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A NEW ASPECT OF CHEBYSHEV'S BIAS FOR ELLIPTIC CURVES OVER FUNCTION FIELDS

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Abstract. This work addresses the prime number races for non-constant elliptic curves E over function fields. We prove that if $\mathrm{rank}(E) > 0$, then there exist Chebyshev biases towards being negative, and otherwise there exist Chebyshev biases towards being positive. The key input is the convergence of the partial Euler product at the centre, which follows from the Deep Riemann Hypothesis over function fields.

1. Introduction

In 1853, Chebyshev noticed in a letter to Fuss that primes congruent to 3 modulo 4 seem to dominate over those congruent to 1 modulo 4. If $\pi(x; q, a)$ is the number of primes $p \leq x$ such that $p \equiv a \pmod{q}$, then the inequality $\pi(x; 4, 3) \geq \pi(x; 4, 1)$ holds for more than 97 % of $x < 10^{11}$. By a classical theorem of Dirichlet, it is expected that the number of the primes of the form $4k+1$ and $4k+3$ should be asymptotically equal. Therefore, the Chebyshev bias indicates that the primes of the form $4k+3$ appear earlier than those of the form $4k+1$. Classical triumphs include the work of Littlewood [12] who established that $\pi(x; 4, 3) - \pi(x; 4, 1)$ changes its sign infinitely often. Knapowski–Turán [9] conjectured that the density of the numbers x for which $\pi(x; 4, 3) \geq \pi(x; 4, 1)$ holds is 1, but Kaczorowski [7] disproved their conjecture conditionally on the Generalised Riemann Hypothesis. Note that they have a logarithmic density via the work of Rubinstein–Sarnak [14], which is approximately $0.9959 \cdots$.

In this fundamental scenario, the next layer of methodological depth came with the introduction of a weighted counting function that allows one to scrutinise the above phenomenon. The work of Aoki–Koyama [2] introduces the counting function

$$\pi_s(x; q, a) := \sum_{\substack{p \leq x \\ p \equiv a \,(\mathrm{mod}\, q)}} \frac{1}{p^s}, \qquad s \geq 0, \tag{1.1}$$

extending $\pi(x; q, a) = \pi_0(x; q, a)$, where the smaller prime p permits a higher contribution to $\pi_s(x; q, a)$ as long as $s > 0$. The function $\pi_s(x; q, a)$ is more

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appropriate than $\pi(x;q,a)$ to represent the phenomenon, because it reflects the size of the primes that $\pi(x;q,a)$ ignores. While the natural density of the set $\{x>0 \mid \pi_s(x;4,3)-\pi_s(x;4,1)>0\}$ does not exist when $s=0$ under the Generalised Riemann Hypothesis, they showed under the Deep Riemann Hypothesis (DRH) that it would be equal to 1 when $s=1/2$. More precisely, the Chebyshev bias could be realised in terms of the asymptotic formula

$$\pi_{\frac{1}{2}}(x;4,3)-\pi_{\frac{1}{2}}(x;4,1)=\frac{1}{2}\log\log x+c+o(1) \tag{1.2}$$

as $x\to\infty$, where c is a constant.

We now formulate the Chebyshev biases for prime ideals $\mathfrak{p}$ of a global field K.

Definition 1.1 (Aoki–Koyama [2])**.** Let $c(\mathfrak{p})\in\mathbb{R}$ be a sequence over prime ideals $\mathfrak{p}$ of K such that

$$\lim_{x\to\infty}\frac{\#\{\mathfrak{p}\mid c(\mathfrak{p})>0,\mathrm{N}(\mathfrak{p})\le x\}}{\#\{\mathfrak{p}\mid c(\mathfrak{p})<0,\mathrm{N}(\mathfrak{p})\le x\}}=1.$$

