Chebyshev's Bias against Splitting and Principal Primes in Global Fields

Miho Aoki and Shin-ya Koyama

Dedicated to Professor Nobushige Kurokawa on his 70th birthday

Abstract

A reason for the emergence of Chebyshev's bias is investigated. The Deep Riemann Hypothesis (DRH) enables us to reveal that the bias is a natural phenomenon for making a well-balanced disposition of the whole sequence of primes, in the sense that the Euler product converges at the center. By means of a weighted counting function of primes, we succeed in expressing magnitudes of the deflection by a certain asymptotic formula under the assumption of DRH, which gives a new formulation of Chebyshev's bias.

For any Galois extension of global fields and for any element σ in the Galois group, we establish a criterion of the bias of primes whose Frobenius elements are equal to σ under the assumption of DRH. As an application we obtain a bias toward non-splitting and non-principle primes in abelian extensions under DRH. In positive characteristic cases, DRH is proved, and all these results hold unconditionally.

1 Introduction

Chebyshev's bias is the phenomenon that there tend to be more primes of the form $4k+3$ than of the form $4k+1$ ($k \in \mathbb{Z}$). In fact, if denoting by $\pi(x;\, q,\, a)$ the number of primes $p \le x$ such that $p \equiv a \pmod{q}$, then the inequality

$$\pi(x;\, 4,\, 3) \ge \pi(x;\, 4,\, 1) \tag{1.1}$$

holds for any x less than 26861, which is the first prime number violating the inequality (1.1). However, the both sides draw equal at the next prime 26863, and $\pi(x;\, 4,\, 3)$ gets ahead again until 616841. It is computed that more than 97% of $x < 10^{11}$ satisfy the inequality (1.1). In spite of that, Littlewood [28] proved that the difference $\pi(x;\, 4,\, 3) - \pi(x;\, 4,\, 1)$ changes its sign infinitely many times. Moreover, Knapowski and Turan [21, (2.4)] conjectured that the limit of the percentage in all positive numbers of the set

$$A_X = \{x < X \mid \pi(x;\,4,\,3) \geq \pi(x;\,4,\,1)\}$$

as $X \to \infty$ would equal 100%, but now it is proved under GRH that the limit does not exist and that the conjecture is false [16].

In place of such a naive density, the logarithmic density plays the role. Define the *logarithmic density* of the set A_X in $[2, X]$ by

$$\delta(A_X) = \frac{1}{\log X} \int_{t \in A_X} \frac{dt}{t}. \tag{1.2}$$

Rubinstein-Sarnak [34] proved that $\lim_{X\to\infty} \delta(A_X) = 0.9959...$ under the assumption of the GRH and the GSH (Grand Simplicity Hypothesis) for the L-functions.

It is known by Dirichlet's prime number theorem in arithmetic progressions that the number of primes of the form $4k + 3$ and $4k + 1$ should asymptotically equal. Then we find from the following insight that Chebyshev's bias means that the primes of the form $4k + 3$ appear earlier than those of the form $4k + 1$.

For example, if, among the first 100 prime numbers, 50 of the first half are of the form $4k+3$ and those of the latter half are of the form $4k+1$, the inequality $\pi(x;\,4,\,3) \geq \pi(x;\,4,\,1)$ always holds in this interval even if their total number of elements are equal, and the maximum difference becomes 50. On the other hand, in the case that primes of the form $4k + 3$ and $4k+1$ appear alternately, the maximum difference may be 1 even if the same inequality holds incessantly. From this discussion, it is effective to apply the structure that "regards smaller primes as heavier elements", reflecting the magnitudes of the primes, in order to elucidate Chebyshev's bias. One of the reasons why the logarithmic density was effective would be that it treated the contribution of smaller numbers as greater ones by virtue of the factor $1/t$ in the integral in (1.2).

Therefore in this paper, we make use of a weighted counting function

$$\pi_s(x;\,q,\,a) = \sum_{\substack{p<x:\,\text{prime} \\ p\equiv a \pmod q}} \frac{1}{p^s} \qquad (s \geq 0),$$

which is a generalization of the counting function $\pi(x;\,q,\,a) = \pi_0(x;\,q,\,a)$. Here, small primes allow higher contribution to $\pi_s(x;\,q,\,a)$, as long as we fix $s > 0$. The function $\pi_s(x;\,q,\,a)$ $(s > 0)$ should be more appropriate to represent the phenomenon, because it reflects the size of primes which $\pi(x;\, q,\, a)$ has ignored.

Indeed, although the natural density of the set

$$A(s) = \{x > 0 \mid \pi_s(x;\, 4,\, 3) - \pi_s(x;\, 4,\, 1) > 0\}$$

did not exist for $s = 0$ under GRH, we show under the assumption of the Deep Riemann Hypothesis (DRH), described below, that it would exist and equal to 1 when $s = 1/2$, that is,

$$\lim_{X\to\infty} \frac{1}{X} \int_{t\in A(1/2)\cap[2,X]} dt = 1.$$

More precisely we would reach under DRH that

$$\pi_{\frac{1}{2}}(x;\, 4,\, 3) - \pi_{\frac{1}{2}}(x;\, 4,\, 1) \sim \frac{1}{2}\log\log x \quad (x \to \infty), \tag{1.3}$$

where $f(x) \sim g(x)$ $(x \to \infty)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. The asymptotic (1.3) suggests a formulation of Chebyshev's bias.

This observation derives from the conditional convergence of the Euler products of Dirichlet L-functions, which is a part of the Deep Riemann Hypothesis (DRH) named by Kurokawa [27]. In case of Dirichlet L-functions $L(s, \chi)$ for non-principal Dirichlet characters χ, DRH states that it holds on $\mathrm{Re}(s) = 1/2$ that

$$\lim_{x\to\infty}\left((\log x)^m \prod_{p\le x:\,\mathrm{prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}\right) = \frac{L^{(m)}(s,\chi)}{e^{m\gamma} m!} \times \begin{cases} \sqrt{2} & (\chi^2 = 1,\ s = \frac{1}{2}) \\ 1 & (\text{otherwise}) \end{cases} \tag{1.4}$$

with γ being the Euler constant and $m = m_\chi = \mathrm{ord}_{s=\frac{1}{2}} L(s,\chi)$. Taking the logarithm of (1.4) for $\chi^2 = 1$ and $s = \frac{1}{2}$, the convergence of the above limit gives that

$$\sum_{p\le x:\,\mathrm{prime}} \frac{\chi(p)}{\sqrt{p}} + \left(\frac{1}{2} + m\right)\log\log x = O(1) \quad (x \to \infty).$$

In particular, if q is an odd prime and $\chi\colon (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ is defined by

$$\chi(a) = \left(\frac{a}{q}\right) = \begin{cases} 1 & (a \in ((\mathbb{Z}/q\mathbb{Z})^\times)^2) \\ -1 & (\text{otherwise}), \end{cases}$$

then we would have

$$\begin{aligned}
&\sum_{\substack{b\in(\mathbb{Z}/q\mathbb{Z})^\times \\ \text{quadratic non-residue}}} \pi_{\frac{1}{2}}(x;\,q,\,b) - \sum_{\substack{a\in(\mathbb{Z}/q\mathbb{Z})^\times \\ \text{quadratic residue}}} \pi_{\frac{1}{2}}(x;\,q,\,a) \\
&= \sum_{\substack{p\le x\colon \text{prime} \\ \text{quadratic non-residue}}} \frac{1}{\sqrt{p}} - \sum_{\substack{p\le x\colon \text{prime} \\ \text{quadratic residue}}} \frac{1}{\sqrt{p}} \\
&\sim \left(\frac{1}{2}+m\right)\log\log x \quad (x\to\infty).
\end{aligned}$$

A more precise result on the difference between individual $\pi_{\frac{1}{2}}(x;\,q,\,b)$ and $\pi_{\frac{1}{2}}(x;\,q,\,a)$ is proved in (3.4) in §3 under DRH. Chebyshev's original case (1.3) is restored by choosing $(q,\,a,\,b)=(4,\,1,\,3)$ in Example 3.4.

Aymone [3] also introduced the weighted counting function $\sum_{p\le x\colon \text{prime}} \frac{\chi(p)}{p^\sigma}$ for $0\le\sigma<1$, and recently showed that its chaiging sign only for a finite number of integers $x\ge 1$ is equivalent to $L(s,\chi)\ne 0$ for $\mathrm{Re}(s)>1-\varepsilon$ for some $\varepsilon>0$. Since $L(s,\chi)$ has zeros on the line $\mathrm{Re}(s)=1/2$, his results are essentially effective in $1/2<\sigma<1$. Then DRH describes the situation as $\sigma\to 1/2$ in a refined form.

Throughout this paper we put K to be a global field. By virtue of the formula (1.3) we reach a formulation of the Chebyshev bias of primes $\mathfrak{p}$ of K as follows.

