The Deep Riemann Hypothesis and Chebyshev's Bias

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1 Introduction

In 1853 Chebyshev observed that there tend to be more primes of the form $4k+3$ than of the form $4k+1$ $(k \in \mathbb{Z})$. In fact, if denoting by $\pi(x;\, q,\, a)$ the number of primes $p \leq x$ such that $p \equiv a \pmod q$, then the following inequality holds for more than 97% of $x < 10^{11}$:

$$\pi(x;\, 4,\, 3) \geq \pi(x;\, 4,\, 1).$$

Littlewood [11], however, proved that the difference $\pi(x;\, 4,\, 3) - \pi(x;\, 4,\, 1)$ changes its sign infinitely many times. In 1962 Knapowski and Turan conjectured that the limit of the percentage in all positive numbers of the set

$$A_X = \{x < X \mid \pi(x;\, 4,\, 3) \geq \pi(x;\, 4,\, 1)\}$$

as $X \to \infty$ would equal 100%. However, now it is proved in [4] under the Generalized Riemann Hypothesis that the limit does not exist and that the conjecture is false.

In place of such a naive density, the logarithmic density takes its place. Define the *logarithmic density* of the set A_X in $[2, X]$ by

$$\delta(A_X) = \frac{1}{\log X} \int_{t \in A_X} \frac{dt}{t}.$$

Rubinstein and Sarnak [12] proved that the limit $\lim_{X\to\infty} \delta(A_X)$ exists and equals 0.9959... under the assumption of the Generalized Riemann Hypothesis and the Grand Simplicity Hypothesis for the Dirichlet L-functions $L(s, \chi)$, which asserts linear independence over $\mathbb{Q}$ of the imaginary parts of all nontrivial zeros of $L(s, \chi)$ in the upper half plane.

It is known by Dirichlet's prime number theorem in arithmetic progressions that the number of primes of the form $4k+3$ and $4k+1$ should equal asymptotically. Therefore, Chebyshev's bias means that the primes of the form $4k+3$ appears "earlier" than those of the form $4k+1$. By this observation, Aoki and Koyama [1] adopted the following weighted counting function generalizing $\pi(x;\, q,\, a) = \pi_0(x;\, q,\, a)$:

$$\pi_s(x;\, q,\, a) = \sum_{\substack{p<x:\, \text{prime} \\ p \equiv a \pmod q}} \frac{1}{p^s} \qquad (s \geq 0).$$

Here the smaller prime p allows higher contribution to $\pi_s(x;\, q,\, a)$, as long as we fix $s > 0$. Although the natural density of the set

$$A(s) = \{x > 0 \mid \pi_s(x;\, 4,\, 3) - \pi_s(x;\, 4,\, 1) > 0\}$$

does not exist when $s = 0$, they showed under the assumption of the DRH that it would exist and equal to 1 when $s = \frac{1}{2}$, that is,

$$\lim_{X\to\infty} \frac{1}{X} \int_{t\in A(\frac{1}{2})\cap[2,X]} dt = 1.$$

More precisely, they proved in [1, Corollary 3.2] that the existence of a constant C such that

$$\pi_{\frac{1}{2}}(x;\, 4,\, 3) - \pi_{\frac{1}{2}}(x;\, 4,\, 1) = \frac{1}{2}\log\log x + C + o(1) \quad (x \to \infty) \tag{1}$$

is equivalent to the convergence of the Euler product of the Euler's L-function $L(s, \chi_{-4})$ at the center $s = 1/2$ with $\chi_{-4}(p) = (-1)^{(p-1)/2}$ for odd primes p. This central covergence is a part of the conjecture named the Deep Riemann Hypothesis (DRH) proposed by Kurokawa [9] that is described in the next section.