We say that $c(\mathfrak{p})$ has a *Chebyshev bias towards being positive* if there exists a constant $C>0$ such that

$$\sum_{\mathrm{N}(\mathfrak{p})\le x}\frac{c(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}}\sim C\log\log x,$$

where $\mathfrak{p}$ runs through prime ideals of K. We say that $c(\mathfrak{p})$ is *unbiased* if

$$\sum_{\mathrm{N}(\mathfrak{p})\le x}\frac{c(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}}=O(1).$$

Definition 1.2 (Aoki–Koyama [2])**.** Assume that the set of all prime ideals $\mathfrak{p}$ of K with $\mathrm{N}(\mathfrak{p})\le x$ is expressed as a disjoint union $P_1(x)\cup P_2(x)$ and that their proportion converges to

$$\delta=\lim_{x\to\infty}\frac{|P_1(x)|}{|P_2(x)|}.$$

We say that there exists a Chebyshev bias towards P_1 or a Chebyshev bias against P_2 if there exists a constant $C>0$ such that

$$\sum_{\mathfrak{p}\in P_1(x)}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}-\delta\sum_{\mathfrak{p}\in P_2(x)}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}\sim C\log\log x.$$

We say that there exist no biases between P_1 and P_2 if

$$\sum_{\mathfrak{p}\in P_1(x)}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}-\delta\sum_{\mathfrak{p}\in P_2(x)}\frac{1}{\sqrt{\mathrm{N}(\mathfrak{p})}}=O(1).$$

Definitions 1.1 and 1.2 differ from those in [1,6,14] and formulate an asymptotic formula for the size of the discrepancy caused by the Chebyshev bias disregarded in the conventional definitions of the length of the interval in terms of the limiting distributions. Definitions 1.1 and 1.2 involve little information on the density distributions and both types of formulations appear not to have any logical connections. They shed some light on the Chebyshev bias from different directions.

Aoki–Koyama [2, Corollary 3.2] proved that the Chebyshev bias (1.2) against the quadratic residue $1\,(\mathrm{mod}\,4)$ is equivalent to the convergence of the Euler product at the centre $s=1/2$ for the Dirichlet L-function $L(s,\chi_{-4})$, where χ_{-4} denotes the non-trivial character modulo 4. This pertains to DRH proposed by Kurokawa [8,11] in 2012.

Aoki–Koyama [2] also observed various instances of Chebyshev biases that certain prime ideals in Galois extensions of global fields have over others, with an emphasis on the biases against splitting and principal prime ideals. Koyama–Kurokawa [10] obtained an analogue of this phenomenon for Ramanujan's τ-function $\tau(p)$ and showed that DRH for the automorphic L-function $L(s+11/2,\Delta)$ implies the bias of $\tau(p)/p^{11/2}$ towards being positive.

This work delves into such phenomena on the prime number races that elliptic curves over function fields give rise to. Let E be an elliptic curve over K, and let E_v be the reduction of E on the residue field k_v at a finite place v of K. If E has good reduction at v, then we define

$$a_v = a_v(E) \coloneqq q_v + 1 - \#E_v(k_v),$$

where $q_v = \#k_v$ and $\#E_v(k_v)$ is the number of k_v-rational points on E_v. The symbol a_v can be extended to all other finite places v:

$$a_v \coloneqq \begin{cases} 1 & \text{if } E \text{ has split multiplicative reduction at } v, \\ -1 & \text{if } E \text{ has non-split multiplicative reduction at } v, \\ 0 & \text{if } E \text{ has additive reduction at } v. \end{cases}$$

We prove the following asymptotic corresponding to the case of $s = 1/2$ in (1.1).

Theorem 1.3. *Assume* $\operatorname{char}(K) > 0$ *and that E is a non-constant elliptic curve over K in the terminology of Ulmer* [17, Definitions 1.1.4]*. If* $\operatorname{rank}(E) > 0$*, then the sequence $a_v/\sqrt{q_v}$ has a Chebyshev bias towards being negative. More precisely, we have that*

$$\sum_{q_v \le x} \frac{a_v}{q_v} = \left(\frac{1}{2} - \operatorname{rank}(E)\right) \log\log x + O(1). \tag{1.3}$$

The proof uses the convergence of the Euler product at the centre, which follows from DRH over function fields due to Conrad [5] and Kaneko–Koyama–Kurokawa [8]; see §2 for further details.

