Commun. Math. Phys. 215, 477 – 486 (2000)

Communications in
Mathematical Physics

Quantum Ergodicity of Eisenstein Series for Arithmetic 3-Manifolds

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Received: 20 April 2000 / Accepted: 10 July 2000

Abstract: We prove the quantum ergodicity for Eisenstein series for $PSL(2, O_K)$, where O_K is the integer ring of an imaginary quadratic field K of class number one.

1. Introduction

Luo and Sarnak [LS] proved the quantum ergodicity of Eisenstein series for $PSL(2, \mathbf{Z})$. It is stated as follows:

Theorem 1.1. *Let A, B be compact Jordan measurable subsets of $PSL(2, \mathbf{Z})\backslash H^2$, then*

$$\lim_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} = \frac{\text{Vol}(A)}{\text{Vol}(B)},$$

where $\mu_t = |E(z, \frac{1}{2} + it)|^2 dV$ with $E(z, s)$ being the Eisenstein series for $PSL(2, \mathbf{Z})$, and dV is the volume element of the upper half plane H^2.

In this paper we will generalize Theorem 1.1 to three dimensional cases $X = PSL(2, O_K)\backslash H^3$, where O_K is the integer ring of an imaginary quadratic field K of class number one, and H^3 is the three dimensional upper half space. Our main theorem is analogously described as follows:

Theorem 1.2. *Let A, B be compact Jordan measurable subsets of X, then*

$$\lim_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} = \frac{\text{Vol}(A)}{\text{Vol}(B)},$$

where $\mu_t = |E(v, 1 + it)|^2 dV$ with $E(v, s)$ being the Eisenstein series for X, and dV is the volume element of H^3.

⋆ Partially supported by Japan Association for Mathematical Sciences

Indeed we show that as $t \to \infty$,

$$\mu_t(A) \sim \frac{2\,\mathrm{Vol}(A)}{\zeta_K(2)} \log t,$$

where $\zeta_K(s)$ is the Dedekind zeta function.

In two dimensional cases numerical examples [HR] suggested that the quantum ergodicity would hold. For higher dimensional cases no numerical examples are known. Theorem 1.2 is the first result along this direction.

2. Three-Dimensional Settings

In this section we introduce some notation on the three-dimensional hyperbolic space.

A point in the hyperbolic three-dimensional space H^3 is denoted by $v = z + yj$, $z = x_1 + x_2 i \in \mathbf{C}$, $y > 0$. We fix an imaginary quadratic field K whose class number is one. Denote its discriminant by D_K and integer ring $O = O_K$. Put $D = |D_K|$. We often regard O as a lattice in $\mathbf{R}^2$, which is denoted by L with the fundamental domain $F_L \subset \mathbf{R}^2$. Also put $\omega = \omega_K = D^{-1/2}$, the inverse different of K. The group $\Gamma = PSL(2, O)$ acts on H^3 and the quotient space $X = \Gamma\backslash H^3$ is a three dimensional arithmetic hyperbolic orbifold. The Laplacian on X is defined by

$$\Delta = -y^2\left(\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} + \frac{d^2}{dy^2}\right) + y\frac{d}{dy}.$$

It has a self-adjoint extension on $L^2(X)$. It is known that the spectra of Δ is composed of both discrete and continuous ones. The eigenfunction for a discrete spectrum is called a cusp form. We denote it by $\phi_j(v)$ with eigenvalue λ_j $(0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots)$. We put $\lambda_j = 1 + r_j^2$. We shall assume the $\phi_j(v)$'s to be chosen so that they are eigenfunctions of the ring of Hecke operators and are L^2-normalized. The Fourier development of $\phi_j(v)$ is given in [S] (2.20):

$$\phi_j(v) = \sum_{n \in O^*/\sim} \rho_j(n) y K_{ir_j}(2\pi|n|y) e(\langle n, z\rangle), \tag{2.1}$$

where $n \sim m$ means that they generate the same ideal in O, and $\langle n, z\rangle$ is the standard inner product in $\mathbf{R}^2$ with K_ν being the K-Bessel function.