Virtue of the formula (1) allows to reach a formulation of the Chebyshev bias of prime ideals $\mathfrak{p}$ of a global field K:

Definition 1.1 (Aoki-Koyama [1])**.** Let $a(\mathfrak{p}) \in \mathbb{R}$ be a sequence over prime ideals $\mathfrak{p}$ of K such that

$$\lim_{x\to\infty} \frac{\#\{\mathfrak{p} \mid a(\mathfrak{p}) > 0,\ N(\mathfrak{p}) \le x\}}{\#\{\mathfrak{p} \mid a(\mathfrak{p}) < 0,\ N(\mathfrak{p}) \le x\}} = 1.$$

The sequence $a(\mathfrak{p})$ has a *Chebyshev bias to being positive*, if there exists $C > 0$ such that

$$\sum_{N(\mathfrak{p})\le x} \frac{a(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}} \sim C \log\log x \quad (x \to \infty),$$

where $\mathfrak{p}$ runs through primes of K. Yet, $a(\mathfrak{p})$ is *unbiased*, if

$$\sum_{N(\mathfrak{p})\le x} \frac{a(\mathfrak{p})}{\sqrt{N(\mathfrak{p})}} = O(1) \quad (x \to \infty).$$

Definition 1.2 (Aoki-Koyama [1])**.** Assume that the set of all primes $\mathfrak{p}$ of K with $N(\mathfrak{p}) \le x$ is expressed as a disjoint union $P_1(x) \cup P_2(x)$ and that their proportion converges to

$$\delta = \lim_{x\to\infty} \frac{|P_1(x)|}{|P_2(x)|}.$$

There exists a *Chebyshev bias toward* P_1 (or *Chebyshev bias against* P_2), if the following asymptotic holds:

$$\sum_{p\in P_1(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{p\in P_2(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} \sim C \log\log x \quad (x \to \infty)$$

for some $C > 0$. On the other hand, there exist *no biases* between P_1 and P_2, if the following holds:

$$\sum_{p \in P_1(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} - \delta \sum_{p \in P_2(x)} \frac{1}{\sqrt{N(\mathfrak{p})}} = O(1) \quad (x \to \infty).$$

Aoki and the author [1] have found various examples of Chebyshev biases of prime ideals in Galois extensions of global fields such as the biases against splitting primes and principal prime ideals. Kurokawa and the author [8] have obtained an analog of this phenomenon for the Ramanujan τ-function $\tau(p)$ and proved that DRH for the automporphic L-function $L(s+\frac{11}{2}, \Delta)$ implies the bias of $\tau(p)/p^{11/2}$ toward positive values. Later Kaneko and the author [5] discovered the Chebyshev bias emerged from elliptic curves.

The main purpose of this article is to survey such recent results as well as to generalize the work of [8] to other weights $k = 12,\ 16,\ 20$ and $k = 18,\ 22,\ 26$. We observe that the bias would exist in all these cases, but its direction would become opposite if the L-function has a zero at $s = 1/2$ (Theorem 4.1).

2 Deep Riemann Hypothesis

Let K be a one-dimensional global field, which is either a number field or a function field in one variable over a finite field. For a place v of K, let $M(v)$ be a unitary matrix of degree $r_v \in \mathbb{N} = \{1, 2, 3, ...\}$. We consider an L-function expressed by the Euler product

$$L(s, M) = \prod_{v:\ \text{finite place}} \det(1 - M(v) q_v^{-s})^{-1}, \tag{2}$$

where q_v is the cardinal of the residue field k_v at v. The product (2) is absolutely convergent in $\Re(s) > 1$. In this paper we assume that $L(s, M)$ has an analytic continuation as an entire function on $\mathbb{C}$ and that a functional equation holds for $s \leftrightarrow 1 - s$. Moreover we put

$$\delta(M) = -\mathrm{ord}_{s=1} L(s, M^2)$$

with $\mathrm{ord}_{s=1}$ signifying the order of the zero at $s = 1$. Here we do not suppose that M is a representation. The square M^2 is interpreted as the Adams operation. Note that since

$$\begin{aligned} L(s, M^2) &= \prod_v \det(1 - M(v)^2 q_v^{-s})^{-1} \\ &= \frac{L(s, \mathrm{Sym}^2 M)}{L(s, \wedge^2 M)}, \end{aligned}$$

it holds that

$$\delta(M) = -\mathrm{ord}_{s=1} L(s, \mathrm{Sym}^2 M) + \mathrm{ord}_{s=1} L(s, \wedge^2 M), \tag{3}$$

where Sym^2 and $\wedge^2$ denote the symmetric and the exterior square of matrices.

When M is an Artin representation

$$\rho: \ \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathbb{C}}(V) \quad (\rho \neq \mathbf{1})$$

with a representation space V, then it holds that

$$\delta(M) = \mathrm{mult}(\mathbf{1}, \mathrm{Sym}^2\rho) - \mathrm{mult}(\mathbf{1}, \wedge^2\rho)$$

with $\mathrm{mult}(\mathbf{1}, \sigma)$ being the multiplicity of the trivial representation $\mathbf{1}$ in σ.