Definition 1.1. Let $a(\mathfrak{p})\in\mathbb{R}$ be a sequence over prime ideals $\mathfrak{p}$ of K such that

$$\lim_{x\to\infty}\frac{\#\{\mathfrak{p}\mid a(\mathfrak{p})>0,\ N(\mathfrak{p})\le x\}}{\#\{\mathfrak{p}\mid a(\mathfrak{p})<0,\ N(\mathfrak{p})\le x\}}=1.$$

We say $a(\mathfrak{p})$ has a *Chebyshev bias to being positive*, if there exists $C>0$ such that

$$\sum_{N(\mathfrak{p})\le x}\frac{a(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}}\sim C\log\log x \quad (x\to\infty),$$

where $\mathfrak{p}$ runs through primes of K. On the other hand we say $a(\mathfrak{p})$ is *unbiased*, if

$$\sum_{N(\mathfrak{p})\le x}\frac{a(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}}=O(1) \quad (x\to\infty).$$

Definition 1.2. Assume that the set of all primes $\mathfrak{p}$ of K with $N(\mathfrak{p}) \le x$ is expressed as a disjoint union $P_1(x) \cup P_2(x)$ and that their proportion converges to

$$\delta = \lim_{x\to\infty} \frac{|P_1(x)|}{|P_2(x)|}.$$

We say there exists a *Chebyshev bias toward* P_1 (or *Chebyshev bias against* P_2), if the following asymptotic holds:

$$\sum_{p\in P_1(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{p\in P_2(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} \sim C \log\log x \quad (x \to \infty)$$

for some $C > 0$. On the other hand we say there exists *no biases* between P_1 and P_2, if

$$\sum_{p\in P_1(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{p\in P_2(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} = O(1) \quad (x \to \infty).$$

Definitions 1.1 and 1.2 are concerning a different aspect from those dealt with by the definitions discovered by Rubinstein-Sarnak [34] and developed by Akbary-Ng-Shahabi [2], Devin [10], and so on. Definitions 1.1 and 1.2 is an estimate of the size of the discrepancy caused by Chebyshev's bias, which was ignored by the conventional definitions on the (logarithmic) length of the interval in terms of limiting distributions. For example, they cannot tell the two extreme cases in the first 100 primes described after (1.2), because the two cases have the same length of the interval where the inequality holds. On the other hand, Definitions 1.1 and 1.2 give little information on the density distributions. Therefore these two types of definitions apparently have no logical implications. Both shed lights on Chebyshev's bias from different directions.

We denote an n-dimensional nontrivial irreducible Artin representation by

$$\rho : \ \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathbb{C}}(V) \quad (\rho \ne \mathbf{1}).$$

The L-function $L_K(s,\rho)$ is defined by the Euler product as

$$L_K(s,\rho) = \prod_{\mathfrak{p}} \det\left(1 - N(\mathfrak{p})^{-s} \rho(\mathrm{Frob}_{\mathfrak{p}} | V^{I_{\mathfrak{p}}})\right)^{-1} \quad (\mathrm{Re}(s) > 1),$$

where $\mathfrak{p}$ runs through the prime ideals of K with $N(\mathfrak{p})$ its norm and $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ the Frobenius element with $I_{\mathfrak{p}}$ the inertia group.

Conjecture 1.1 (Deep Riemann Hypothesis (DRH))**.** Put $m = m_\rho = \mathrm{ord}_{s=\frac{1}{2}} L_K(s,\rho)$. Then the limit

$$\lim_{x\to\infty}\left((\log x)^m \prod_{N(\mathfrak{p})\le x} \det\left(1-\rho(\mathrm{Frob}_\mathfrak{p}|V^{I_\mathfrak{p}})N(\mathfrak{p})^{-\frac{1}{2}}\right)^{-1}\right) \quad (1.5)$$

satisfies the following:

DRH(A) The limit (1.5) exists and is nonzero.

DRH(B) The limit (1.5) satisfies the following identity:

$$\lim_{x\to\infty}\left((\log x)^m \prod_{N(\mathfrak{p})\le x} \det\left(1-\rho(\mathrm{Frob}_\mathfrak{p}|V^{I_\mathfrak{p}})N(\mathfrak{p})^{-\frac{1}{2}}\right)^{-1}\right) = \frac{\sqrt{2}^{\nu(\rho)} L_K^{(m)}(1/2,\rho)}{e^{m\gamma} m!},$$

where $\nu(\rho) = \mathrm{mult}(\mathbf{1}, \mathrm{sym}^2\rho) - \mathrm{mult}(\mathbf{1}, \wedge^2\rho) \in \mathbb{Z}$ with $\mathrm{mult}(\mathbf{1},\sigma)$ being the multiplicity of the trivial representation $\mathbf{1}$ in σ.

Obviously DRH(B) implies DRH(A). We also note that if $m > 0$, DRH(A) implies that

$$\lim_{x\to\infty} \prod_{N(\mathfrak{p})\le x} \det\left(1-\rho(\mathrm{Frob}_\mathfrak{p}|V^{I_\mathfrak{p}})N(\mathfrak{p})^{-\frac{1}{2}}\right)^{-1} = L_K(1/2,\rho) = 0.$$

Justification of DRH may depend on the pair (K,ρ). In what follows in this paper, whenever we assume DRH, we will specify the types of (K,ρ) on which $L_K(s,\rho)$ should satisfy DRH. However, since we fix K throughout this paper, we sometimes abbreviate "DRH for $L_K(s,\rho)$" to "DRH for ρ".

Conjecture 1.1 originates from the Birch and Swinnerton-Dyer [5, p.79 (A)]. They conjectured for the representation ρ_E attached to an elliptic curve $E/\mathbb{Q}$ that the Euler product of $L_\mathbb{Q}(s,\rho_E)$ should converge at the center, and that in case the limit is zero, it asymptotically equals $C(\log x)^g$ with $g = \mathrm{rank}(E)$. Goldfeld [14] proved their conjecture implies the GRH for $L_\mathbb{Q}(s,\rho_E)$ and that if their conjecture is correct, then it holds $g = m$.

It is also discovered by Conrad [9, Theorems 3.3 and 6.3] that DRH implies both the convergence of the Euler product in $\mathrm{Re}(s) \ge 1/2$ and the Generalized Riemann Hypothesis (GRH) for $L_K(s,\rho)$.

When $\mathrm{char}(K) > 0$, it is proved in [19] that both DRH(A) and (B) are true. The main theorems in this paper will give a criterion of emergence of the Chebyshev biases under the assumption of DRH, which means the criterion holds unconditionally for $\mathrm{char}(K) > 0$.

In this paper we examine Chebyshev biases existing in the primes of global fields. Let L/K be a finite Galois extension of global fields. The main theorem is described in terms of the set S of all primes in K and its subset S_σ of unramified primes whose Frobenius element $\left(\frac{L/K}{\mathfrak{p}}\right)$ is equal to $\sigma \in \mathrm{Gal}(L/K)$:

Theorem 1.1 (a part of Theorem 2.2)**.** *Let L/K be a finite Galois extension of global fields. The following (i) and (ii) are equivalent:*

(i) DRH(A) for all non trivial irreducible representations of Gal(L/K).

(ii) For any $\sigma \in$Gal(L/K) it holds that

$$\sum_{\substack{\mathfrak{p}\in S \\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - \frac{[L:K]}{|c_\sigma|} \sum_{\substack{\mathfrak{p}\in S_\sigma \\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} = C \log\log x + c + o(1) \qquad (x \to \infty)$$

for some constants C and c depending on σ.

Here C is expressed in terms of $\nu(\rho)$ and m_ρ in Conjecture 1.1. Calculating such constants for specific cases under the assumption of (i), we obtain various examples of Chebyshev biases. We will give some examples at the end of this section. Note that Chebyshev's bias is a problem beyond the Riemann Hypothesis, since it is essentially equivalent to DRH(A).

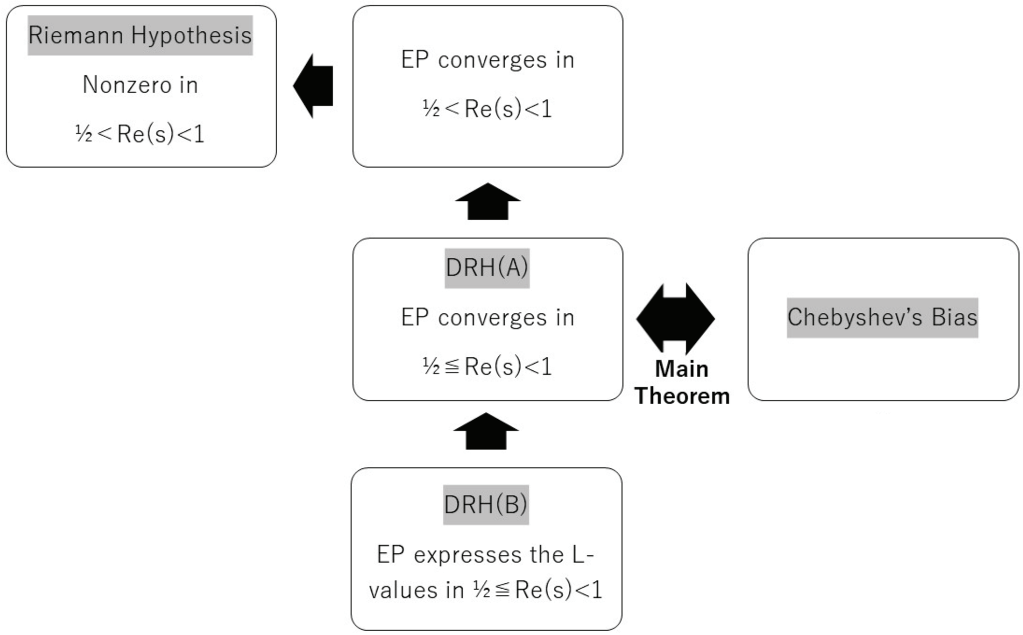

Figure 1: Deep Riemann Hypothesis (DRH) and Chebyshev's Bias
(in case of entire L-functions)

We illustrate this situation for the case of entire L-functions[1] in Figure 1. Here we denote the Generalized Riemann Hypothesis (GRH) by just the Riemann Hypothesis (RH), because our interest is focused on whether the hypothesis is deep or not. By this notation, the Riemann Hypothesis is located in the upper left box, which follows from the upper middle box, the convergence of the Euler product in $\frac{1}{2} < \mathrm{Re}(s) < 1$, because convergence of an infinite product implies by definition that the limit is nonzero. Extension of the region of convergence to the boundary $\mathrm{Re}(s) = \frac{1}{2}$ is in the middle box, which is DRH(A). The theme of this paper is the essential equivalence between Chebyshev's bias and DRH(A). Although it is weaker than the full Deep Riemann Hypothesis (DRH(B)), it is still stronger than the Riemann Hypothesis. For numerical evidence of DRH, see [20, Table I].

We conclude this section by introducing three simple examples.

Example 1.1 (Bias against splitting primes (Example 3.3))**.** Assume $[L :$

[1] In case the L-functions have a pole at $s = 1$, DRH needs modifications by Akatsuka [1].

$K] = 2$ and let χ be the nontrivial character of $\mathrm{Gal}(L/K)$. The following (i) and (ii) are equivalent:

(i) DRH(A) for $L_K(s, \chi)$ holds.