2. Deep Riemann Hypothesis

Let K be a one-dimensional global field that is either a number field or a function field in one variable over a finite field. For a place v of K, let $M(v)$ denote a unitary matrix of degree $r_v \in \mathbb{N}$. We consider an L-function expressed as an Euler product

$$L(s, M) = \prod_{v<\infty} \det(1 - M(v) q_v^{-s})^{-1}, \tag{2.1}$$

where q_v is the cardinal of the residue field k_v at v. The product (2.1) is absolutely convergent for $\operatorname{Re}(s) > 1$. In this paper, we assume that $L(s, M)$ has an analytic continuation as an entire function over $\mathbb{C}$ and a functional equation relating values at s and $1-s$. Write

$$\delta(M) = -\operatorname*{ord}_{s=1} L(s, M^2),$$

where $\operatorname{ord}_{s=1}$ is the order of the zero at $s = 1$. We here do not presuppose that M is a representation. The square M^2 is interpreted as an Adams operation. Note that since

$$L(s, M^2) = \prod_{v<\infty} \det(1 - M(v)^2 q_v^{-s})^{-1} = \frac{L(s, \mathrm{Sym}^2 M)}{L(s, \wedge^2 M)},$$

we derive

$$\delta(M) = -\operatorname*{ord}_{s=1} L(s, \mathrm{Sym}^2 M) + \operatorname*{ord}_{s=1} L(s, \wedge^2 M), \tag{2.2}$$

where Sym^2 and $\wedge^2$ denote the symmetric and the exterior squares, respectively. If M is an Artin representation

$$\rho\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}_{\mathbb{C}}(V), \qquad \rho \neq \mathbb{1}$$

for a representation space V, then

$$\delta(M) = \mathrm{mult}(\mathbb{1}, \mathrm{Sym}^2 \rho) - \mathrm{mult}(\mathbb{1}, \wedge^2 \rho),$$

where $\mathrm{mult}(\mathbb{1}, \sigma)$ is the multiplicity of the trivial representation $\mathbb{1}$ in σ.

We are now in a position to formulate DRH due to Kurokawa [8, 11].

Conjecture 2.1 (Deep Riemann Hypothesis)**.** *Keep the assumptions and notation as above. Let $m = \mathrm{ord}_{s=1/2} L(s, M)$. Then the limit*

$$\lim_{x\to\infty} \left((\log x)^m \prod_{q_v \le x} \det\left(1 - M(v) q_v^{-\frac{1}{2}}\right)^{-1} \right) \tag{2.3}$$

satisfies the following conditions:

DRH (A): *The limit* (2.3) *exists and is non-zero.*
DRH (B): *The limit* (2.3) *satisfies*

$$\lim_{x\to\infty} \left((\log x)^m \prod_{q_v \le x} \det\left(1 - M(v) q_v^{-\frac{1}{2}}\right)^{-1} \right) = \frac{\sqrt{2}^{\delta(M)}}{e^{m\gamma} m!} \cdot L^{(m)}\left(\frac{1}{2}, M\right).$$

DRH (B) implies DRH (A). Nonetheless, DRH (A) is still meaningful since it is essentially equivalent to the Chebyshev biases. The following examples clarify this situation; we direct the reader to the work of Aoki–Koyama [2] for more details.

Example 2.2 (Aoki–Koyama [2])**.** Let $K = \mathbb{Q}$, and let $v = p$. If $r_p = 1$ for any p and $M(p) = \chi_{-4}(p)$ is the non-trivial Dirichlet character modulo 4, then $L(s, M) = L(s, \chi_{-4})$ and $\delta(M) = 1$. DRH (A) for $L(s, \chi_{-4})$ is equivalent to the original form of the Chebyshev bias (1.2).

