Abstract

We systematically study the interrelations between all possible variations of $\Delta^0_1$ variants of the law of excluded middle and related principles in the context of intuitionistic arithmetic and analysis.

Keywords: $\Delta^0_1$, weak logical principle, intuitionistic arithmetic and analysis, extended frame.

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1 Introduction

1.1 Motivation

In these two decades, the arithmetical hierarchy of logical principles has been studied extensively (see [1,7]). One motivation of this line of research comes from so-called constructive reverse mathematics (see e.g. [2, 3, 10]). Proofs in mathematics which are noneffective by making use of logical principles in most concrete cases use only rather restricted forms of the classical principle, e.g., the logical principle restricted to formulas of $\Sigma^1_1$ or $\Pi^1_1$ form. By showing the equivalence between a mathematical statement and a restricted logical principle over intuitionistic theory, constructive reverse mathematics reveals that the use of the restricted logical principle is essential for the proof of the mathematical statement in question. Then the study of the arithmetical hierarchy of logical principles provides a fine measure for future development of constructive reverse mathematics.

On the one hand, some of such restricted logical principles are mutually equivalent intuitionistically despite it is not clear at first glance. On the other hand, it is also the case that some restricted logical principle does

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not imply another one over an intuitionistic theory. Thus the study of the arithmetical hierarchy of the logical principles is interesting in itself.

In addition, “weak” fragments of logical principles are also interesting from a viewpoint of the foundation of constructive mathematics. Although it is our common view that constructive mathematics can be seen “roughly” as mathematics based on intuitionistic logic, it is still debatable whether the reasoning in constructive mathematics can be captured “exactly” by intuitionistic logic. In fact, the weak logical principles treated in this paper can be seen as constructive principles in the sense that those are realizable (e.g. by the modified realizability interpretation). It is expected to find some mathematical statements which are intuitionistically equivalent to our weak logical principles.

1.2 Framework

In this paper, we show derivability and underrivability results over intuitionistic arithmetic HA and intuitionistic analysis EL. For the description of HA and EL as well as the basic facts on these theories, we refer the reader to see [13, Section 1.3] and [13, Section 1.9.10] respectively. Recall that the language contains function symbols for all primitive recursive functions, and hence, every formula only with bounded quantifiers is equivalent to some quantifier-free formulas in our theories.

**Notation 1.** Let \( \varphi_{qf} \) denote a quantifier-free formula of the theory in question. In addition, FV (\( \varphi \)) denotes the set of free variables in \( \varphi \). Recall that HA has variables and quantifiers only for natural numbers but EL has those also for functions over natural numbers. The letters \( x, y, z, x', k \) etc. range over objects of type \( N \) (a type for natural numbers). For logical principles, we basically follow the notation in [6] while the same (or equivalent) principles are sometimes denoted by different notations in the literature (cf. [6, Section 2.1]). For notations on Kripke semantics, we basically follow those in [5].

1.3 Our targets

A particular focus of this paper is on \( \Delta^0_1 \) variants of the logical principles, which have not comprehensively studied so far (some of them were already studied in the literature as mentioned below). The class \( \Delta^0_1 \) of formulas is usually mentioned in the context of classical theories as follows: a set \( X \) is \( \Delta^0_1 \)-definable (then \( X \) is said to be in \( \Delta^0_1 \)) if the set \( X \) is \( \Sigma^0_1 \)-definable and its complement \( X^c \) is also \( \Sigma^0_1 \)-definable. In particular, it is well-known that a set \( X \) of natural numbers is \( \Delta^0_1 \)-definable if and only if \( X \) is computable. Reflecting this fundamental fact, the most popular base system RCA_0 of classical reverse mathematics [12] has set-existence axiom only for \( \Delta^0_1 \)-definable sets. Since the law of excluded middle scheme LEM for formulas in a class \( \Gamma \) allows us to obtain (characteristic functions of) \( \Gamma \)-definable
sets in an intuitionistic theory containing countable choice principle AC°0:
\[ \forall x \in N \exists y \in N \varphi(x, y) \rightarrow \exists \alpha \rightarrow N \forall x \in N \varphi(x, \alpha(x)), \]
our classification of several variations of \( \Delta^0_1 \)-LEM can be seen as a classification of classically computable sets from constructive standpoint.