(ii) There exists a Chebyshev bias against splitting primes with the asymptotic

$$\sum_{\substack{\mathfrak{p}\colon \text{nonsplit} \\ N(\mathfrak{p}) \le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - \sum_{\substack{\mathfrak{p}\colon \text{split} \\ N(\mathfrak{p}) \le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} = \left(\frac{1}{2} + m_\chi\right) \log\log x + c + o(1) \qquad (x \to \infty)$$

for some constant c.

Example 1.2 (Bias against quadratic residues (Corollary 3.2))**.** Let q be a positive integer, $t(q)$ the number of distinct prime numbers dividing q, and

$$t := \begin{cases} t(q) - 1 & (2 || q), \\ t(q) & (4 || q \text{ or } 2 \nmid q), \\ t(q) + 1 & (8 | q). \end{cases}$$

Assume that $L(\frac{1}{2}, \chi) \neq 0$ for all Dirichlet characters χ modulo q. The following (i) and (ii) are equivalent:

(i) DRH(A) holds for $L(s, \chi)$ for any Dirichlet character χ modulo q.

(ii) There exists a Chebyshev bias against quadratic residues modulo q with the asymptotic

$$\pi_{\frac{1}{2}}(x; q, b) - \pi_{\frac{1}{2}}(x; q, a) = \frac{2^{t-1}}{\varphi(q)} \log\log x + c + o(1) \quad (x \to \infty)$$

for some constant c and for any pair (a, b) of a quadratic residue a and a non-residue b, and there exist no biases for all other types of pairs (a, b).

Example 1.3 (Bias against principal ideals (Corollary 3.5))**.** We denote by $\widetilde{K}$ the Hilbert class field of K. The ideal class group is expressed as $\mathrm{Cl}_K \simeq \mathrm{Gal}(\widetilde{K}/K)$. An ideal class $[\mathfrak{a}] \in \mathrm{Cl}_K$ corresponds to $\sigma_{\mathfrak{a}} := \left(\frac{\widetilde{K}/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(\widetilde{K}/K)$. Assume DRH(A) for $L_K(s, \chi)$ and that $L_K(\frac{1}{2}, \chi) \neq 0$ for any character χ of Cl_K. If $|\mathrm{Cl}_K|$ is even, then in the whole set of prime ideals of K, there exists a Chebyshev bias against principal ideals. Namely,

$$\sum_{\substack{\mathfrak{p}:\ \text{nonprincipal} \\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - (h_K - 1) \sum_{\substack{\mathfrak{p}:\ \text{principal} \\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}}$$
$$= \frac{|\mathrm{Cl}_K/\mathrm{Cl}_K^2| - 1}{2} \log\log x + c + o(1) \quad (x \to \infty)$$

with $h_K = |\mathrm{Cl}_K|$.

2 Bias of primes in global fields

Let K be a global field, that is, a number field or an algebraic function field of one variable over a finite field $\mathbb{F}_q$ (q is a power of a prime number), and L/K a finite Galois extension. For a prime ideal $\mathfrak{p}$ of K which does not divide the relative discriminant $D_{L/K}$, we denote by $\left(\frac{L/K}{\mathfrak{p}}\right)$ the Frobenius element of $\mathrm{Gal}(L/K)$. For any $\sigma \in \mathrm{Gal}(L/K)$, we know from the Chebotarev density theorem that the density of prime ideals $\mathfrak{p}$ of K satisfying $\mathfrak{p} \nmid D_{L/K}$ and $\left(\frac{L/K}{\mathfrak{p}}\right) \sim \sigma$ is $|c_\sigma|/[L:K]$, where c_σ is the conjugacy class of σ (See [33, p125, Theorem 9.13 A] for function fields). Let S be the set of all prime ideals of K and put $S_\sigma := \left\{\mathfrak{p} \in S \,\middle|\, \mathfrak{p} \nmid D_{L/K},\ \left(\frac{L/K}{\mathfrak{p}}\right) \sim \sigma\right\}$ for $\sigma \in \mathrm{Gal}(L/K)$.

Theorem 2.1 (Chebotarev density theorem)**.** *For any finite Galois extension L/K of global fields and $\sigma \in \mathrm{Gal}(L/K)$, we have*

$$\lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in S_\sigma} N(\mathfrak{p})^{-s}}{\sum_{\mathfrak{p}\in S} N(\mathfrak{p})^{-s}} = \frac{|c_\sigma|}{[L:K]}.$$

We will examine biases of prime ideals toward or against the set S_σ depending on σ, under the assumption of DRH(A) in the case of number fields and unconditionally in the case of algebraic function fields. For $x \in \mathbb{R}_{>0}$, we put

$$\pi_{s,K}(x) := \sum_{\substack{\mathfrak{p}\in S \\ N(\mathfrak{p})\le x}} \frac{1}{N(\mathfrak{p})^s},$$
$$\pi_s(x;\sigma) := \sum_{\substack{\mathfrak{p}\in S_\sigma \\ N(\mathfrak{p})\le x}} \frac{1}{N(\mathfrak{p})^s},$$

for any $\sigma \in$Gal(L/K) and

$$\pi_s(x;H) := \sum_{\sigma \in H} \pi_s(x;\sigma)$$

for any $H \subset \mathrm{Gal}(L/K)$.

Theorem 2.2. *Let L/K be a finite Galois extension of global fields. For any $\sigma \in$Gal(L/K) we put*

$$M(\sigma) := \frac{1}{2} \sum_{\substack{\rho \neq \mathit{1} \\ \text{irred.}}} \chi_\rho(\sigma)\nu(\rho)$$

with χ_ρ being the character of ρ and

$$m(\sigma) := \sum_{\substack{\rho \neq \mathit{1} \\ \text{irred.}}} \chi_\rho(\sigma) m_\rho,$$

where ρ runs through all nontrivial irreducible representations of Gal(L/K). The following (i),(ii) and (iii) are equivalent:

(i) DRH(A) for all non trivial irreducible representations of Gal(L/K).

(ii) For all $\sigma \in$Gal(L/K) it holds that

$$\pi_{\frac{1}{2},K}(x) - \frac{[L:K]}{|c_\sigma|}\pi_{\frac{1}{2}}(x;\sigma) = (M(\sigma)+m(\sigma))\log\log x + c + o(1) \qquad (x \to \infty)$$

for some constant c.

(iii) For all pairs σ, $\tau \in$Gal(L/K) it holds that

$$\begin{aligned} &\frac{1}{|c_\tau|}\pi_{\frac{1}{2}}(x;\tau) - \frac{1}{|c_\sigma|}\pi_{\frac{1}{2}}(x;\sigma) \\ &\quad = \frac{1}{[L:K]}\left(M(\sigma) - M(\tau) + m(\sigma) - m(\tau)\right)\log\log x + c + o(1) \end{aligned} \qquad (x \to \infty)$$

for some constant c.

Remark 2.1. For any irreducible representation ρ, the complex conjugate representation $\overline{\rho}$ is irreducible and we have $\nu(\rho) = \nu(\overline{\rho})$, $m_\rho = m_{\overline{\rho}}$ and that $\chi_{\overline{\rho}}(\sigma) = \overline{\chi_\rho(\sigma)}$. Therefore it holds that $M(\sigma), m(\sigma) \in \mathbb{R}$.

Example 2.1 (Quaternion extension)**.** Let $L/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(L/\mathbb{Q})$ isomorphic to the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. There are five irreducible representations of $\mathrm{Gal}(L/\mathbb{Q})$. We denote the non-trivial 1-dimensional representations by χ_1, χ_2 and χ_3, and the 2-dimensional representation by ρ. In this case, we have $\nu(\chi_1) = \nu(\chi_2) = \nu(\chi_3) = 1$ and $\nu(\rho) = -1$. If we assume $m_{\chi_1} = m_{\chi_2} = m_{\chi_3} = 0$, then the coefficients $M(\sigma)$ and $m(\sigma)$ are given as follows.

σ	1	-1	$\pm i$	$\pm j$	$\pm k$
$M(\sigma)$	$1/2$	$5/2$	$-1/2$	$-1/2$	$-1/2$
$m(\sigma)$	$2m_\rho$	$-2m_\rho$	0	0	0

Fröhlich [11] proved that there exist infinitely many quaternion extensions $L/\mathbb{Q}$ satisfying $m_\rho \neq 0$. We have $M(1) - M(-1) + m(1) - m(-1) = -2 + 4m_\rho$ from the table. Therefore, if $m_\rho = \mathrm{ord}_{s=\frac{1}{2}} L_{\mathbb{Q}}(s, \rho) = 0$ (resp. $m_\rho \neq 0$), then there exists a bias toward prime numbers p satisfying $p \nmid D_{L/\mathbb{Q}}$ and $\left(\frac{L/\mathbb{Q}}{(p)}\right) = 1$ $\left(\text{resp. } \left(\frac{L/\mathbb{Q}}{(p)}\right) = -1\right)$. In other words, the direction of the bias is determined by the order of the central zero of the L-function. This principle is also referred to by Bailleul [4, Theorem A] as "the most important feature" of his result. An analogous phenomenon is also observed by Kaneko-Koyama-Kurokawa [19, Theorem 6.9] for holomorphic modular forms, where the direction of the bias depends on the central zero of the automorphic L-functions.

To prove the theorem, we show the following proposition.

Proposition 2.1. *Let K be a global field and ρ a nontrivial irreducible Artin representation of Gal(K^{sep}/K). The following (i) and (ii) are equivalent.*

(i) DRH(A) for ρ.