For a Maass-Hecke cusp form $\phi_j(v)$ with its Fourier development given by (2.1), we have the Rankin-Selberg convolution L-function $L(s, \phi_j \times \phi_j)$ and the second symmetric power L-function $L^{(2)}(s, \phi_j)$ which satisfy the following:

$$L(s, \phi_j \times \phi_j) = \zeta_K(2s) \sum_{n \in O^*/\sim} \frac{|\lambda_j(n)|^2}{N(n)^s},$$

$$L^{(2)}(s, \phi_j) = \sum_{n \in O^*/\sim} \frac{c_j(n)}{N(n)^s} = \zeta_K(s)^{-1} L(s, \phi_j \times \phi_j),$$

with $\rho_j(n) = \sqrt{\frac{\sinh \pi r_j}{r_j}}\, v_j(n)$, $v_j(n) = v_j(1)\lambda_j(n)$ and $c_j(n) = \sum_{l^2 k = n} \lambda_j(k^2)$. It is known that the both functions converge in $\mathrm{Re}(s) > 1$. The functional equation of

$L(s, \phi_j \times \phi_j)$ is inherited from the Eisenstein series by our unfolding the integral. We compute that

$$\int_X |\phi_j(v)|^2 E(v, 2s) dv = |\rho_j(1)|^2 \frac{L(s, \phi_j \times \phi_j)}{\zeta_K(2s)} \frac{\Gamma(s + ir_j)\Gamma(s - ir_j)\Gamma(s)^2}{8\pi^{2s}\Gamma(2s)}$$

is invariant under changing the variable s to $1 - s$. We normalize such that $\|\phi_j\| = 1$ with respect to the Petersson inner product

$$\langle f, g \rangle = \frac{1}{\mathrm{vol}}(X) \int_X f(v)\overline{g(v)} dv.$$

The residue R_j of $L(s, \phi_j \times \phi_j)$ at its unique simple pole $s = 1$ is equal to

$$\frac{8\pi\zeta_K(2)}{|v_j(1)|^2} \operatorname{Res}_{s=2} E(v, s) = \frac{8\pi\zeta_K(2)\operatorname{Vol}(F_L)}{|v_j(1)|^2 \operatorname{Vol}(X)}, \tag{2.2}$$

where $\operatorname{Res}_{s=2} E(v, s) = \operatorname{Vol}(F_L)/\operatorname{Vol}(X)$ is known by Sarnak [S], Lemma 2.15.

3. Proofs

In this section we prove Theorem 1.2. We first define the Eisenstein series by

$$E(v, s) = \sum_{\Gamma_\infty \backslash \Gamma} y(\gamma v)^s, \tag{3.1}$$

where $y(v) = y$ for $v = z + jy \in H^3$ and $\mathrm{Re}(s) > 2$. Here the group Γ_∞ is given by

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;:\; n \in O \right\}.$$

The Fourier development of $E(v, s)$ is known by Asai [A] and Elstrodt et al. [E]:

$$\begin{aligned} E(v, s) = y^s + y^{2-s} \frac{\xi_K(s-1)}{\xi_K(s)} \\ + \frac{2}{\xi_K(s)} \sum_{n \in O^*/\sim} |n|^{s-1} \sigma_{2(1-s)}(n) e^{4\pi i \operatorname{Re}(n\omega z)} K_{s-1}(4\pi |n\omega| y) y, \end{aligned} \tag{3.2}$$

where $\sigma_s(n) = \sum_{d|n} |d|^s$ and $\xi_K(s) = (\frac{\sqrt{D}}{2\pi})^s \Gamma(s) \zeta_K(s)$.