In [1], Akama et al. already studied a version of \( \Delta^0_1 \) variants of LEM in connection with \( \Sigma^0_1 \) variants of LEM, the weak law of excluded middle WLEM and the double negation elimination DNE in the context of HA. More recently, Ishihara, Nemoto and the author [4] systematically studied several variations of \( \Delta^0_1 \) variants of LEM (including the one in [1]), WLEM and DNE in the context of EL. In the present paper, extending the study in [4], we explore the interrelations of all possible variations of \( \Delta^0_1 \) variants of LEM and related principles in the context of HA and EL in parallel. It is remarkable that the principles studied in this paper are extremely weak. In particular, most of them are even weaker than the weakest principles in [1] and [4]. Then the traditional methods for separating weak logical principles (e.g. in [1]) do not work anymore in our case. For separating such extremely weak principles, we employ a technique recently developed in [5].

2 Preliminary

First of all, we discuss all possible formulations of being \( \Delta^0_1 \) over an intuitionistic theory. Let \( \Sigma^0_1 \) (which is often written as \( \Sigma_1 \) in the context of HA) denote the class of formulas of form \( \exists x \varphi_{qf} \). Note that \( \varphi \rightarrow \neg \varphi \), \( \neg \neg (\varphi \leftrightarrow \psi) \leftrightarrow (\neg \varphi \leftrightarrow \neg \psi) \) and \( \neg \neg \varphi \rightarrow \neg \varphi \) are intuitionistically provable but \( \neg \neg \exists x \varphi_{qf} \rightarrow \exists x \varphi_{qf} \) is not so.

Definition 2.1. Let \( T \) be an intuitionistic theory HA or EL. For \( X \in \{a, b, c, ab\} \), \( T \)-formula \( \varphi \in \Sigma^0_1 \) is \( \Delta^0_1 \) (in \( T \)) if there is some \( T \)-formula \( \varphi' \in \Sigma^0_1 \) such that \( T \) proves \( \delta_X(\varphi, \varphi') \), where
\[
\delta_a(\varphi, \varphi') : \varphi \leftrightarrow \neg \varphi' ;
\delta_b(\varphi, \varphi') : \neg \varphi \leftrightarrow \varphi' ;
\delta_c(\varphi, \varphi') : \neg \neg \varphi \leftrightarrow \neg \varphi' ;
\delta_{ab}(\varphi, \varphi') : (\varphi \leftrightarrow \neg \varphi') \wedge (\neg \varphi \leftrightarrow \varphi').
\]
Remark 2.2. Let \( \varphi \) be a formula in \( \Sigma^0_1 \). If \( \varphi \) is \( \Delta_a \) or \( \Delta_b \), then it is \( \Delta_c \) trivially. On the other hand, \( \varphi \) is \( \Delta_a \) and \( \Delta_b \) if and only if it is \( \Delta_{ab} \): if \( T \) proves \( \varphi \leftrightarrow \neg \exists x \varphi_{qf} \) and \( \neg \varphi \leftrightarrow \exists x \varphi'_{qf} \), then \( T \) proves \( \neg \psi \leftrightarrow \neg \psi \) and \( \neg \varphi \leftrightarrow \psi \) for \( \psi : \equiv \exists x, x' \left( \varphi_{qf} \vee \varphi'_{qf} \right) \in \Sigma^0_1 \) (the two existential quantifiers of type \( N \) can be contracted into one by using a standard pairing function definable in HA). The converse direction is obvious.
Remark 2.2 shows that Definition 2.1 covers all possible formulations of being $\Delta^0_1$. In addition, it shows that the implications described in Figure 1 hold for our logical principles $P$ (cf. Definition 2.3).