(ii) There exists a constant c such that

$$\sum_{N(\mathfrak{p}) \leq x} \frac{\chi_\rho(\mathrm{Frob}_{\mathfrak{p}})}{N(\mathfrak{p})^{\frac{1}{2}}} = -\left(\frac{\nu(\rho)}{2} + m_\rho\right) \log\log x + c + o(1) \quad (x \to \infty).$$

Proof. By taking the logarithm of (1.5), DRH(A) for ρ is equivalent to the existence of a constant $L \neq 0$ such that

$$m_\rho \log\log x + \underbrace{\sum_{N(\mathfrak{p}) \leq x} \sum_{k=1}^{\infty} \frac{\chi_\rho(\mathrm{Frob}_{\mathfrak{p}}^k)}{kN(\mathfrak{p})^{\frac{k}{2}}}}_{(*)} = L + o(1) \quad (x \to \infty). \tag{2.1}$$

We decompose the double sum $(*)$ into three subseries:

$$(*) = \sum_{N(\mathfrak{p})\le x} \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})^{\frac{1}{2}}} + \sum_{N(\mathfrak{p})\le x} \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p}^2)}{2N(\mathfrak{p})} + \sum_{N(\mathfrak{p})\le x} \sum_{k=3}^{\infty} \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p}^k)}{kN(\mathfrak{p})^{\frac{k}{2}}}. \quad (2.2)$$

Since

$$\sum_{N(\mathfrak{p})\le x} \sum_{k=3}^{\infty} \left| \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p}^k)}{kN(\mathfrak{p})^{\frac{k}{2}}} \right| < \frac{4\dim\rho}{3} \zeta_K(3/2) < \infty,$$

the last term of (2.2) is absolutely convergent. Denote the limit by

$$C_1 = \lim_{x\to\infty} \sum_{N(\mathfrak{p})\le x} \sum_{k=3}^{\infty} \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p}^k)}{kN(\mathfrak{p})^{\frac{k}{2}}}. \quad (2.3)$$

Next the central term of (2.2) is estimated by means of the Generalized Mertens' theorem ([32, Theorem 5], [19, Lemma 5.3]):

$$\sum_{N(\mathfrak{p})\le x} \frac{\chi_\tau(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})} = \mathrm{mult}(\mathbf{1};\tau) \log\log x + c_\tau + o(1) \quad (x\to\infty)$$

with a constant c_τ depending on the representation τ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Thus the central term in (2.2) is expressed as

$$\begin{aligned} \sum_{N(\mathfrak{p})\le x} \frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p}^2)}{2N(\mathfrak{p})} &= \frac{1}{2} \sum_{N(\mathfrak{p})\le x} \frac{\chi_{\mathrm{sym}^2\rho}(\mathrm{Frob}_\mathfrak{p}) - \chi_{\wedge^2\rho}(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})} \\ &= \frac{\nu(\rho)}{2} \log\log x + C_2 + o(1) \qquad (2.4) \end{aligned}$$

with some constant C_2.

If we assume (i), then (ii) follows from (2.1) (2.3) (2.4) with $c = L - C_1 - C_2$. Conversely, if (ii) holds with some c, then (2.1) holds with $L = c + C_1 + C_2$. □

Conrad [9, Proposition 2.1] obtains an analogous result under the assumption of the second moment hypothesis.

Corollary 2.1. *Let L be a quadratic extension of a global field K, and χ the nontrivial character of Gal(L/K). The following (i) and (ii) are equivalent:*

(i) DRH(A) for $L_K(s,\chi)$ holds.

(ii) There exists a Chebyshev bias toward the primes $\mathfrak{p}$ such that $\chi(\mathrm{Frob}_\mathfrak{p}) = -1$ with the asymptotic

$$\sum_{N(\mathfrak{p})\le x} \frac{\chi(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})^{\frac{1}{2}}} = -\left(\frac{1}{2} + m_\chi\right)\log\log x + c + o(1) \quad (x \to \infty) \tag{2.5}$$

with some constant c.

Now we are ready to show Theorem 2.2.

Proof of Theorem 2.2. First we prove that (i) implies (ii). By the orthogonality relations of characters and Proposition 2.1, we have

$$\begin{aligned}
&[L:K]\pi_{\frac{1}{2}}(x\,;\sigma)\\
&= [L:K] \sum_{\substack{\mathfrak{p}\in S_\sigma\\ N(\mathfrak{p})\le x}} \frac{1}{N(\mathfrak{p})^{\frac{1}{2}}}\\
&= |c_\sigma| \sum_{\substack{\mathfrak{p}\in S\\ N(\mathfrak{p})\le x}} \sum_{\rho:\text{irred.}} \frac{\chi_\rho\left(\left(\frac{L/K}{\mathfrak{p}}\right)\right)\overline{\chi_\rho(\sigma)}}{N(\mathfrak{p})^{\frac{1}{2}}}\\
&= |c_\sigma| \left\{\pi_{\frac{1}{2},K}(x) - \sum_{\rho\neq\mathbf{1}} \overline{\chi_\rho(\sigma)}\left(\frac{\nu(\rho)}{2} + m_\rho\right)\log\log x\right\} + c + o(1)\\
&= |c_\sigma|\{\pi_{\frac{1}{2},K}(x) - (M(\sigma) + m(\sigma))\log\log x\} + c + o(1).
\end{aligned}$$

Next we show that (ii) implies (i). Let ρ be a nontrivial irreducible representation of $\mathrm{Gal}(L/K)$. Multiplying (2.5) by $\chi_\rho(\sigma)$ and adding them for all $\sigma \in \mathrm{Gal}(L/K)$, we obtain

$$\begin{aligned}
&\pi_{\frac{1}{2},K}(x)\sum_{\sigma\in G}\chi_\rho(\sigma) - [L:K]\sum_{N(\mathfrak{p})\le x}\frac{\chi_\rho(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})^{\frac{1}{2}}}\\
&\qquad = \left(\sum_{\sigma\in G}\chi_\rho(\sigma)M(\sigma) + \sum_{\sigma\in G}\chi_\rho(\sigma)m(\sigma)\right)\log\log x + c + o(1)\\
&\qquad\qquad (x\to\infty)
\end{aligned} \tag{2.6}$$

for some constant c. By the orthogonality relations of characters, we have $\sum_{\sigma\in G}\chi_\rho(\sigma)=0$ and

$$\sum_{\sigma\in G}\chi_\rho(\sigma)M(\sigma)=\frac{1}{2}\sum_{\substack{\rho'\neq\mathbf{1}\\ \text{irred.}}}\nu(\rho')\sum_{\sigma\in G}\chi_\rho(\sigma)\chi_{\rho'}(\sigma)=\frac{|G|}{2}\nu(\rho),$$

$$\sum_{\sigma\in G}\chi_\rho(\sigma)m(\sigma)=\sum_{\substack{\rho'\neq\mathbf{1}\\ \text{irred.}}}m_{\rho'}\sum_{\sigma\in G}\chi_\rho(\sigma)\chi_{\rho'}(\sigma)=|G|m_\rho.$$

From these results, (2.6) and Proposition 2.1, we reach DRH(A) for ρ.

The equivalence between (ii) and (iii) follows from the fact that $\sum_{\sigma\in G}M(\sigma)=\sum_{\sigma\in G}m(\sigma)=0$. □

3 Bias of primes in abelian extensions

3.1 General theory

In this section, we consider biases of prime ideals in finite abelian extensions L/K of global fields. We put $G:=\mathrm{Gal}(L/K)$. In the case of abelian extensions, the coefficient $M(\sigma)$ for $\sigma\in G$ in Theorem 2.2 is given by

$$M(\sigma)=\begin{cases}\frac{1}{2}(|G/G^2|-1) & (\sigma\in G^2),\\ -\frac{1}{2} & (\sigma\notin G^2).\end{cases}$$

Hence we have the following.

Theorem 3.1. *Let L/K be a finite abelian extension of global fields. The following (i),(ii) and (iii) are equivalent:*

(i) DRH(A) holds for any nontrivial character of G.

(ii) For any $\sigma\in G$, there exists a constant c such that

$$\pi_{\frac{1}{2},K}(x)-[L:K]\pi_{\frac{1}{2}}(x;\sigma)=\begin{cases}\left(\frac{|G/G^2|-1}{2}+m(\sigma)\right)\log\log x+c+o(1) & (\sigma\in G^2)\\ \left(-\frac{1}{2}+m(\sigma)\right)\log\log x+c+o(1) & (\sigma\notin G^2)\end{cases}\qquad (x\to\infty)$$

where $m(\sigma):=\sum_{\chi\neq\mathbf{1}}\chi(\sigma)m_\sigma$ with χ running through all nontrivial characters of G.

(iii) For any $\sigma, \tau \in G$, there exists a constant c such that

$$\pi_{\frac{1}{2}}(x;\tau) - \pi_{\frac{1}{2}}(x;\sigma) = \begin{cases} \frac{1}{[L:K]}\left(\frac{|G/G^2|}{2} + m(\sigma) - m(\tau)\right)\log\log x + c + o(1) \\ \hfill (\sigma \in G^2, \tau \notin G^2) \\ -\frac{1}{[L:K]}\left(\frac{|G/G^2|}{2} - m(\sigma) + m(\tau)\right)\log\log x + c + o(1) \\ \hfill (\sigma \notin G^2, \tau \in G^2) \\ \frac{1}{[L:K]}(m(\sigma) - m(\tau))\log\log x + c + o(1) \\ \hfill (\sigma, \tau \in G^2 \text{ or } \sigma, \tau \notin G^2) \end{cases} \quad (x \to \infty).$$

Remark 3.1. From Theorem 3.1 we see that if $G \neq G^2$, then there exists a Chebyshev bias against primes $\mathfrak{p}$ such that $\left(\frac{L/K}{\mathfrak{p}}\right) \in G^2$. See Example 3.1 for details. Note that G/G^2 is the Galois group of the composite of all quadratic extensions over K contained in L, and that $|G/G^2| = 1$ is equivalent to $|G| = [L : K]$ being odd.

Corollary 3.1. *Let L/K be a finite abelian extension of global fields. The following (i) and (ii) are equivalent:*

(i) DRH(A) holds for any nontrivial character of G.