Example 2.3 (Koyama–Kurokawa [10])**.** Let $K = \mathbb{Q}$ and $r_p = 2$ for any p, and let $\tau(p) \in \mathbb{Z}$ be Ramanujan's τ-function defined for $q = e^{2\pi i z}$ with $\mathrm{Im}(z) > 0$ by

$$\Delta(z) := q \prod_{k=1}^{\infty} (1 - q^k)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

If $M(p) = \begin{pmatrix} e^{i\theta_p} & 0 \\ 0 & e^{-i\theta_p} \end{pmatrix}$ for the Satake parameters $\theta_p \in [0, \pi] \cong \mathrm{Conj}(\mathrm{SU}(2))$ defined by $\tau(p) = 2p^{11/2} \cos(\theta_p)$, then the associated L-function

$$L(s, M) = \prod_{p} (1 - 2\cos(\theta_p) p^{-s} + p^{-2s})^{-1}$$

satisfies a functional equation relating $s \leftrightarrow 1-s$. If we define Ramanujan's L-function by

$$L(s,\Delta) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

which satisfies a functional equation relating $s \leftrightarrow 12-s$, then $L(s,M) = L(s+11/2,\Delta)$ and $\delta(M) = -1$. DRH (A) for $L(s,M) = L(s+11/2,\Delta)$ implies that there exists a Chebyshev bias for the sequence $\tau(p)/p^{11/2}$ towards being positive.

Conjecture 2.1 is known to hold when the characteristic is positive. The proof was given by Conrad [5, Theorems 8.1 and 8.2] under the second moment hypothesis, and the full proof was given by Kaneko–Koyama–Kurokawa [8, Theorem 5.5]. We record their result as follows.

Theorem 2.4. *Conjecture* 2.1 *holds for* $\mathrm{char}(K) > 0$.

3. Proof of Theorem 1.3

Let E be an elliptic curve over a global field K, and let a_v be the same as in the introduction. We define the parameter $\theta_v \in [0,\pi] \cong \mathrm{Conj}(\mathrm{SU}(2))$ by $a_v = 2\sqrt{q_v}\cos(\theta_v)$. If we write

$$r_v = \begin{cases} 2 & \text{if } v \text{ is good,} \\ 1 & \text{if } v \text{ is bad,} \end{cases}$$

and

$$M(v) = M_E(v) = \begin{cases} \begin{pmatrix} e^{i\theta_v} & 0 \\ 0 & e^{-i\theta_v} \end{pmatrix} & \text{if } v \text{ is good,} \\ a_v & \text{if } v \text{ is bad,} \end{cases}$$

then the L-function (2.1) is equal to

$$L(s,M) = L(s,M_E) = \prod_{v\colon \text{good}} (1 - 2\cos(\theta_v) q_v^{-s} + q_v^{-2s})^{-1} \prod_{v\colon \text{bad}} (1 - a_v q_v^{-s})^{-1}.$$

This Euler product is absolutely convergent for $\mathrm{Re}(s) > 1$ and has a meromorphic continuation to $\mathbb{C}$ with a functional equation relating $s \leftrightarrow 1-s$.

The L-function $L(s,M_E)$ is expressed in terms of an Artin-type L-function in the following fashion. Fixing ℓ and K with $\ell \neq \mathrm{char}(K)$, we obtain the representation

$$\rho_E\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\ell(E) \otimes \mathbb{Q}_\ell), \tag{3.1}$$

where $T_\ell(E) = \varprojlim_n E[\ell^n]$ is the ℓ-adic Tate module of E/K. The Artin L-function associated to the Galois representation ρ_E is defined by the Euler product

$$L(s,\rho_E) = \prod_{v<\infty} \det(1 - q_v^{-s}\rho_E(\mathrm{Frob}_v | V^{I_v}))^{-1}.$$

The Euler factors of $L(s,M_E)$ are in accordance with those of $L(s+1/2,\rho_E)$ for all places v at which E has good reduction. In other words, the L-function $L(s,M_E)$ equals $L(s+1/2,\rho_E)$ up to finite factors from bad places.