Figure 1: Relations between $\Delta^X$-variants of logical principles where $X \in \{a, b, c, ab\}$

Recall the following logical principles:

- **LEM**: $\varphi \lor \neg \varphi$;
- **WLEM**: $\neg \varphi \lor \neg \neg \varphi$;
- **DNE**: $\neg \neg \varphi \rightarrow \varphi$;
- **DML**: $\neg (\varphi \land \psi) \rightarrow \neg \varphi \lor \neg \psi$;
- **WDML**: $\neg (\neg \varphi \land \neg \psi) \rightarrow \neg \neg \varphi \lor \neg \neg \psi$;
- **IP**: $(\varphi \rightarrow \exists x \xi) \rightarrow \exists x (\varphi \rightarrow \xi)$ where $x \notin FV(\varphi)$;

Here DML and IP stand for De Morgan’s law and the independence of premise respectively. All of them are classically valid but known to be unprovable in intuitionistic logic. For each principle $P \in \{\text{LEM, WLEM, DNE, DML, WDML, IP}\}$, $\Sigma^0_1-P$ is the restriction of $P$ to $\varphi, \psi \in \Sigma^0_1$. These $\Sigma^0_1$-variants of LEM, WLEM, DNE, DML and WDML play an important role in constructive reverse mathematics [3,10]. Note that,

\[ \Sigma^0_1\text{-DNE} : \neg \neg \exists x \varphi_{qf} \rightarrow \exists x \varphi_{qf} \]

and

\[ \Sigma^0_1\text{-WDML} : \neg (\neg \exists x \varphi_{qf} \lor \neg \exists y \psi_{qf}) \rightarrow \neg \neg \exists x \varphi_{qf} \lor \neg \neg \exists y \psi_{qf} \]

are equivalent to the $\Delta^0_1$ variants of LEM and WLEM in the sense of $\Delta^c$ respectively (see Propositions 3.1 and 3.2).

**Definition 2.3.** Let $X, X' \in \{a, b, c, ab\}$. For a logical principle $P \in \{\text{LEM, WLEM, DNE, IP}\}$, $\Delta^X_1-P$ denotes the following weakening of the $\Sigma^0_1-P$: instances of $\Delta^X_1-P$ are obtained from the instances of $\Sigma^0_1-P$ (namely, $\varphi \in \Sigma^0_1$) by inserting “$\delta^X_1(\varphi, \varphi') \rightarrow$” (in Definition 2.1) in front with some $\varphi' \in \Sigma^0_1$. For $Q \in \{\text{DML, WDML}\}$, $(\Delta^X_1, \Delta^{X'}_1)-Q$ is defined in the same manner by inserting “$\delta^X_1(\varphi, \varphi') \land \delta^{X'}_1(\psi, \psi') \rightarrow$” with some $\varphi', \psi' \in \Sigma^0_1$. In addition, $\Delta^X_1-Q$ denotes $(\Delta^X_1, \Delta^X_1)-Q$. 

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The interrelations between $\Delta_X$-variants of LEM, WLEM and DNE where $X \in \{a, b, c\}$ were completely solved already in [4]. Note that the principles in Definition 2.3 with allowing function parameters are equivalent over $\mathbf{EL}$ to the corresponding principles in [4] defined with function quantifiers $\forall \alpha, \beta$ respectively. In fact, $\Delta_c$-LEM (under the name of IIIa) is shown to be the weakest principle (among them) which is not provable in $\mathbf{EL}$ [4, Theorem 2.1]. In addition, $\Delta_a$-WLEM and $\Delta_b$-WLEM (under the names of IVa and IVb respectively) are shown to be equivalent to $\Delta_a$-LEM [4, Proposition 3]. After [4], Kohlenbach and the author [6] studied the the double negated variants of the logical principles in connection with some weak variants of the double negation shift principle. In particular, it follows from [6, Proposition 3.2] and [6, Theorem 4.9] that the double negated variant $\neg\neg\Delta_a$-LEM of $\Delta_a$-LEM does not derive $\Delta_a$-LEM. In addition, it follows from [6, Theorem 4.11] that $\neg\neg\Delta_a$-LEM is not provable in $\mathbf{EL}$. Thus $\neg\neg\Delta_a$-LEM is a principle strictly weaker than $\Delta_a$-LEM but still unprovable in $\mathbf{EL}$. As far as the author knows, however, this is only an example of principles $Q$ such that $Q$ are strictly weaker than $\Delta_a$-LEM but still unprovable in $\mathbf{EL}$. Thus such a weak principle $Q$ has not been found except the double negated variant of $\Delta_a$-LEM. On the other hand, each $\Delta_{ab}$-variant are weaker than its $\Delta_a$-variant and $\Delta_b$-variant (cf. Figure 1). Thus $\Delta_{ab}$-LEM and $\Delta_{ab}$-WLEM are the strong candidates of such $Q$ (while $\Delta_{ab}$-DNE is intuitionistically provable since $\Delta_a$-DNE is already so). Furthermore, since LEM derives IP and WLEM derives DML and WDML, it is also natural to ask the status of the $\Delta_X$-variants of IP, DML and WDML where $X \in \{a, b, c\}$ (interestingly, it will be found in the end that not only the $\Delta_{ab}$-variants of LEM, WLEM, DML, WDML, IP, but also $\Delta_a$-DML, $\Delta_b$-DML, $\Delta_c$-WDML and $\Delta_b$-WDML are examples of that kind). In the following sections, we comprehensively study the interrelations between $\Delta_X$-variants of LEM, WLEM, DNE, DML, WDML and IP where $X \in \{a, b, c, ab\}$. The $\Delta_X$-variants of DML, WDML and IP where $X \in \{a, b, c, ab\}$, as well as $\Delta_{ab}$-variants of LEM, WLEM and DNE, are first studied systematically in this paper.