(ii) For any subset H of G, there exists a constant c such that

$$\begin{aligned} &|H|\,\pi_{\frac{1}{2}}(x; G \setminus H) - |G \setminus H|\,\pi_{\frac{1}{2}}(x;H) \\ =&\frac{\log\log x}{2|G^2|}(|G \setminus H|\,|H \cap G^2| - |G^2 \setminus H|\,|H|) \\ &- \log\log x \sum_{\sigma \in H} m(\sigma) + c + o(1) \quad (x \to \infty). \end{aligned} \tag{3.1}$$

Proof. Assume (i), and it follows from Theorem 3.1 that

$$\begin{aligned} \pi_{\frac{1}{2}}(x;H) =& \sum_{\sigma \in H \cap G^2} \pi_{\frac{1}{2}}(x;\sigma) + \sum_{\sigma \in H \setminus G^2} \pi_{\frac{1}{2}}(x;\sigma) \\ =&\frac{1}{|G|}\Bigg(|H|\,\pi_{\frac{1}{2},K}(x) - \frac{\log\log x}{2}(|H \cap G^2|\,|G/G^2| - |H|) \\ &- \log\log x \sum_{\sigma \in H} m(\sigma)\Bigg) + c + o(1) \quad (x \to \infty), \end{aligned}$$

and that

$$
\begin{aligned}
&\pi_{\frac{1}{2}}(x; G \setminus H) \\
=&\sum_{\sigma \in G^2 \setminus H} \pi_{\frac{1}{2}}(x;\sigma) + \sum_{\sigma \in G \setminus (H \cup G^2)} \pi_{\frac{1}{2}}(x;\sigma) \\
=&\frac{1}{|G|}\left(|G \setminus H|\, \pi_{\frac{1}{2},K}(x) - \frac{\log\log x}{2}(|G^2 \setminus H|\,|G/G^2| - |G \setminus H|) \right. \\
&\left. - \log\log x \sum_{\sigma \in G \setminus H} m(\sigma) \right) + c + o(1) \quad (x \to \infty).
\end{aligned}
$$

The assertion (ii) follows from these two identities and $\sum_{\sigma \in G} m(\sigma) = 0$.

Conversely, assume (ii). We can show Theorem 3.1 (ii) by considering the subset $H = \{\sigma\}$ for (3.1) for any $\sigma \in G$. $\square$

Example 3.1. Let L/K be a finite abelian extension of global fields. Assume DRH(A) for any nontrivial character of G. Then there exists a constant c such that

$$
\begin{aligned}
&\pi_{\frac{1}{2}}(x; G \setminus G^2) - (|G/G^2| - 1)\pi_{\frac{1}{2}}(x; G^2) \\
&= \left(\frac{|G/G^2| - 1}{2} - \frac{1}{|G^2|}\sum_{\sigma \in G^2} m(\sigma) \right) \log\log x + c + o(1)
\end{aligned}
\tag{3.2}
$$

as $x \to \infty$. Let P_2 be the set of primes $\mathfrak{p}$ of K such that $\mathfrak{p} \nmid D_{L/K}$ and $\left(\frac{L/K}{\mathfrak{p}}\right) \in G^2$ and P_1 be the complement of P_2 in the set of primes $\mathfrak{p}$ of K satisfying $\mathfrak{p} \nmid D_{L/K}$. By Theorem 2.1, we put a positive constant δ as

$$
\delta = \frac{|G \setminus G^2|}{|G^2|} = \lim_{x \to \infty} \frac{\#\{\mathfrak{p} \in P_1 \mid N(\mathfrak{p}) \le x\}}{\#\{\mathfrak{p} \in P_2 \mid N(\mathfrak{p}) \le x\}}.
$$

Then the left hand side of (3.2) is equal to

$$
\sum_{\mathfrak{p} \in P_1,\ N(\mathfrak{p}) \le x} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{\mathfrak{p} \in P_2,\ N(\mathfrak{p}) \le x} \frac{1}{\sqrt{N(\mathfrak{p})}}.
$$

Thus we conclude that if $G \neq G^2$ and $m(\sigma) = 0$ for any σ, there exists a Chebyshev bias against primes $\mathfrak{p}$ such that $\left(\frac{L/K}{\mathfrak{p}}\right) \in G^2$.

Example 3.2. Let L/K be a quadratic extension of global fields with $G = \mathrm{Gal}(L/K) = \langle \tau \rangle$. The following (i) and (ii) are equivalent:

(i) DRH(A) for $L_K(s, \chi)$ holds for the nontrivial character χ of G.

(ii) There exists a Chebyshev bias toward the primes $\mathfrak{p}$ such that $\left(\frac{L/K}{\mathfrak{p}}\right) = \tau$ with the asymptotic

$$\pi_{\frac{1}{2}}(x;\tau) - \pi_{\frac{1}{2}}(x;1) = \left(\frac{1}{2} + m_\chi\right) \log\log x + c + o(1) \qquad (x \to \infty)$$

for some constant c.

In the following sections we will apply the above results to various finite quotients of the idele class group C_K of K. The class field theory asserts that there is a one-to-one correspondence between finite abelian extensions over K and open subgroups of finite index of C_K, and there is an isomorphism called the Artin map

$$\psi : C_K/N_{L/K}C_L \longrightarrow \mathrm{Gal}(L/K)$$

for any finite abelian extension L/K. Let $K_\mathfrak{p}$ be the completion of K for a prime ideal $\mathfrak{p}$ of K, and $\pi_\mathfrak{p}$ a prime element of $K_\mathfrak{p}$. Let $[\pi_\mathfrak{p}] \in C_K$ be the image of a natural map $K_\mathfrak{p}^\times \to C_K/N_{L/K}C_L$. For a prime ideal $\mathfrak{p}$ which does not divide the relative discriminant $D_{L/K}$ of L/K, we have $\psi([\pi_\mathfrak{p}]) = \left(\frac{L/K}{\mathfrak{p}}\right)$, where $\left(\frac{L/K}{\mathfrak{p}}\right)$ is the Frobenius element in $\mathrm{Gal}(L/K)$.

3.2 Bias against splitting primes

We denote by S_D the set of all prime ideals of K which split completely and let $S_N := S \setminus S_D$. For $x \in \mathbb{R}_{>0}$, we define

$$\pi_s(x; L/K)_D := \sum_{\substack{\mathfrak{p} \in S_D \\ N(\mathfrak{p}) \le x}} \frac{1}{N(\mathfrak{p})^s},$$

$$\pi_s(x; L/K)_N := \sum_{\substack{\mathfrak{p} \in S_N \\ N(\mathfrak{p}) \le x}} \frac{1}{N(\mathfrak{p})^s}.$$

Since a prime ideal $\mathfrak{p}$ of K with $\mathfrak{p} \nmid D_{L/K}$ decomposes completely in L if and only if $\left(\frac{L/K}{\mathfrak{p}}\right) = 1$, we have $\pi_s(x; L/K)_D = \pi_s(x;1)$ and hence from Theorem 3.1 we obtain

$$\begin{aligned} &\pi_{\frac{1}{2},K}(x) - [L:K]\pi_{\frac{1}{2}}(x; L/K)_D \\ &\qquad = \left(\frac{|G/G^2| - 1}{2} + m(1)\right) \log\log x + c + o(1) \quad (x \to \infty) \end{aligned}$$

for some constant c under the assumption of DRH(A) in the case of number fields. This asymptotic suggests that if $G \neq G^2$ and $m(1) = 0$, then there exists a Chebyshev bias against primes which split completely. Indeed we obtain the following theorem.

Theorem 3.2. *Let L/K be a finite abelian extension of global fields. We assume DRH(A) for any nontrivial characters of G in the case of number fields. We have the following asymptotic for some constant c:*

$$\begin{aligned}\pi_{\frac{1}{2}}(x; L/K)_N - ([L:K]-1)\pi_{\frac{1}{2}}(x; L/K)_D \\ = \left(\frac{|G/G^2|-1}{2} + m(1)\right)\log\log x + c + o(1) \\ (x \to \infty).\end{aligned}$$

Proof. The assertion follows from Corollary 3.1 and the following identities:

$$\pi_s(x; L/K)_D = \pi_s(x; 1),$$
$$\pi_s(x; L/K)_N = \pi_s(x; G \setminus \{1\}) + \sum_{\substack{\mathfrak{p}\in S,\, \mathfrak{p}|D_{L/K} \\ N(\mathfrak{p})\le x}} \frac{1}{N(\mathfrak{p})^s}.$$

□

Example 3.3. Let L/K be a quadratic extension of global fields. The following (i) and (ii) are equivalent:

(i) DRH(A) for $L_K(s,\chi)$ holds with χ the nontrivial character of $G = \mathrm{Gal}(L/K)$.

(ii) There exists a Chebyshev bias against splitting primes with the asymptotic:

$$\pi_{\frac{1}{2}}(x; L)_N - \pi_{\frac{1}{2}}(x; L)_D = \left(\frac{1}{2} + m_\chi\right)\log\log x + c + o(1) \quad (x \to \infty)$$

for some constant c.

3.3 Cyclotomic fields

Let $L = \mathbb{Q}(\zeta_q)$ $(q \geq 3)$ be a cyclotomic field. The Galois group $G := \mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ and $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ corresponds to $\sigma_a = \left(\frac{L/\mathbb{Q}}{(a)}\right) \in G$, where $\left(\frac{L/\mathbb{Q}}{\cdot}\right)$ is the Artin symbol. The group of characters of G is isomorphic to the group of Dirichlet characters modulo q. Let $t(q)$ be the number of distinct prime numbers dividing q, and

$$t := \begin{cases} t(q) - 1 & (2||q), \\ t(q) & (4||q \text{ or } 2 \nmid q), \\ t(q) + 1 & (8|q). \end{cases}$$

Then we have $|G/G^2| = 2^t$. For $x \in \mathbb{R}_{>0}$ and $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, let

$$\pi_s(x; q, a) := \sum_{\substack{p<x:\,\text{prime} \\ p\equiv a \pmod q}} \frac{1}{p^s}.$$

Under the assumption of DRH(A) for any nontrivial character of G, it follows from Theorem 3.1 that there exists a constant c such that

$$\begin{aligned} &\pi_{\frac{1}{2},\mathbb{Q}}(x) - \varphi(q)\pi_{\frac{1}{2}}(x; q, a) \\ &= \begin{cases} \left(\frac{2^t-1}{2} + m(\sigma_a)\right) \log\log x + c + o(1) & \\ \qquad\qquad\qquad\qquad (a \text{ is a quadratic residue modulo } q) & \\ \left(-\frac{1}{2} + m(\sigma_a)\right) \log\log x + c + o(1) \quad (\text{otherwise}) & \end{cases} \\ &\qquad\qquad\qquad\qquad\qquad\qquad (x \to \infty), \end{aligned} \tag{3.3}$$

where $\varphi(q)$ is Euler's totient function. It is expected [8] that $L(\frac{1}{2}, \chi) \neq 0$ for any Dirichlet character χ, that is $m(\sigma) = 0$ for any $\sigma \in G$. Under this assumption we have the following.