Conjecture 2.1 originates from the Birch–Swinnerton-Dyer conjecture in the following form.

Conjecture 3.1 (Birch–Swinnerton-Dyer [3, Page 79 (A)])**.** *Let E be an elliptic curve E over K. Then there exists a constant $A > 0$ dependent on E such that*

$$\prod_{\substack{q_v \leq x \\ v:\ good}} \frac{\#E(k_v)}{q_v} \sim A(\log x)^r, \tag{3.2}$$

where $r = \mathrm{rank}(E)$. Furthermore, r is equal to the order of vanishing of $L(s, M_E)$ at $s = 1/2$.

Since the left-hand side of (3.2) matches the Euler product over good places of the L-function $L(s, M_E)$ at $s = 1/2$, Conjecture 3.1 implies DRH (A) for $L(s, M_E)$.

Theorem 3.2. *Keep the notation as above. The following conditions are equivalent.*

(i) DRH (A) *holds for $L(s, M)$.*
(ii) *There exists a constant c such that*

$$\sum_{q_v \leq x} \frac{\mathrm{tr}(M(v))}{\sqrt{q_v}} = -\left(\frac{\delta(M)}{2} + m\right) \log\log x + c + o(1),$$

where $m = \mathrm{ord}_{s=1/2} L(s, M)$.

Proof. Define

$$\begin{aligned}
\mathrm{I}(x) &\coloneqq \sum_{q_v \leq x} \frac{\mathrm{tr}(M(v))}{\sqrt{q_v}}, \\
\mathrm{II}(x) &\coloneqq \frac{1}{2} \sum_{q_v \leq x} \frac{\mathrm{tr}(M(v)^2)}{q_v}, \\
\mathrm{III}(x) &\coloneqq \sum_{k=3}^{\infty} \frac{1}{k} \sum_{q_v \leq x} \frac{\mathrm{tr}(M(v)^k)}{q_v^{k/2}}.
\end{aligned}$$

Because

$$\mathrm{I}(x) + \mathrm{II}(x) + \mathrm{III}(x) = \log\left(\prod_{q_v \leq x} \det\left(1 - M(v) q_v^{-\frac{1}{2}}\right)^{-1}\right),$$

the condition (i) is equivalent to the claim that there exists a constant L such that

$$m \log\log x + \mathrm{I}(x) + \mathrm{II}(x) + \mathrm{III}(x) = L + o(1). \tag{3.3}$$

The generalised Mertens theorem (see [13, Theorem 5] and [8, Lemma 5.3]) gives

$$\mathrm{II}(x) = \frac{\delta(M)}{2} \log\log x + C_1 + o(1) \tag{3.4}$$

for some constant C_1. It is straightforward to see that there exists a constant C_2 such that

$$\mathrm{III}(x) = C_2 + o(1). \tag{3.5}$$

Therefore, the estimates (3.3)–(3.5) lead to

$$\mathrm{I}(x) = -\left(\frac{\delta(M)}{2} + m\right) \log\log x + L - C_1 - C_2 + o(1).$$

If we assume (i), then the condition (ii) holds with $c = L - C_1 - C_2$. Conversely, if we assume (ii), then (3.3) holds with $L = c + C_1 + C_2$. This completes the proof of Theorem 3.2. $\square$

To study the asymptotic behaviour of a sum over v with $q_v \le x$ as $x \to \infty$, it suffices to restrict ourselves to places v at which E has good reduction. When v is good, the *n-th symmetric power matrix* $\mathrm{Sym}^n M$ of size $n+1$ is given by

$$(\mathrm{Sym}^n M)(v) = \mathrm{diag}(e^{in\theta_v}, e^{i(n-2)\theta_v}, \cdots, e^{-i(n-2)\theta_v}, e^{-in\theta_v}).$$

We calculate

$$\mathrm{tr}(\mathrm{Sym}^n M)(v) = \frac{\sin((n+1)\theta_v)}{\sin \theta_v}.$$