Our goal is to prove the equidistribution of the measure $\mu_t = |E(v, 1 + it)|^2 dV(v)$, where $dV(v) = \frac{dx_1 dx_2 dy}{y^3}$. We consider its inner product with various functions spanning $L^2(X)$. We begin with inner products with Maass cusp forms ϕ_j.

Proposition 3.1. *For any fixed* ϕ_j,

$$\lim_{t \to \infty} \int_X \phi_j d\mu_t = 0.$$

Proof. Set

$$J_j(t) = \int_X \phi_j d\mu_t = \int_X \phi_j(v) E(v, 1+it) E(v, 1-it) \frac{dx_1 dx_2 dy}{y^3} \tag{3.3}$$

with $z = x_1 + x_2 i$. To investigate this we first consider

$$I_j(s) = \int_X \phi_j(v) E(v, 1+it) E(v, s) \frac{dx_1 dx_2 dy}{y^3}. \tag{3.4}$$

All of the above integrals converge since ϕ_j is a cusp form. We unfold the integral (3.4) to get

$$I_j(s) = \int_0^\infty \int_{F_L} \phi_j(v) E(v, 1+it) y^s \frac{dx_1 dx_2 dy}{y^3}. \tag{3.5}$$

Denote the conjugate of $v = z + yj \in H^3$ by $\overline{v} = z - yj$. As is well-known in the two dimensional case, the space of the Maass cusp forms is expressed as a direct sum of spaces of even and odd cusp forms. Here even (resp. odd) cusp forms are ones satisfying $\phi_j(1 - \overline{v}) = \epsilon \phi_j(v)$ with $\epsilon = 1$ (resp. -1). Since $E(v, s) = E(1 - \overline{v}, s)$, it follows that $I_j(s) \equiv 0$ if ϕ_j odd. So we may assume that ϕ_j is even. In this case the Fourier development (2.1) is written as

$$\phi_j(v) = y \sum_{n \in O^*/\sim} \rho_j(n) K_{ir_j}(2\pi |n| y) \cos(2\pi i \langle n, z \rangle), \tag{3.6}$$

where $1 + r_j^2 = \lambda_j$. Normalizing the coefficients by $\rho_j(n) = \rho_j(1)\lambda_j(n)$, the multiplicative relations are satisfied by $\lambda_j(n)$. These amount to

$$L(\phi_j, s) := \sum_{n \in O^*/\sim} \frac{\lambda_j(n)}{N(n)^s} = \prod_{(p):\text{prime ideal}} \left(1 - \frac{\lambda_j(p)}{N(p)^s} + \frac{1}{N(p)^{2s}}\right)^{-1}. \tag{3.7}$$

By substituting (3.2) and (3.6) into (3.5) we have

$$\begin{aligned} I_j(s) = \int_0^\infty \int_{F_L} & \left(y \sum_{n \in O^*/\sim} \rho_j(n) K_{ir_j}(2\pi |n| y) \cos(2\pi \langle n, z \rangle) \right) \\ & \left(y^{1+it} + y^{1-it} \frac{\xi_K(it)}{\xi_K(1+it)} \right. \\ + \frac{2y}{\xi_K(1+it)} & \left. \sum_{m \in O^*/\sim} |m|^{it} \sigma_{-2it}(m) e^{4\pi i \,\mathrm{Re}(m\omega z)} K_{it}(4\pi |m\omega y) \right) y^s \frac{dx_1 dx_2 dy}{y^3}. \end{aligned} \tag{3.8}$$

Now we have

$$\int_{F_L} \cos(2\pi i \langle n\omega, z \rangle) dv = \begin{cases} 0 & n \in O - \{0\} \\ 1 & n = 0 \end{cases}.$$

