^{\times 2}\}$: quadratic nonresidues

Corollary 1 (Bias against quadratic residues)

Assume DRH for $L(s,\chi)$ with χ a nontrivial Dirichlet character mod q.
Assume that $L(\frac{1}{2},\chi) \neq 0$ (Chowla's Conjecture).

- If $(a,b) \in R_q \times N_q$, $(q,\ a,\ b) = (4,\ 1,\ 3)$, $\frac{|G/G^2|}{2\varphi(q)} = \frac{1}{2}$: Chebyshev's case

$$\pi_{\frac{1}{2}}(x;\,q,\,b) - \pi_{\frac{1}{2}}(x;\,q,\,a) = \frac{|G/G^2|}{2\varphi(q)} \log\log x + O(1) \quad (x \to \infty).$$

- If either $(a,b) \in R_q \times R_q$ or $(a,b) \in N_q \times N_q$,

$$\pi_{\frac{1}{2}}(x;\,q,\,b) - \pi_{\frac{1}{2}}(x;\,q,\,a) = O(1) \quad (x \to \infty).$$

Examples (under DRH)

Example 1 (Bias mod 8)

$(\mathbb{Z}/8\mathbb{Z})^\times = \{1,\ 3,\ 5,\ 7 \pmod 8\}.\quad R_8 = \{1\},\quad N_8 = \{3, 5, 7\}$

For $j = 3, 5, 7$, it holds that

$$\pi_{\frac{1}{2}}(x; 8, j) - \pi_{\frac{1}{2}}(x; 8, 1) = \frac{1}{2} \log\log x + O(1) \qquad (x \to \infty).$$

For all pairs of $j, k \in \{3, 5, 7\}$, it holds that

$$\pi_{\frac{1}{2}}(x; 8, j) - \pi_{\frac{1}{2}}(x; 8, k) = O(1) \qquad (x \to \infty).$$

Bias in the polynomial ring $\mathbb{F}_q[T]$

Let $f \in \mathbb{F}_q[T]$ and put $F_f = \mathbb{F}_q[T]/(f)$. $\quad N(f) := q^{\deg f}$

Assume that $F_f^\times$ is a cyclic group of even order.

$$R_f := \{g \in F_f^\times \mid g \in (F_f^\times)^2\} \quad \text{(quadratic residues)}$$

$$N_f := \{g \in F_f^\times \mid g \notin (F_f^\times)^2\} \quad \text{(quadratic nonresidues)}$$

Corollary 2 (Bias in irreducible polynomials over $\mathbb{F}_q$)

$$\exists C_f > 0 \qquad \sum_{\substack{h \in N_f\colon \text{irreducible} \\ \deg h \le n}} \frac{1}{\sqrt{N(h)}} - \sum_{\substack{h \in R_f\colon \text{irreducible} \\ \deg h \le n}} \frac{1}{\sqrt{N(h)}} \sim C_f \log n \quad (n \to \infty)$$

Example 2 ($q = 2,\ f = T^2$) $\quad F_f^\times = (\mathbb{F}_2[T]/(T^2))^\times = \{1,\ T+1\}$

The bias is towards $T+1 \in N_f$ (against $1 \in R_f$). If $h(T) = \sum_{j=0}^{n} a_j T^j$ ($a_j \in \mathbb{F}_2$) is irreducible, then polynomials with $a_1 = 1$ appear "earlier" than those with $a_1 = 0$.

Bias against splitting primes

L/K: a finite abelian extension, $S_D := \{\mathfrak{p} \in S \mid \text{splitting in } L\}$, $S_N := S \setminus S_D$.

$$\pi_s(x; L)_D := \sum_{\substack{\mathfrak{p} \in S_D \\ N(\mathfrak{p}) \le x}} \frac{1}{N(\mathfrak{p})^s}, \quad \pi_s(x; L)_N := \sum_{\substack{\mathfrak{p} \in S_N \\ N(\mathfrak{p}) \le x}} \frac{1}{N(\mathfrak{p})^s}.$$

If $\mathfrak{p} \nmid D_{L/K}$, $\mathfrak{p}$ splits in $L \iff \left(\frac{L/K}{\mathfrak{p}}\right) = 1$. $\therefore \pi_s(x; L)_D = \pi_s(x; 1)$.

Theorem 2 (Aoki-Koyama, JNT 2022) (Bias against splitting primes)

Let L/K be a quadratic extension (for simplicity).

Under DRH, it holds as $x \to \infty$ that

$$\pi_{\frac{1}{2}}(x; L)_N - \pi_{\frac{1}{2}}(x; L)_D = \left(\tfrac{1}{2} + m_\chi\right) \log\log x + O(1).$$

(In the paper we prove for general abelian extensions.)

There exists a bias towards nonsplitting (i.e. against splitting) primes.