Extending the definition of $(\mathrm{Sym}^n M)(v)$ to all places v by setting $(\mathrm{Sym}^n M)(v) = a_v^n$ for bad places v, we can define the *n-th symmetric power L-function* $L(s, \mathrm{Sym}^n M)$. With the standard notation for the Galois representation $\rho = \rho_E$ in (3.1), we have the normalisation

$$L(s, \mathrm{Sym}^n M) = L\left(s + \frac{n}{2}, \mathrm{Sym}^n \rho\right). \tag{3.6}$$

If E is a non-constant elliptic curve in the terminology of Ulmer [17, Definitions 1.1.4], then $L(s, \mathrm{Sym}^n M)$ is a polynomial in $q^{-n/2-s}$ (see [4, 16]) and the absolute values of its roots are equal to $q^{-(n+1)/2}$ with the normalisation (3.6) in mind. Therefore, all the zeros of (3.6) lie on the critical line $\mathrm{Re}(s) = 1/2$ and there holds

$$\operatorname*{ord}_{s=1} L(s, \mathrm{Sym}^n M) = 0, \qquad n \in \mathbb{N}. \tag{3.7}$$

Lemma 3.3. *If* $\mathrm{char}(K) > 0$ *and E is a non-constant elliptic curve over K, then* $\delta(M) = -1$ *for* $M = M_E$.

Proof. Because M is a unitary matrix of size 2, the exterior square matrix $\wedge^2 M$ is trivial. Thus $\mathrm{ord}_{s=1} L(s, \wedge^2 M) = -1$. The claim follows from (2.2) and (3.7). $\square$

In what follows, we abbreviate $m_n = \mathrm{ord}_{s=1/2} L(s, \mathrm{Sym}^n M)$.

Theorem 3.4. *If* $\mathrm{char}(K) > 0$ *and E is a non-constant elliptic curve over K, then we have that*

$$\sum_{q_v \le x} \frac{a_v}{q_v} = \left(\frac{1}{2} - m_1\right) \log\log x + O(1). \tag{3.8}$$

In particular, if $\mathrm{rank}(E) > 0$*, then the sequence $a_v/\sqrt{q_v}$ has a Chebyshev bias towards being negative.*

Proof. In [16, §3.1.7] and [17, Theorem 9.3], Ulmer proved that $L(s, M_E)$ is a polynomial in q^{-s} for any non-constant elliptic curve E, and thus it is entire and satisfies the assumption of Conjecture 2.1. By Theorem 2.4, DRH holds for $L(s, M_E)$. Now (3.8) follows from Theorem 3.2 and Lemma 3.3. To justify the second assertion, we use [17, Theorem 12.1 (1)], which states that $\mathrm{rank}(E) \le m_1$. This yields $m_1 \ge 1$, and hence $C < 0$. This completes the proof of Theorem 3.4. $\square$

Theorem 3.4 is in harmony with the prediction of Sarnak [15, page 5] that $\mathrm{rank}(E) > 0$ implies the existence of a bias towards being negative, although he considered a_v instead of $a_v/\sqrt{q_v}$. He also stated that $\mathrm{rank}(E) = 0$ implies the existence of a bias towards being positive. We verify this phenomenon conditionally on the Birch–Swinnerton-Dyer conjecture.

Corollary 3.5. *Assume that* $L(1, M_E) \neq 0$. *If* $\mathrm{char}(K) > 0$ *and* E *is a non-constant elliptic curve over* K *such that* $\mathrm{rank}(E) = 0$, *then we have that*

$$\sum_{q_v \le x} \frac{a_v}{q_v} = \frac{1}{2} \log\log x + O(1). \tag{3.9}$$

In other words, the sequence $a_v/\sqrt{q_v}$ *has a Chebyshev bias towards being positive.*

Proof. The Birch–Swinnerton-Dyer conjecture asserts that $m_1 = \mathrm{rank}(E)$, implying $m_1 = 0$. Hence, the asymptotic formula (3.8) gives the desired result. □