In the expansion of (3.8), we appeal to the formula $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$. Only the terms with $n = m$ remain as follows:

$$I_j(s) = \frac{2}{\xi_K(1+it)} \int_0^\infty \sum_{n \in O^*/\sim} |n|^{it} \sigma_{-2it}(n) K_{it}(2\pi|n|y) \rho_j(n) K_{ir_j}(2\pi|n|y) y^s \frac{dy}{y}$$
$$= \frac{2}{\xi_K(1+it)} \sum_{n \in O^*/\sim} \frac{|n|^{it}\sigma_{-2it}(n)\rho_j(n)}{|n|^s} \int_0^\infty K_{it}(2\pi y) K_{ir_j}(2\pi y) y^s \frac{dy}{y}.$$

An evaluation of the integral involving Bessel functions [GR] yields

$$I_j(s) = \frac{2\pi^{-s}}{\xi_K(1+it)} \frac{\Gamma(\frac{s+ir_j+it}{2})\Gamma(\frac{s+ir_j-it}{2})\Gamma(\frac{s-ir_j+it}{2})\Gamma(\frac{s-ir_j-it}{2})}{\Gamma(s)} R(s)$$

with

$$R(s) = \sum_{n \in O^*/\sim} \frac{|n|^{it}\sigma_{-2it}(n)\rho_j(n)}{|n|^s}.$$

We compute $R(s)$ as follows:

$$R(s) = \frac{1}{\rho_j(1)} \prod_{(p):\text{prime ideal}} \sum_{k=0}^\infty \frac{\lambda_j(p^k)|p|^{ikt}\sigma_{-2it}(p^k)}{|p|^{ks}}$$
$$= \frac{1}{\rho_j(1)} \prod_{(p)} \sum_{k=0}^\infty \frac{\lambda_j(p^k)|p|^{ikt}}{|p|^{ks}} \sum_{l=0}^k |p|^{-2itl}$$
$$= \frac{1}{\rho_j(1)} \prod_{(p)} \sum_{k=0}^\infty \frac{\lambda_j(p^k)|p|^{ikt}}{|p|^{ks}} \frac{1-|p|^{-2it(k+1)}}{1-|p|^{-2it}}$$
$$= \frac{1}{\rho_j(1)(1-|p|^{-2it})} \prod_{(p)} \left(\sum_{k=0}^\infty \lambda_j(p^k)|p|^{-k(s-it)} - |p|^{-2it} \sum_{k=0}^\infty \lambda_j(p^k)|p|^{-k(s+it)} \right)$$
$$= \frac{1}{\rho_j(1)(1-|p|^{-2it})}$$
$$\prod_{(p)} \left(\frac{1}{1-\lambda_j(p)|p|^{-(s-it)}+|p|^{-2(s-it)}} - \frac{|p|^{-2it}}{1-\lambda_j(p)|p|^{-(s+it)}+|p|^{-2(s+it)}} \right)$$
$$= \frac{1}{\rho_j(1)}$$
$$\prod_{(p)} \frac{1-|p|^{-2s}}{(1-\lambda_j(p)|p|^{-(s-it)}+|p|^{-2(s-it)})(1-\lambda_j(p)|p|^{-(s+it)}+|p|^{-2(s+it)})}$$
$$= \frac{1}{\rho_j(1)} \frac{L(\phi_j, \frac{s-it}{2})L(\phi_j, \frac{s+it}{2})}{\zeta_K(s)}. \tag{3.9}$$

Therefore

$$J_j(t) = I_j(1-it)$$
$$= \frac{2\pi^{-1+it}}{\xi_K(1+it)} \frac{\Gamma(\frac{1+ir_j}{2})\Gamma(\frac{1+ir_j-2it}{2})\Gamma(\frac{1-ir_j}{2})\Gamma(\frac{1-ir_j-2it}{2})}{\Gamma(1-it)} R(1-it). \tag{3.10}$$