Corollary 3.2. *Assume DRH(A) for $L(s, \chi)$ and that $L(\frac{1}{2}, \chi) \neq 0$ for Dirichlet characters χ. If $m(\sigma_a) = 0$ for any $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, there exists a Chebyshev bias against primes which are quadratic residues modulo q.*

Moreover the size of the bias grows as the number of distinct prime divisors of q increases. Indeed under the assumption of $m(\sigma_a) = m(\sigma_b) = 0$, it follows from (3.3) that

$$\pi_{\frac{1}{2}}(x; q, b) - \pi_{\frac{1}{2}}(x; q, a) = \frac{2^{t-1}}{\varphi(q)} \log\log x + c + o(1) \quad (x \to \infty) \tag{3.4}$$

for some constant c with a and b being a quadratic residue and a non-residue modulo q, respectively. We also have

$$\pi_{\frac{1}{2}}(x; q, b) - \pi_{\frac{1}{2}}(x; q, a) = c + o(1) \quad (x \to \infty),$$

when both a and b are quadratic residues or non-residues modulo q under the assumption of $m(\sigma_a) = m(\sigma_b) = 0$.

Corollary 3.3. *Assume DRH(A) for $L(s,\chi)$ and that $L(\frac{1}{2},\chi) \neq 0$ for Dirichlet characters χ. If $m(\sigma_a) = 0$ for any $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, there exist no biases between any pair of quadratic residues, and no biases between any pair of quadratic non-residues.*

Example 3.4 (Chebyshev's original case $q = 4$)**.** The group $(\mathbb{Z}/4\mathbb{Z})^\times$ has two elements 1 and 3 (mod 4), which are quadratic residue and non-residue, respectively. From Proposition 2.1, DRH(A) for $L_{\mathbb{Q}}(s,\chi)$ with χ the nontrivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$ is equivalent to the asymptotic

$$\pi_{\frac{1}{2}}(x;\, 4,\, 3) - \pi_{\frac{1}{2}}(x;\, 4,\, 1) = \frac{1}{2}\log\log x + c + o(1) \qquad (x \to \infty).$$

This elucidates the original question of Chebyshev.

Example 3.5 ($q = 8$)**.** The group $(\mathbb{Z}/8\mathbb{Z})^\times$ has four elements 1, 3, 5 and 7 (mod 8), all of which except 1 are quadratic non-residues. Thus DRH(A) for $L_{\mathbb{Q}}(s,\chi)$ for all nontrivial characters χ of $(\mathbb{Z}/8\mathbb{Z})^\times$ is equivalent to the following asymptotics: for $j = 3,\, 5,\, 7$

$$\pi_{\frac{1}{2}}(x;\, 8,\, j) - \pi_{\frac{1}{2}}(x;\, 8,\, 1) = \frac{1}{2}\log\log x + c + o(1) \qquad (x \to \infty),$$

and for any pairs of $j,\, k \in \{3,\, 5,\, 7\}$

$$\pi_{\frac{1}{2}}(x;\, 8,\, j) - \pi_{\frac{1}{2}}(x;\, 8,\, k) = c + o(1) \qquad (x \to \infty).$$

3.4 Cyclotomic function fields

Let $K = \mathbb{F}_q(T)$ and $L = K(\Lambda_M)\,(M \in \mathbb{F}_q[T], M \neq 0)$ be a cyclotomic function field introduced by Carlitz [6]. The set Λ_M is given by $\Lambda_M := \{u \in \overline{K} \mid u^M = 0\}$ where $u^M := M(\varphi+\mu)(u)$, and φ and μ the endomorphisms of $\mathbb{F}_q$-module $\overline{K}$ defined by $\varphi(u) := u^q$ and $\mu(u) := Tu$. Hayes [15] developed Carlitz' ideas to an explicit class field theory for algebraic function fields. The Galois group $G = \mathrm{Gal}(L/K)$ is isomorphic to the multiplicative group $(\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times$. A residue class $A \in (\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times$ corresponds to $\sigma_A := \left(\frac{L/K}{(A)}\right) \in G$, where $\left(\frac{L/K}{\cdot}\right)$ is the Artin symbol. For a positive integer n and $A \in (\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times$, let

$$\pi_{s,K}(q^n) := \sum_{\deg P \leq n} \frac{1}{N(P)^s},$$

$$\pi_s(q^n; M, A) := \sum_{\substack{\deg P \leq n \\ P \equiv A \pmod{M}}} \frac{1}{N(P)^s},$$

where P runs through all monic irreducible polynomials in $\mathbb{F}_q[T]$. It follows from Theorem 3.1 that there exists a constant c such that

$$\pi_{\frac{1}{2},K}(q^n) - \Phi(M)\pi_{\frac{1}{2}}(q^n; M, A)$$
$$= \begin{cases} \left(\frac{2^t-1}{2} + m(\sigma_A)\right)\log n + c + o(1) \\ \qquad\qquad\qquad\qquad (A \text{ is a quadratic residue modulo } M) \\ \left(-\frac{1}{2} + m(\sigma_A)\right)\log n + c + o(1) \quad (n \to \infty) \quad (\text{otherwise}) \end{cases}$$
$$(n \to \infty),$$

where $\Phi(M) := |(\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times|$ and $t := \dim_{\mathbb{F}_2}(G/G^2)$. Note that if M is a product of different r irreducible polynomials, then t is given by $t = \begin{cases} 1 & (q \text{ is a power of } 2) \\ 2^r & (\text{otherwise}) \end{cases}.$

Corollary 3.4. *If $m(\sigma_A) = 0$ for any $A \in (\mathbb{F}_q[T]/M\mathbb{F}_q[T])^\times$, there exists a Chebyshev bias against irreducible polynomials which are quadratic residues modulo M.*

Cha [7, p. 1365] also obtains an analogous asymptotic formula as a bias against quadratic residues over funciton fields. Our result improves it in the sense that it clarifies the size of the discrepancy in terms of the group structure of G/G^2.

Example 3.6. Put $q = 2$ and $M = T^2$. The quotient ring $\mathbb{F}_2[T]/(T^2)$ has two invertible elements 1 and $T+1$. Since $m(\sigma_1) = m(\sigma_{T+1}) = 0$, there exists a Chebyshev bias toward irreducible polynomials congruent to the quadratic non-residue $T+1$. If we denote an irreducible polynomial by $h = \sum_{j=0}^{n} a_j T^j$ $(a_j \in \mathbb{F}_2)$, there exists a Chebyshev bias toward those with $a_1 = 1$.

3.5 Bias against principal ideals and divisors

Let K be a global field. In this section, we apply Theorem 3.1 for finite abelian extensions corresponding to the ideal class group and the divisor class group of degree 0 which are quotients of the idele class group of K. Let I_K be the group of fractional ideals of K, P_K its subgroup of principal ideals, $\mathrm{Cl}_K := I_K/P_K$ the ideal class group and $h_k := |\mathrm{Cl}_K|$ the class number. We denote by $\widetilde{K}$ the Hilbert class field of K, hence we have $\mathrm{Cl}_K \simeq \mathrm{Gal}(\widetilde{K}/K)$. An ideal class $[\mathfrak{a}] \in \mathrm{Cl}_K$ corresponds to $\sigma_{\mathfrak{a}} := \left(\frac{\widetilde{K}/K}{\mathfrak{a}}\right) \in \mathrm{Gal}(\widetilde{K}/K)$.

In what follows, we assume DRH(A) if K is a number field. From Theorem 3.1 we obtain

$$\sum_{N(\mathfrak{p})\le x} \frac{1}{\sqrt{N(\mathfrak{p})}} - h_K \sum_{\substack{\mathfrak{p}\in P_K\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}}$$
$$= \left(\frac{|Cl_K/\mathrm{Cl}_K^2|-1}{2} + m(1)\right)\log\log x + c + o(1) \quad (x\to\infty)$$

for some constant c, where $\mathfrak{p}$ runs through prime ideals of K. Furthermore, it follows that

$$\sum_{\substack{\mathfrak{p}\notin P_K\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - (h_K-1) \sum_{\substack{\mathfrak{p}\in P_K\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}}$$
$$= \left(\frac{|\mathrm{Cl}_K/\mathrm{Cl}_K^2|-1}{2} + m(1)\right)\log\log x + c + o(1) \qquad (x\to\infty)$$

for some constant c.

Hence we have the following.

Corollary 3.5. *Assume DRH(A) for $L_K(s,\sigma_{\mathfrak{a}})$ and that $m(\sigma_{\mathfrak{a}})=0$ for any $[\mathfrak{a}]\in\mathrm{Cl}_K$. If $|\mathrm{Cl}_K|$ is even, then in the whole set of prime ideals of K, there exists a Chebyshev bias against principal ideals.*

Example 3.7. Let $K=\mathbb{Q}(\sqrt{D})$ be a quadratic field with the discriminant D. Let P_K^+ be the group of totally positive principal ideals and $\mathrm{Cl}_K^+ := I_K/P_K^+$ the narrow ideal class group of K and $h_K^+ := |\mathrm{Cl}_K^+|$. We apply Theorem 3.1 for the finite abelian extension L/K corresponding to Cl_K^+, hence we have $\mathrm{Cl}_K^+ \simeq \mathrm{Gal}(L/K)$. From the genus theory we obtain

$$\sum_{N(\mathfrak{p})\le x} \frac{1}{\sqrt{N(\mathfrak{p})}} - h_K^+ \sum_{\substack{\mathfrak{p}\in P_K^+\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}}$$
$$= \left(\frac{2^{t(D)-1}-1}{2} + m(1)\right)\log\log x + c + o(1) \quad (x\to\infty)$$

for some constant c, where $t(D)$ is the number of distinct prime numbers dividing D.