Remark 1. Cha–Fiorilli–Jouve [4, Theorem 1.7] proved that there exist infinitely many elliptic curves $E/\mathbb{F}_q(T)$ such that the sequence a_v is *unbiased* in that

$$\lim_{X\to\infty} \frac{1}{X} \sum_{\substack{x \le X \\ T_E(x) > 0}} 1 = \frac{1}{2}$$

with

$$T_E(x) = -\frac{x}{q^{x/2}} \sum_{\substack{\deg(v) \le x \\ v\colon \text{good}}} 2\cos\theta_v = -\frac{x}{q^{x/2}} \sum_{\substack{\deg(v) \le x \\ v\colon \text{good}}} \frac{a_v}{q^{\deg(v)/2}}.$$

Their work discusses the Chebyshev bias for the sequence a_v, which is different from the sequence $a_v/\sqrt{q_v}$ in Theorem 3.4, and we strongly believe that these are constant elliptic curves whose L-functions have a pole at $s = 1$ and do not obey DRH.

We obtain other types of biases for the Satake parameters θ_v by considering the symmetric square L-function.

Theorem 3.6. *If* $\mathrm{char}(K) > 0$ *and* E *is a non-constant elliptic curve over* K, *then we have that*

$$\sum_{q_v \le x} \frac{(2\cos\theta_v - 1)(\cos\theta_v + 1)}{\sqrt{q_v}} = (1 - m_1 - m_2)\log\log x + O(1). \tag{3.10}$$

In particular, if both $L(1/2, M) \neq 0$ *and* $L(1/2, \mathrm{Sym}^2 M) \neq 0$ *are true, then the sequence* $(2\cos\theta_v - 1)(\cos\theta_v + 1)$ *has a Chebyshev bias towards being positive.*

Proof. Applying Theorem 3.2 to $L(s, M)$ and $L(s, \mathrm{Sym}^2 M)$ yields

$$\sum_{q_v \le x} \frac{2\cos\theta_v}{\sqrt{q_v}} = \left(\frac{1}{2} - m_1\right) \log\log x + O(1) \tag{3.11}$$

and

$$\sum_{q_v \le x} \frac{2\cos 2\theta_v}{\sqrt{q_v}} = \left(\frac{1}{2} - m_2\right) \log\log x + O(1). \tag{3.12}$$

It follows that (3.12) + (3.11) equals

$$\begin{aligned} \sum_{q_v \le x} \frac{2(\cos\theta_v + \cos 2\theta_v)}{\sqrt{q_v}} &= \sum_{q_v \le x} \frac{2(2\cos\theta_v - 1)(\cos\theta_v + 1)}{\sqrt{q_v}} \\ &= (1 - m_1 - m_2)\log\log x + O(1). \end{aligned}$$

This completes the proof of Theorem 3.6. □

We also adduce unbiased sequences constructed from the Satake parameters θ_v.

Theorem 3.7. *If* $\mathrm{char}(K) > 0$ *and* E *is a non-constant elliptic curve over* K*, then we have that*

$$\sum_{q_v \le x} \frac{2(2\cos\theta_v + 1)(\cos\theta_v - 1)}{\sqrt{q_v}} = (m_1 - m_2)\log\log x + O(1). \tag{3.13}$$

In particular, if $m_1 = m_2$*, then the sequence* $(2\cos\theta_v + 1)(\cos\theta_v - 1)$ *is unbiased.*

Proof. It follows that (3.12) − (3.11) equals

$$\begin{aligned}\sum_{q_v \le x} \frac{2(\cos 2\theta_v - \cos\theta_v)}{\sqrt{q_v}} &= \sum_{q_v \le x} \frac{2(2\cos\theta_v + 1)(\cos\theta_v - 1)}{\sqrt{q_v}} \\ &= (m_1 - m_2)\log\log x + O(1).\end{aligned}$$

This completes the proof of Theorem 3.7. □

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