By Stirling's formula $|\Gamma(\sigma+it)| \sim e^{-\pi t/2}|t|^{\sigma-\frac{1}{2}}$, we see

$$\text{the gamma factors in (3.10)} \ll |t|^{-1} \tag{3.11}$$

as $t \to \infty$. It is known that the Dedekind zeta function in (3.10) is estimated as

$$t^{-\epsilon} \ll |\zeta_K(1+it)| \ll t^{\epsilon}. \tag{3.12}$$

Estimating the automorphic L-functions in (3.10) was recently done successfully by Sarnak and Petridis [SP]. They proved there exists $\delta > 0$ such that for any $\epsilon > 0$,

$$L(\phi_j, \frac{1}{2}+it) \ll_{j,\epsilon} |t|^{1-\delta+\epsilon} \tag{3.13}$$

as $|t| \to \infty$. The estimates (3.11)–(3.13) yield

$$J_j(t) \ll |t|^{-\delta+\epsilon}. \tag{3.14}$$

This implies Proposition 3.1. □

We now turn to inner products of μ_t with incomplete Eisenstein series. Let $h(y)$ be a rapidly decreasing function at 0 and ∞, that is $h(y) = O_N(y^N)$ as $y \to \infty$ or 0 and $N \in \mathbf{Z}$. Let $H(s)$ be its Mellin transform

$$H(s) = \int_0^\infty h(y) y^{-s} \frac{dy}{y}.$$

Clearly $H(s)$ is entire in s and is of Schwartz class in t for each vertical line $\sigma + it$. The inversion formula gives

$$h(y) = \frac{1}{2\pi i} \int_{(\sigma)} H(s) y^s ds$$

for any $\sigma \in \mathbf{R}$. For such an h we form the convergent series

$$F_h(v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(y(\gamma v)) = \frac{1}{2\pi i} \int_{(3)} H(s) E(v,s) ds,$$

which we call incomplete Eisenstein series.

Proposition 3.2. *For incomplete Eisenstein series $F(v)$, we have*

$$\int_X F(v) d\mu_t(v) \sim \frac{2}{\zeta_K(2)} \left(\int_X F(v) dV(v) \right) \log t$$

as $t \to \infty$.

Proof. Incomplete Eisenstein series decrease rapidly as $y \to \infty$ and belong to $C^\infty(X)$. Hence

$$\begin{aligned}
\int_X F_h(v) d\mu_t(v) &= \int_X F_h(v)|E(v, 1+it)|^2 \frac{dzdy}{y^3} \\
&= \frac{1}{2\pi i} \int_X \int_{(3)} H(s)E(v,s)ds|E(v,1+it)|^2 \frac{dzdy}{y^3} \\
&= \frac{1}{2\pi i} \int_0^\infty \int_{(3)} H(s) y^s ds \int_{F_L} |E(v,1+it)|^2 \frac{dzdy}{y^3} \\
&= \frac{1}{2\pi i} \int_0^\infty \int_{(3)} H(s) y^s ds \left(\left| y^{1+it} + y^{1-it} \frac{\xi_K(it)}{\xi_K(1+it)} \right|^2 \right. \\
&\quad \left. + \left| \frac{2y}{\xi_K(1+it)} \right|^2 \sum_{n \in O^*/\sim} |\sigma_{-2it}(n) K_{it}(4\pi |n| \omega y)|^2 \right) \frac{dy}{y^3} \\
&= F_1(t) + F_2(t),
\end{aligned}$$

where we put

$$F_1(t) = \frac{1}{2\pi i} \int_0^\infty \int_{(3)} H(s) y^s ds \left| y^{1+it} + y^{1-it} \frac{\xi_K(it)}{\xi_K(1+it)} \right|^2 \frac{dy}{y^3}.$$

Since $\left| \frac{\xi_K(it)}{\xi_K(1+it)} \right| = 1$, we have

$$F_1(t) = 2 \int_0^\infty h(y) \frac{dy}{y} + (\text{a rapidly decreasing function of } t), \tag{3.15}$$

whereas

$$\begin{aligned}
F_2(t) = \ & \frac{2}{\pi i |\xi_K(1+it)|^2} \int_{(3)} H(s) \sum_{n \in O^*/\sim} \frac{|\sigma_{-2it}(n)|^2}{|n|^s} \\
& \int_0^\infty |K_{it}(4\pi\omega y)|^2 y^s \frac{dy}{y} ds.
\end{aligned} \tag{3.16}$$