Example 3.8. Let K be an algebraic function field of one variable over $\mathbb{F}_q$. Let D_K^0 be the group of divisors of degree 0, H_K its subgroup of principal divisors, $\mathfrak{Cl}_K := D_K^0/H_K$ the divisor class group of degree 0 and $\mathfrak{h}_K := |\mathfrak{Cl}_K|$. We apply Theorem 3.1 for the finite abelian extension L/K corresponding to $\mathfrak{Cl}_K$, hence we have $\mathfrak{Cl}_K \simeq \mathrm{Gal}(L/K)$. We obtain

$$\begin{aligned}&\sum_{\deg P \leq n} \frac{1}{\sqrt{N(P)}} - \mathfrak{h}_K \sum_{\substack{P \in H_K \\ \deg P \leq n}} \frac{1}{\sqrt{N(P)}} \\ &= \left(\frac{|\mathfrak{Cl}_K/\mathfrak{Cl}_K^2| - 1}{2} + m(1) \right) \log n + c + o(1) \quad (x \to \infty)\end{aligned}$$

for some constant c, where the sums are taken over monic irreducible polynomials in $\mathbb{F}_q[T]$.

4 Perspectives

The method of the proof of Theorem 2.2 can be applied to more general L-functions. Here we will introduce two examples as applications of our technique as well as future prospects on Selberg zeta functions.

4.1 Automorphic L-functions

Let $\tau(n)$ be the Ramanujan's function defined by

$$\Delta(z) = q \prod_{k=1}^{\infty} (1 - q^k)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Kurokawa and the second author obtain a bias on its signature as follows:

Theorem 4.1 (Koyama-Kurokawa [24])**.** *Assume DRH(A) for $L(s+\frac{11}{2}, \Delta)$. Then the sequence $a(p) = \tau(p)p^{-11/2}$ has a Chebyshev bias to being positive. More precisely, it holds that*

$$\sum_{\substack{p \leq x \\ prime}} \frac{\tau(p)}{p^6} = \frac{1}{2} \log \log x + c + o(1) \quad (x \to \infty)$$

with some constant c.

Here Ramanujan's L-function is defined as

$$L(s, \Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \quad \left(\mathrm{Re}(s) > \frac{13}{2}\right).$$

Theorem 4.1 suggests that the Satake parameters $\theta(p) \in [0, \pi]$ have a bias to being in $[0, \pi/2]$, where we define $\theta(p)$ as $\tau(p) = 2p^{\frac{11}{2}} \cos(\theta(p))$. In general, we may obtain biases of the Satake parameters for automorphic forms on $GL(n)$. For example, it is announced in [23] that under DRH(A) for the symmetric square L-function of $L(s, \Delta)$, which is an L-function for $GL(3)$, they found a bias to being *negative* for a sequence over squares of primes. Namely,

$$\sum_{p \le x} \frac{\tau(p^2)}{p^{\frac{23}{2}}} = -\frac{1}{2} \log \log x + c + o(1) \quad (x \to \infty)$$

with some constant c.

Sarnak [35] also reached a similar prediction under the assumption of the Generalized Riemann Hypothesis as well as the Grand Simplicity Hypothesis for $L(s, \Delta)$, which asserts linear independence over $\mathbb{Q}$ of the imaginary parts of all nontrivial zeros of $L(s, \Delta)$ in the upper half plane. He has pointed out that the sum

$$S(x) = \sum_{\substack{p \le x \\ \text{prime}}} \frac{\tau(p)}{p^{\frac{11}{2}}}$$

has a bias to being positive, in the sense that the mean of the measure μ defined by

$$\frac{1}{\log X} \int_2^X f\left(\frac{\log x}{\sqrt{x}} S(x)\right) \frac{dx}{x} \to \int_{\mathbb{R}} f(x) d_\mu(x) \quad (x \to \infty)$$

for $f \in C(\mathbb{R})$ is equal to 1. In the proof he closely examines the logarithmic derivative of $L(s, \Delta)$ to find that the second term in its expansion is the cause of the bias. While our above discussion deals with the logarithm instead of its derivative, we have also reached the point that the bias derives from the second term in the expansion. Although our conclusion has a common cause of the bias with what is given in [35], our proof is straightforward enough to simplify the proofs.

4.2 L-functions of elliptic curves

Let E be an elliptic curve over K. For any finite place v of K, we put

$$a_v = a_v(E) = q_v + 1 - \#E_v^{\mathrm{ns}}(k_v),$$

where q_v is the cardinal of the residue field k_v, and $\#E_v^{\mathrm{ns}}(k_v)$ is the number of k_v-rational points on the nonsingular locus of the reduction E_v of E at v. Defining $\theta_v \in [0, \pi]$ as $a_v = 2q_v^{\frac{1}{2}} \cos(\theta_v)$, it holds that the L-function

$$L(s, M_E) = \prod_{v\text{: good}} (1 - 2\cos(\theta_v) q_v^{-s} + q_v^{-2s})^{-1} \prod_{v\text{: bad}} (1 - a_v q_v^{-s})^{-1}$$

satisfies a functional equation for $s \leftrightarrow 1 - s$.

By fixing ℓ and K with $\ell \neq \mathrm{char}(K)$, we are equipped with a representation

$$\rho_E : \ \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\ell(E) \otimes \mathbb{Q}_\ell),$$

where $T_\ell(E) = \varprojlim_n E[\ell^n]$ is the ℓ-adic Tate module of E/K. It holds that $L(s, M_E) = L_K(s, \rho_E)$. If ρ_E is irreducible and DRH(A) is true for $L_K(s, \rho_E)$, we obtain a bias of the sequence $a_v/\sqrt{q_v}$. Ulmer [36] proves that when $\mathrm{char}(K) > 0$ and E is nonconstant, it follows that ρ_E is irreducible. By using this result, Kaneko and the second author obtain the following theorem.

Theorem 4.2 (Kaneko-Koyama [18])**.** *Assume* $\mathrm{char}(K) > 0$ *and that* E *is a non-constant elliptic curve over* K*. If* $\mathrm{rank}(K) > 0$*, then the sequence* $a_v/\sqrt{q_v}$ *has a Chebyshev bias to being negative. More precisely, it holds that*

$$\sum_{q_v \le x} \frac{a_v}{q_v} = C \log\log x + O(1) \quad (x \to \infty), \tag{4.1}$$

where $C = \frac{1}{2} - \mathrm{rank}(K)$*.*

The proof uses the DRH(A) proved in [19] and a part of the Birch-Swinnerton-Dyer Conjecture proved by Ulmer in [36] for $\mathrm{char}(K) > 0$.

Theorem 4.2 shows that the size of the discrepancy can be arbitrarily big if $\mathrm{rank}(K) \to \infty$. It coincides with the phenomenon which Fiorilli [12] observed under some assumptions including the Riemann Hypothesis for elliptic curve L-functions.

4.3 Selberg zeta functions

In positive characteristic cases, the Selberg zeta functions for congruence subgroups of $PGL(2, \mathbb{F}_q[T])$ attached to nontrivial representations satisfy both DRH(A) and (B) (Koyama-Suzuki [25]). From this we can consider biases of closed paths in the Ramanujan diagrams (i.e. infinite Ramanujan graphs) corresponding to the congruence subgroups.

When a Ramanujan diagram X is a cover of X_0 of finite degree, a Chebyshev bias would exist in the set of closed paths in X_0 toward those coming from X.

For Riemann surfaces with characteristic zero, analogous phenomena are expected to emerge. For this we need to prove DRH(A) for Selberg zeta functions. Kaneko-Koyama [17] proved the convergence of the Euler products of Selberg zeta functions for arithmetic subgroups of $PSL(2, \mathbb{R})$ in the whole zero-free region. So in the case of congruence subgroups of $PSL(2, \mathbb{Z})$, the Euler product converges in $\frac{1}{2} < \mathrm{Re}(s) < 1$ under the assumption of the Selberg $\frac{1}{4}$-conjecture. If the convergence region is extended to the boundary, then DRH(A) holds and we would be able to obtain biases of prime geodesics on arithmetic surfaces toward those coming from covering spaces.

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Appendix: Numerical Evidence of the Chebyshev Biases

Miho Aoki, Shin-ya Koyama and Takamasa Yoshida

Abstract

We give numerical evidences for the facts which were proved in the joint paper of the first and the second authors [A1] under the assumption of the Deep Riemann Hypothesis (DRH).

A Introduction

In this appendix we show numerical evidences for the Deep Riemann Hypothesis (DRH), by verifying the asymptotics in [A1] numerically. We fulfilled the calculations for $p \le 10^{10}$.

We used the software MATLAB R2021b (Mathworks, Natick, MA) and illustrated the results by plotting the points in the following manner for the purpose of suppressing the load to the computer system:

- Showed all values for $p \le 10^5$.
- Showed every ten values for $10^5 < p \le 10^{10}$.

B Chebyshev's original case

Let $q \ge 3$ be an integer. Denote by $\pi_{\frac{1}{2}}(x; q, a)$ the sum of the reciprocals of the square roots of primes $p \le x$ satisfying $p \equiv a \pmod q$. Under the assumption of the Deep Riemann Hypothesis (DRH) and non-vanishing of the Dirichlet L-function $L(s, \chi)$ at $s = 1/2$ for any nontrivial Dirichlet character χ modulo q, it follows from (3.4) in [A1] that

$$\pi_{\frac{1}{2}}(x; q, b) - \pi_{\frac{1}{2}}(x; q, a) \sim \frac{2^{t-1}}{\varphi(q)} \log\log x \quad (x \to \infty) \tag{B.1}$$

with a and b being a quadratic residue and a non-residue modulo q, respectively, where the integer t is defined by $|G/G^2| = 2^t$ with $G = \mathrm{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$.