Bias against principal prime ideals

I_K: the ideal group of K,

P_K: the principal ideal group

$Cl_K := I_K/P_K$: $\qquad h_K := |Cl_K|$

$\widetilde{K}$: the Hilbert class field of K (i.e. $Cl_K \ni [\mathfrak{a}] \overset{\sim}{\mapsto} \sigma_{\mathfrak{a}} := \left(\frac{\widetilde{K}/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(\widetilde{K}/K)$)

Theorem 3 (Aoki-Koyama, JNT 2022) (Bias against principal prime ideals)

Assume $h_K = 2$ (for simplicity). Put $\mathrm{Gal}(\widetilde{K}/K)\hat{} = \{1, \chi\}$.

Under DRH, it holds as $x \to \infty$ that

$$\sum_{\substack{\mathfrak{p} \notin P_K \\ N(\mathfrak{p}) \leq x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - \sum_{\substack{\mathfrak{p} \in P_K \\ N(\mathfrak{p}) \leq x}} \frac{1}{\sqrt{N(\mathfrak{p})}} \sim \left(\tfrac{1}{2} + m(\chi)\right) \log\log x \quad (x \to \infty).$$

(In the paper we prove for general class numbers.)

There exists a bias towards nonprincipal (i.e. against principal) prime ideals.

Bias of Ramanujan's $\tau(p)$

$$L(s,\Delta)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s},\qquad \Delta(z)=q\prod_{k=1}^{\infty}(1-q^k)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n\qquad (-2p^{\frac{11}{2}}\le\tau(p)\le 2p^{\frac{11}{2}}).$$

Theorem 4 (Koyama-Kurokawa, PJA 2022) (Bias to $0<\tau(p)<2p^{\frac{11}{2}}$)

Under DRH of $L(s+\frac{11}{2},\Delta)$, it holds that

$$\sum_{p\le x}\frac{\tau(p)}{p^6}\sim\frac{1}{2}\log\log x\quad (x\to\infty).$$

In other words, the sequence $\frac{\tau(p)}{p^{11/2}}$ has a Chebyshev bias to being positive.

- The distribution of θ_p is "uniform" in the sense of the Sato-Tate Conjecture (proved), which corresponds to the PNT in arithmetic progressions.

Bias of $\tau(p^2)$

Theorem 5 (Koyama-Kurokawa, 2022) (Bias of $\tau(p^2)$ to being negative)

Under DRH for $L(s+\frac{17}{2}, \mathrm{sym}^2\Delta) = \prod_p \det\left(1-\left(\mathrm{sym}^2 M(p)\right) p^{-s}\right)^{-1}$

with $\mathrm{sym}^2 M(p) = \begin{pmatrix} e^{2i\theta_p} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2i\theta_p} \end{pmatrix}$, it holds as $x \to \infty$ that

$$\sum_{p\le x} \frac{\tau(p^2)}{p^{\frac{23}{2}}} = \sum_{p\le x} \frac{\left(\tau(p)-p^{\frac{11}{2}}\right)\left(\tau(p)+p^{\frac{11}{2}}\right)}{p^{\frac{23}{2}}} = \sum_{p\le x} \frac{(2\cos\theta_p-1)(2\cos\theta_p+1)}{\sqrt{p}} \sim -\frac{1}{2}\log\log x$$

In other words, the sequence $\frac{\tau(p^2)}{p^{11}}$ has a Chebyshev bias to being negative.

- It also suggests a bias of the Satake parameters $\theta_p \in [0,\pi]$ towards $-\frac{1}{2} \le \cos\theta_p \le \frac{1}{2}$ (i.e. $\frac{\pi}{3} \le \theta_p \le \frac{2}{3}\pi$), which is compared to the bias towards $0 \le \cos\theta_p \le 1$ (i.e. $0 \le \theta_p \le \frac{\pi}{2}$) in Theorem 4
- Other types of biases may be discovered by higher symmetric powers.

Bias of a_v for elliptic curves

K: a function field, v: a finite place, k_v: the residue field, $q_v = |k_v|$.

For an elliptic curve E/K, put $a_v := q_v + 1 - \#E_v(k_v)$.

$$L(s, E) = \prod_{v:\text{ good}} (1 - 2a_v q_v^{-s} + q_v^{1-2s})^{-1} \prod_{v:\text{ bad}} (1 - a_v q_v^{-s})^{-1}$$

Theorem 6 (Kaneko-Koyama, 2023) (Bias of a_v)

Assume E/K is not isotrivial ($\Leftrightarrow$ ρ_E is reducible containing the identity rep).