Conjecture 2.1 (Deep Riemann Hypothesis (DRH))**.** Assume that $L(s, M)$ has an analytic continuation as an entire function on $\mathbb{C}$ and that a functional equation holds with $s \leftrightarrow 1-s$. Let $m = \mathrm{ord}_{s=1/2} L(s, M)$. Then it holds that

$$\lim_{x\to\infty} \left((\log x)^m \prod_{q_v \leqslant x} \det(1 - M(v) q_v^{-1/2})^{-1} \right) = \frac{\sqrt{2}^{\delta(M)} L^{(m)}(\frac{1}{2}, M)}{e^{m\gamma} m!}. \tag{4}$$

Conjecture 2.2 (Convergence Conjecture (CC))**.** The limit in (4) exists.

Obviously DRH implies CC. Yet CC is meaningful since it is essentially equivalent to the Chebyshev biases. Several examples of works along this direction will be explained in the next section.

Conjecture 2.1 is known to be true for positive characteristic cases as in the following theorem. The proof is substantially given by Conrad [2, Theorems 8.1 and 8.2] in a different context under the assumption of the second moment hypothesis, and the full proof is recently given by Kaneko-Koyama-Kurokawa [6, Theorem 5.5].

Theorem 2.1 (Kaneko-Koyama-Kurokawa [6])**.** Conjecture 2.1 holds for $\mathrm{char}(K) > 0$.

3 Applications of DRH

This section illustrates applications of DRH to Chebyshev's bias discovered by Aoki-Koyama [1] and generalized by Koyama-Kurokawa [7, 8]. It is revealed by the DRH that the bias is a natural phenomenon for making a well-balanced disposition of the whole sequence of primes, in the sense of the convergence of the Euler product at the center. By means of a weighted counting function of primes, Aoki and the author succeed in expressing magnitudes of the deflection by a certain asymptotic formula under the assumption of DRH, which is a new formulation of Chebyshev's bias. Here, their main ideas are described.

Note from Conrad's theorem [2, Theorem 5.3] that Conjecture 2.2 implies the convergence in $\Re(s) > \frac{1}{2}$. Thus Conjecture 2.2 is stronger than the RH.

The proof of the equivalence between (1) and Conjecture 2.2 is simply illustrated as follows. It is known that $m = 0$ when $M = \rho = \chi_{-4}$. Conjecture 2.2 is equivalent to

$$\prod_{p \le x} \left(1 - \chi_{-4}(p) p^{-\frac{1}{2}}\right)^{-1} = L + o(1) \qquad (x \to \infty) \tag{5}$$

with $L \neq 0$. Then it equivalently has a bounded logarithm:

$$\sum_{p \leq x} \log \left(1 - \chi_{-4}(p) p^{-\frac{1}{2}}\right)^{-1} = \log L + o(1) \quad (x \to \infty).$$

When expanding the left hand side as

$$\sum_{k=1}^{\infty} \sum_{p \leq x} \frac{\chi_{-4}(p)^k}{k p^{\frac{k}{2}}},$$

the subseries over $k \geq 3$ is absolutely convergent as $x \to \infty$. On the other hand, the subseries over $k = 2$ satisfies by Euler's theorem that

$$\sum_{p \leq x} \frac{\chi_{-4}(p)^2}{2p} = \sum_{p \leq x} \frac{1}{2p} = \frac{1}{2} \log\log x + c + o(1) \quad (x \to \infty) \tag{6}$$

for some $c \in \mathbb{R}$. Then from (5) and (6) the behavior of the remaining part $k = 1$ is obtained:

$$\sum_{p \leq x} \frac{\chi_{-4}(p)}{\sqrt{p}} = -\frac{1}{2} \log\log x + \log L - c + o(1) \quad (x \to \infty).$$

This completes the proof of the equivalence.

Let L/K be a finite Galois extension of global fields. In [1] Aoki and Koyama examine various Chebyshev biases existing in the primes of K. Here we introduce some simplest examples. Let S be the set of all primes in K and $S_\sigma \subset S$ be the subset of unramified primes whose Frobenius element $\left(\frac{L/K}{\mathfrak{p}}\right)$ is equal to $\sigma \in \mathrm{Gal}(L/K)$.