The series is computed as follows:

$$\begin{aligned}
\sum_{n \in O^*/\sim} \frac{|\sigma_a(n)|^2}{|n|^s} &= \prod_{(p)\text{: prime ideal}} \sum_{k=0}^\infty \frac{\sigma_a(p^k)\sigma_{-a}(p^k)}{|p|^{ks}} \\
&= \prod_{(p)} \sum_{k=0}^\infty \frac{1}{|p|^{ks}} \left(\frac{1-|p|^{a(k+1)}}{1-|p|^a} \right) \left(\frac{1-|p|^{-a(k+1)}}{1-|p|^{-a}} \right)^2 \\
&= \prod_{(p)} \frac{1}{(1-|p|^a)(1-|p|^{-a})} \\
&\qquad \sum_{k=0}^\infty \left(2|p|^{-ks} - |p|^{(a-s)k+a} + |p|^{(-a-s)k-a} \right)
\end{aligned}$$

$$= \prod_{(p)} \frac{1}{(1-|p|^a)(1-|p|^{-a})} \tag{3.17}$$

$$\left(\frac{2}{1-|p|^{-s}} - \frac{|p|^a}{1-|p|^{a-s}} - \frac{|p|^{-a}}{1-|p|^{-a-s}}\right)$$

$$= \prod_{(p)} \frac{1+p^{-s}}{(1-p^{-s})(1-p^{-(s-a)})(1-p^{-(s+a)})}$$

$$= \frac{\zeta_K(\frac{s}{2})^2\zeta_K(\frac{s-a}{2})\zeta_K(\frac{s+a}{2})}{\zeta_K(s)}. \tag{3.18}$$

The y-integral in (3.16) is evaluated in terms of the Γ function as before. We obtain

$$\begin{aligned} F_2(t) &= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} H(s) \sum_{n\in O^*/\sim} \frac{|\sigma_{-2it}(n)|^2}{|n|^s}\int_0^\infty |K_{it}(4\pi\omega y)|^2 y^s \frac{dy}{y} ds \\ &= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} \frac{H(s)\zeta_K(\frac{s}{2})^2|\zeta_K(\frac{s}{2}+it)\Gamma(\frac{s}{2}+it)|^2\Gamma(\frac{s}{2})^2}{(4\pi\omega)^s\zeta_K(s)\Gamma(s)} ds \\ &= \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(3)} B(s)ds, \end{aligned} \tag{3.19}$$

where we put

$$B(s) = \frac{H(s)\zeta_K(\frac{s}{2})^2|\zeta_K(\frac{s}{2}+it)\Gamma(\frac{s}{2}+it)|^2\Gamma(\frac{s}{2})^2}{(4\pi\omega)^s\zeta_K(s)\Gamma(s)}. \tag{3.20}$$

By Stirling's formula to estimate the gamma factors and from the fact that $H(\sigma+it)$ is rapidly decreasing in t, we can shift the integral in (3.18) to $\mathrm{Re}(s)=1$:

$$F_2(t) = \frac{4\,\mathrm{Res}_{s=2} B(s)}{|\xi_K(1+it)|^2} + \frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(1)} B(s)ds. \tag{3.21}$$

The second term in (3.20) is evaluated by Heath-Brown [H] as

$$\zeta_K\left(\frac{1}{2}+it\right) \ll t^{\frac{1}{3}+\epsilon}$$

for any fixed $\epsilon > 0$. We find that

$$\frac{2}{\pi i|\xi_K(1+it)|^2}\int_{(1)} B(s)ds \ll_\epsilon t^{-\frac{1}{3}+\epsilon}.$$

This corresponds to the bound (3.14).