We restore Chebyshev's original case by choosing $q = 4$. Denote by χ the nontrivial character of $G = \mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$. Then (B.1) tells that DRH for $L(s, \chi)$ implies

$$\pi_{\frac{1}{2}}(x; 4, 3) - \pi_{\frac{1}{2}}(x; 4, 1) \sim \frac{1}{2} \log\log x \quad (x \to \infty). \tag{B.2}$$

The graph of the left hand side of (B.2), which we put as $S(x;4,1)$, is drawn in blue in Figure B with the horizontal axis being with the logarithmic scale, while the right hand side of (B.2) is shown in brown.

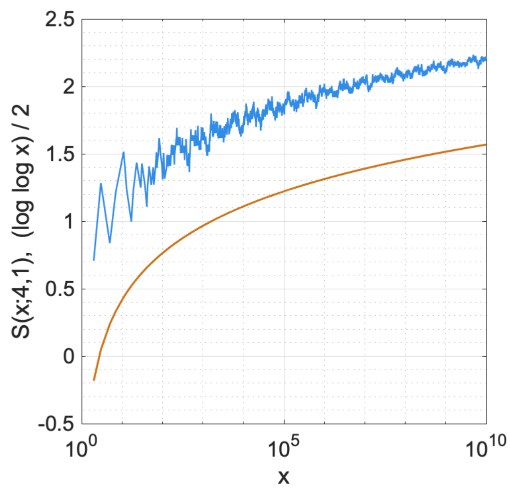

Figure B: Chebyshev's original case

C Other cyclotomic fields

For abelian extension of global fields L/K, we put

$$\pi_{s,K}(x) := \sum_{N(\mathfrak{p})\le x} \frac{1}{N(\mathfrak{p})^s}.$$

Assume DRH and $L(1/2,\chi) \neq 0$ for any nontrivial Dirichlet character modulo q. It follows from (3.3) that under such assumptions

$$\begin{aligned} &\pi_{\frac{1}{2},\mathbb{Q}}(x) - \varphi(q)\pi_{\frac{1}{2}}(x;q,a) \\ &\sim \begin{cases} \frac{2^t-1}{2}\log\log x & (a \text{ is a quadratic residue modulo } q) \\ -\frac{1}{2}\log\log x & (\text{otherwise}) \end{cases} \quad (x\to\infty). \end{aligned} \tag{C.1}$$

Let $q = 60$. We have $\varphi(60) = 16$ and $t = 3$. There exist two quadratic residues $a = 1, 49$ and fourteen non-residues $a = 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59$. Denoting the left hand side of (C.1) by $S(x;q,a)$, we show the graphs of $S(x;60,1)$ and $S(x;60,7)$ in Figure C in blue. The right hand side of (C.1) is shown in brown in each graph.

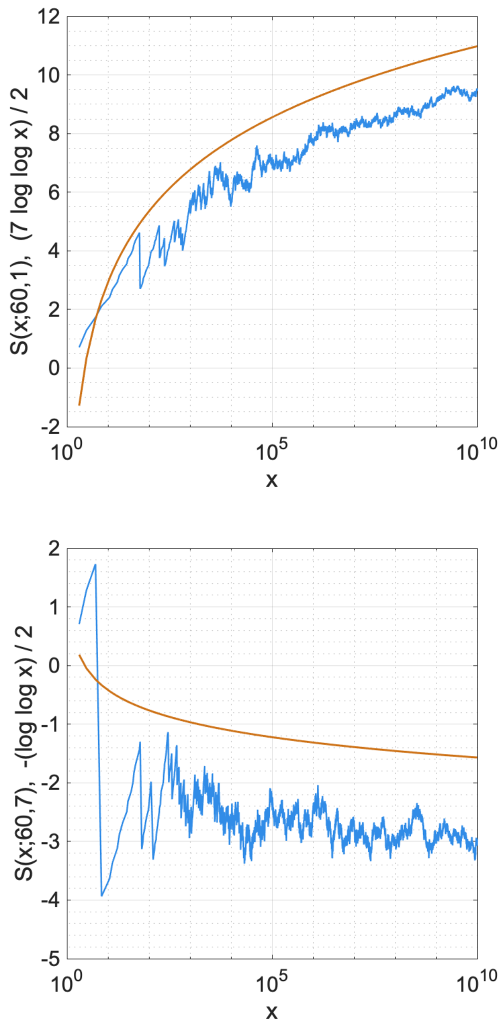

Figure C: Quadratic residue and non-residue modulo 60

D Unbiased cases

Each $\sigma \in G$:=Gal(L/K) is equipped with

$$S_\sigma := \left\{ \mathfrak{p} \,\middle|\, \mathfrak{p} \nmid D_{L/K}, \left(\frac{L/K}{\mathfrak{p}} \right) = \sigma \right\}$$

and

$$\pi_s(x; L/K, \sigma) := \sum_{\substack{\mathfrak{p} \in S_\sigma \\ N(\mathfrak{p}) \le x}} \frac{1}{N(\mathfrak{p})^s}.$$

We choose $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, which is an intermediate field in $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ with $[\mathbb{Q}(\zeta_7) : L] = 2$ and $[L : \mathbb{Q}] = 3$. The group G :=Gal$(L/\mathbb{Q})$ is of order 3, and we put $G := \{1, \sigma, \tau\}$ with

$$S_1 := \left\{ p \,\middle|\, p \neq 7,\ \left(\frac{L/\mathbb{Q}}{(p)} \right) = 1 \right\} = \{ p \mid p = \pm 1 \pmod 7 \},$$

$$S_\sigma := \left\{ p \,\middle|\, p \neq 7,\ \left(\frac{L/\mathbb{Q}}{(p)} \right) = \sigma \right\} = \{ p \mid p = \pm 2 \pmod 7 \},$$

$$S_\tau := \left\{ p \,\middle|\, p \neq 7,\ \left(\frac{L/\mathbb{Q}}{(p)} \right) = \tau \right\} = \{ p \mid p = \pm 3 \pmod 7 \}.$$

Putting $S(x; L/K, \sigma) := \pi_{\frac{1}{2},K}(x) - [L : K]\pi_{\frac{1}{2}}(x; L/K, \sigma)$, we will draw the graphs of the following three functions

$$S(x; L/\mathbb{Q}, 1) := \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3\ \pi_{\frac{1}{2}}(x; L/\mathbb{Q}, 1) = \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3 \sum_{\substack{p \le x:\text{prime} \\ p \equiv 1,6 \pmod 7}} \frac{1}{\sqrt{p}}$$

$$S(x; L/\mathbb{Q}, \sigma) := \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3\ \pi_{\frac{1}{2}}(x; L/\mathbb{Q}, \sigma) = \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3 \sum_{\substack{p \le x:\text{prime} \\ p \equiv 2,5 \pmod 7}} \frac{1}{\sqrt{p}}$$

$$S(x; L/\mathbb{Q}, \tau) := \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3\ \pi_{\frac{1}{2}}(x; L/\mathbb{Q}, \tau) = \pi_{\frac{1}{2},\mathbb{Q}}(x) - 3 \sum_{\substack{p \le x:\text{prime} \\ p \equiv 3,4 \pmod 7}} \frac{1}{\sqrt{p}}$$

under the assumptions of DRH and $L(1/2, \chi) \neq 0$ for any nontrivial Dirichlet character χ modulo 7. By Remark 3.1 in [A1], there would exist no biases when the order of the Galois group is odd. We actually have almost flat graphs in Figure D.

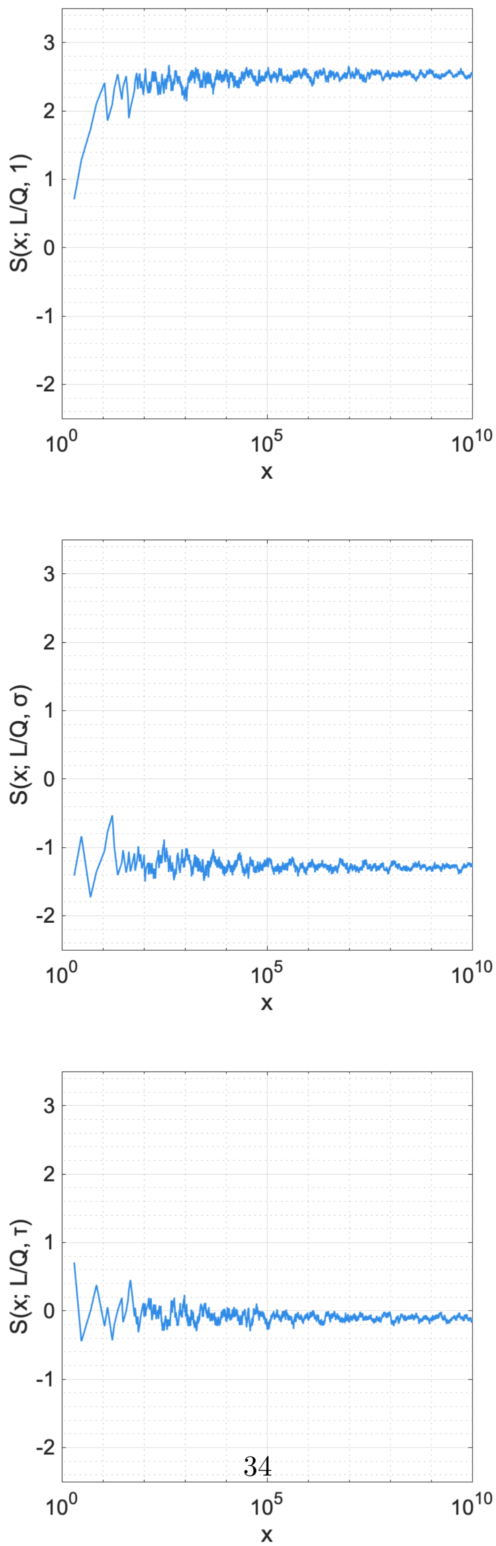

Figure D: Unbiased cases

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