Theorem 3.1 (a part of Theorem 2.2 [1]). Let L/K be a finite Galois extension of global fields. The following (i) and (ii) are equivalent:

(i) Conjecture 2.2 holds for $L(s, \rho)$ for all non trivial irreducible representations ρ of $\mathrm{Gal}(L/K)$.

(ii) For any $\sigma \in$Gal(L/K) it holds that

$$\sum_{\substack{\mathfrak{p} \in S \\ N(\mathfrak{p}) \leq x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - \frac{[L:K]}{|c_\sigma|} \sum_{\substack{\mathfrak{p} \in S_\sigma \\ N(\mathfrak{p}) \leq x}} \frac{1}{\sqrt{N(\mathfrak{p})}} = C \log\log x + c + o(1) \quad (x \to \infty)$$

for some constants C and c depending on σ.

Here C is expressed in terms of ν and m in Conjecture 2.1. Calculating such constants for specific cases under the assumption of (i), the following examples of Chebyshev biases are obtained.

Example 3.1 (Bias against splitting primes (Example 3.3 [1])). Assume $[L : K] = 2$ and let χ be the nontrivial character of $\mathrm{Gal}(L/K)$. The following (i) and (ii) are equivalent:

(i) Conjecture 2.2 for $L(s,\chi)$ holds.

(ii) There exists a Chebyshev bias against splitting primes with the asymptotic

$$\sum_{\substack{\mathfrak{p}:\ \text{nonsplit}\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - \sum_{\substack{\mathfrak{p}:\ \text{split}\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} = \left(\frac{1}{2}+m_\chi\right)\log\log x + c + o(1) \qquad (x\to\infty)$$

for some constant c.

Example 3.2 (Bias against quadratic residues (Corollary 3.2 [1])). Let q be a positive integer. Assume that $L(\frac{1}{2},\chi)\neq 0$ for all Dirichlet characters χ modulo q. The following (i) and (ii) are equivalent:

(i) Conjecture 2.2 holds for $L(s,\chi)$ for any Dirichlet character χ modulo q.

(ii) There exists a Chebyshev bias against quadratic residues modulo q with the asymptotic

$$\pi_{\frac{1}{2}}(x;q,b) - \pi_{\frac{1}{2}}(x;q,a) = \frac{2^{t-1}}{\varphi(q)}\log\log x + c + o(1) \quad (x\to\infty)$$

for some constant c and for any pair (a,b) of a quadratic residue a and a non-residue b, and there exist no biases for all other types of pairs (a,b).

Example 3.3 (Bias against principal ideals (Corollary 3.5 [1])). We denote by $\widetilde{K}$ the Hilbert class field of K. The ideal class group is expressed as $\mathrm{Cl}_K \simeq \mathrm{Gal}(\widetilde{K}/K)$. An ideal class $[\mathfrak{a}]\in \mathrm{Cl}_K$ corresponds to $\sigma_{\mathfrak{a}} := \left(\frac{\widetilde{K}/K}{\mathfrak{a}}\right)\in \mathrm{Gal}(\widetilde{K}/K)$. Assume Conjecture 2.2 for $L(s,\chi)$ and that $L(\frac{1}{2},\chi)\neq 0$ for any character χ of Cl_K. In the case that $|\mathrm{Cl}_K|$ is even, there exists a Chebyshev bias against principal ideals in the whole set of prime ideals of K. Namely, the following holds with $h_K = |\mathrm{Cl}_K|$:

$$\sum_{\substack{\mathfrak{p}:\ \text{nonprincipal}\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} - (h_K-1)\sum_{\substack{\mathfrak{p}:\ \text{principal}\\ N(\mathfrak{p})\le x}} \frac{1}{\sqrt{N(\mathfrak{p})}} = \frac{|\mathrm{Cl}_K/\mathrm{Cl}_K^2|-1}{2}\log\log x + c + o(1) \quad (x\to\infty).$$

Kurokawa and the author [8] obtained a bias of Ramanujan's τ-function by applying the ideas of Aoki-Koyama [1] to automorphic L-functions.