Next we deal with the residue term in (3.20), which is more complicated. Write $B(s)$ as $\zeta_K(\frac{s}{2})^2G(s)$ where $G(s)$ is holomorphic at $s=2$. Put

$$\zeta_K(s/2) = \frac{A_{-1}}{s-2} + A_0 + O(s-2) \quad (s\to 2).$$

In the expansion of

$$B(s) = \left(\frac{A_{-1}}{s-2} + A_0 + O(s-2)\right)^2 \left(G(2) + G'(2)(s-2) + O(s-2)^3\right),$$

the coefficient of $(s-2)^{-1}$ gives the residue

$$\operatorname{Res}_{s=2} B(s) = G(2)A_{-1}\left(2A_0 + A_{-1}\frac{G'}{G}(2)\right).$$

A simple calculation gives

$$G(2) = \frac{H(2)|\zeta_K(1+it)\Gamma(1+it)|^2\Gamma(\frac{1}{2})^2}{(4\pi\omega)^2\zeta_K(2)} = \frac{H(2)|\xi_K(1+it)|^2}{4\zeta_K(2)}$$

and

$$\frac{G'}{G}(2) = \frac{H'}{H}(2) + \frac{\zeta_K'(1+it)}{2\zeta_K(1+it)} + \frac{\zeta_K'(1-it)}{2\zeta_K(1-it)} + \frac{\Gamma'(1+it)}{2\Gamma(1+it)} + \frac{\Gamma'(1-it)}{2\Gamma(1-it)} + C$$

with C being independent of t. For the Weyl–Hadamard–De La Vallée Poussin bound [T, (6.15.3)] and its generalization to Dirichlet L-functions by Landau, we have

$$\frac{\zeta_K'(1+it)}{\zeta_K(1+it)} \ll \frac{\log t}{\log\log t}.$$

This together with $\frac{\Gamma'}{\Gamma}(1+it) \sim \log t$ gives

$$\operatorname{Res}_{s=2} B(s) = \frac{H(2)|\xi_K(1+it)|^2}{2\zeta_K(2)}\log t + O\left(\frac{\log t}{\log\log t}\right).$$

Finally the first term of (3.20) is evaluated as

$$\frac{4\operatorname{Res}_{s=2} B(s)}{|\xi_K(1+it)|^2} = \frac{2H(2)}{\zeta_K(2)}\log t + O(1).$$

Taking into account that

$$H(2) = \int_0^\infty h(y)\frac{dy}{y^3} = \int_X F_h(z)\frac{dzdy}{y^3},$$

we reach the conclusion. □

Proposition 3.3. *Let F be a continuous function of compact support in X. Then*

$$\int_X F(v)d\mu_t(v) \sim \frac{2}{\zeta_K(2)}\left(\int_X F(v)dV(v)\right)\log t$$

as $t \to \infty$.

Proof. The space of all incomplete Eisenstein series and cusp forms is dense in the space of continuous functions vanishing in the cusp. For any $\epsilon > 0$, we can find $G = G_1 + G_2$ with G_1 the finite sum of cusp forms and G_2 in the space of incomplete Eisenstein series, such that $\|G - F\|_\infty < \epsilon$. The difference $H = G - F$ is sufficiently small and rapidly decreasing in the cusp. Namely, it is majorized in terms of another incomplete Eisenstein series

$$H_1(v) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h_1(y(\gamma v))$$

as

$$H_1(v) \geq |H(v)|$$

satisfying

$$\int_X H_1(v) dV(v) < C(K)\epsilon$$

with some constant $C(K)$ depending only on the field K. Hence the conclusion. □

Propositions 2.3 implies Theorem 1.1 by standard approximation arguments.

Acknowledgements. The author would like to express his thanks to Professor Peter Sarnak, who introduced the author to the subject.

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Communicated by P. Sarnak