It holds under the BSD conjecture that

$$\sum_{q_v \le x} \frac{a_v}{q_v} = \left(\frac{1}{2} - \mathrm{rk}(E)\right) \log\log x + O(1) \quad (x \to \infty).$$

Then $\frac{a_v}{\sqrt{q_v}}$ has a bias to being positive if $\mathrm{rk}(E) = 0$, and negative if $\mathrm{rk}(E) > 0$.

The red part holds unconditionally (without BSD).

Main Theorem (Bias of a_v for algebraic curves)

K: a global field, v: a place, k_v: the residue field, $q_v = |k_v|$.

C/K: an algebraic curve with the Galois representation $\rho = \rho_C$

Put $a_v = q_v + 1 - \#C(k_v) = \sqrt{q_v}\mathrm{tr}(\rho(\mathrm{Frob}_v))$.

Theorem 7 (Bias of a_v)

Assume DRH for the Hasse-Weil L-function $L(s, C)$ and put $m = \mathrm{ord}_{s=1/2} L(s, C)$.

$\delta(C) = [$the order of the pole of $L^{(2)}(s, C)$ at $s = 1]$ with
$L^{(2)}(s, C) =$ the second moment L-function.

Then the following holds

$$\sum_{q_v \leq x} \frac{a_v}{q_v} = -\left(\frac{\delta(C)}{2} + m\right) \log\log x + O(1) \quad (x \to \infty).$$

Proof of Main Theorem

Putting $\rho(\mathrm{Frob}_v) = M(v)$, the normalized L-functions is given as

$$L(s, M) = L(s, M_C) = \prod_{v\text{: good}} \det\left(I - M(v)q^{-s}\right)^{-1} \times \text{(bad factors)}.$$

$$\log\left((\log x)^m \prod_{q_v \le x} \det\left(1 - M(v)q_v^{-\frac{1}{2}}\right)^{-1}\right) = \mathsf{I}(x) + \mathsf{II}(x) + \mathsf{III}(x) = O(1) \quad (x \to \infty)$$

with

$$\mathsf{I}(x) = \sum_{q_v \le x} \frac{\mathrm{tr}(M(v))}{\sqrt{q}_v}, \quad \mathsf{II}(x) = \frac{1}{2}\sum_{q_v \le x} \frac{\mathrm{tr}(M(v)^2)}{q_v}, \quad \mathsf{III}(x) = \sum_{k \ge 3} \frac{1}{k} \sum_{q_v \le x} \frac{\mathrm{tr}(M(v)^k)}{q_v^{k/2}}.$$

Proof of Main Theorem

The generalization of Mertens theorem (Kaneko-Koyama-Kurokawa, 2022) gives

$$\mathsf{II}(x) \sim \frac{\delta(C)}{2} \log\log x \qquad (x \to \infty).$$

On the other hand it is easily seen that $\mathsf{III}(x) = O(1)$ $(x \to \infty)$.

Therefore

$$\begin{aligned}\mathsf{I}(x) = \sum_{q_v \le x} \frac{\mathrm{tr}(M(v))}{\sqrt{q_v}} &= \sum_{q_v \le x} \frac{a_v}{q_v} \\ &\sim -\left(m + \frac{\delta(C)}{2}\right) \log\log x \qquad (x \to \infty).\end{aligned}$$

Example (Fermat curves of prime degree)

Theorem 8 (Okumura 2023)

$C/\mathbb{Q}$: the Fermat curve $X^\ell + Y^\ell = Z^\ell$, ℓ an odd prime,
$a_p(C) := p + 1 - \#C(\mathbb{F}_p)$

Put $C_F = C \times_{\mathbb{Q}} F$ with $F = \mathbb{Q}(\mu_\ell)$, μ_ℓ a primitive ℓ-th root of 1

Assume DRH for the Hasse-Weil L-function $L(s, C_F)$.

Then the following holds

$$\sum_{p \leq x} \frac{a_p}{p} = \frac{g-m}{\ell-1} \log\log x + O(1) \quad (x \to \infty),$$

where $g = (\ell-1)(\ell-2)/2$.

Under DRH, $\frac{a_p}{\sqrt{p}}$ has a bias to being positive, if $g - m > 0$.

Example (the case $\ell = 4$)

C: $x^4 + y^4 = z^4$

We can calculte $\#C(\mathbb{F}_p)$ by the Davenport-Hasse theorem:

x	100	200	300	400	500	600	700
$\#\{p \mid a_p > 0\}$	6	9	13	19	24	27	29
$\#\{p \mid a_p < 0\}$	5	12	16	18	20	24	30
$S_x = \sum_{q_v \le x} \frac{a_v}{q_v}$	0.5567	3.3412	-0.1160	0.2871	5.9287	6.0637	6.0438

- The numbers of primes with $a_p > 0$ and $a_p < 0$ are almost equal.
- But S_x is positive and increasing.

It suggests the bias of a_p to being positive.