Theorem 3.2 (Koyama-Kurokawa [8]). Assume Conjecture 2.2 for $L(s+\frac{11}{2},\Delta)$. Then the sequence $a(p)=\tau(p)p^{-11/2}$ has a Chebyshev bias to being positive. More precisely, the following holds with constant c:

$$\sum_{\substack{p\le x\\ \text{prime}}} \frac{\tau(p)}{p^6} = \frac{1}{2}\log\log x + c + o(1) \quad (x\to\infty).$$

Here Ramanujan's L-function is defined as

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \qquad \left(\Re(s) > \frac{13}{2}\right).$$

Theorem 3.2 suggests that the Satake parameters $\theta(p) \in [0, \pi]$ have a bias to being in $[0, \pi/2]$, where we define $\theta(p)$ as $\tau(p) = 2p^{\frac{11}{2}} \cos(\theta(p))$.

Sarnak [13] also reached a similar prediction under the assumption of the Generalized Riemann Hypothesis in addition to the Grand Simplicity Hypothesis for $L(s, \Delta)$, which asserts linear independence over $\mathbb{Q}$ of the imaginary parts of all nontrivial zeros of $L(s, \Delta)$ in the upper half plane. He has pointed out that the sum

$$S(x) = \sum_{\substack{p \leq x \\ \text{prime}}} \frac{\tau(p)}{p^{\frac{11}{2}}}$$

has a bias to being positive, in the sense that the mean of the measure μ defined by

$$\frac{1}{\log X} \int_2^X f\left(\frac{\log x}{\sqrt{x}} S(x)\right) \frac{dx}{x} \to \int_{\mathbb{R}} f(x) d_\mu(x) \quad (x \to \infty)$$

for $f \in C(\mathbb{R})$ is equal to 1. In the proof, he closely examines the logarithmic derivative of $L(s, \Delta)$ to find that the second term in its expansion is the cause of the bias. While our above discussion deals with the logarithm instead of its derivative, we have also reached the point that the bias is derived from the second term in the expansion. Although the cause of the bias discovered by us in Aoki-Koyama [1] is the same as that in [13], our proof is more straightforward and simplified thanks to the DRH.

4 Main Theorem

In this section we extend Theorem 3.2 to general weights k such that $\dim_{\mathbb{C}} S_k(\Gamma) = 1$, that is $k = 12, 16, 18, 20, 22, 26$, where $\Gamma = PSL(2, \mathbb{Z})$ and $S_k(\Gamma)$ is the space of holomorphic cusp forms of weight k for Γ. Let

$$\Delta_k(z) = \sum_{n=1}^{\infty} \tau_k(n) e^{2\pi i n z} \in S_k(\Gamma)$$

be the holomorphic cusp form of weight k normalized as $\tau_k(1) = 1$. Define the L-function by

$$L(s, \Delta_k) := \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}.$$

Then putting $a_k(n) = \tau_k(n) n^{-\frac{k-1}{2}}$, we have

$$L\left(s + \frac{k-1}{2}, \Delta_k\right) = \prod_p (1 - a_k(p) p^{-s} + p^{-2s})^{-1}.$$

Theorem 4.1 (A generalization of Theorem 3.2)**.** Assume Conjecture 2.2 for $L(s + \frac{k-1}{2}, \Delta_k)$. Then the sequence $a_k(p) = \tau_k(p) p^{-\frac{k-1}{2}}$ has a Chebyshev bias to being positive for $k = 12, 16, 20$ and to being negative for $k = 18, 22, 26$.

More precisely, it holds that

$$\sum_{\substack{p \le x \\ \text{prime}}} \frac{\tau_k(p)}{p^{k/2}} = \begin{cases} \frac{1}{2} \log\log x + c + o(1) & (k = 12, 16, 20) \\ -\frac{1}{2} \log\log x + c + o(1) & (k = 18, 22, 26) \end{cases} \tag{7}$$

as $x \to \infty$ with some constants c.

Proof. By Aoki-Koyama [1, Proposition 2.1], we have under the assumption of Conjecture 2.2 that

$$\sum_{\substack{p \le x \\ \text{prime}}} \frac{\tau_k(p)}{p^{k/2}} = \left(\frac{1}{2} - m_k\right) \log\log x + c + o(1),$$

where $m_k = \mathrm{ord}_{s=k/2} L(s, \Delta_k)$. Hence the result (7) follows from the theorem of Kurokawa-Tanaka [10, Theorem 1.2]:

$$m_k = \begin{cases} 0 & (k = 12, 16, 20) \\ 1 & (k = 18, 22, 26). \end{cases}$$

The proof is complete. □

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