3 Derivability results

In this section, we work in $\mathbf{HA}$ and $\mathbf{EL}$ in parallel. All of the proofs in Section 3.1 can be formalized in both of the contexts.

Throughout this section, let $\varphi, \varphi', \psi, \psi'$ denote formulas in $\Sigma^0_1$ of the theory in question. In addition, Fml denotes the class of all formulas of the theory in question.

First we recall the results established already in [4]:

**Proposition 3.1** ([4, Proposition 1]). $\Delta_c$-LEM, $\Delta_b$-LEM, $\Delta_c$-DNE and $\Delta_b$-DNE are equivalent to $\Sigma^0_1$-DNE, which is known as Markov’s principle in constructive mathematics.
Proposition 3.2 (4 Proposition 2). \( \Delta_c\)-WLEM is equivalent to \( \Sigma^0_1\)-WDML, which is called MP in [4] (and called \( \Pi^0_1\)-DML in [2]).

Proposition 3.3 (4 Proposition 3). \( \Delta_a\)-WLEM and \( \Delta_b\)-WLEM are equivalent to \( \Delta_a\)-LEM, which is called \( \Pi_0\) in [4].

Note that \( \Sigma^0_1\)-DNE implies \( \Sigma^0_1\)-WDML (and not vice versa), \( \Sigma^0_1\)-WDML implies \( \Delta_a\)-LEM (and not vice versa), and \( \Delta_a\)-LEM is not provable (cf. Section 3.2).

Proposition 3.4 (4 Proposition 5). \( \Delta_a\)-DNE is provable. Then \( \Delta_{ab}\)-DNE is also provable.

3.1 Derivability proofs

Lemma 3.5. Let \( X \in \{a, b, c, ab\} \).

1. \( \Delta_X\)-LEM derives \( \Delta_X\)-WLEM;
2. \( \Delta_X\)-WLEM derives \( (\Delta_X, \text{Fml})\)-DML;
3. \( \Delta_X\)-WLEM derives \( (\Delta_X, \text{Fml})\)-WDML;
4. \( \Delta_X\)-LEM derives \( \Delta_X\)-IP;
5. \( \Delta_X\)-IP derives \( \Delta_X\)-LEM.

(2): Assume that \( \varphi \) is \( \Delta_X\) and \( \neg (\varphi \land \psi) \). By \( \Delta_X\)-WLEM, we have \( \neg \varphi \lor \neg \neg \varphi \). In the latter case, we have \( \neg \psi \) by \( \neg (\varphi \land \psi) \). Thus we have \( \neg \varphi \lor \neg \psi \) as required.
(3): Assume that \( \varphi \) is \( \Delta_X\) and \( \neg (\neg \varphi \land \neg \psi) \). By \( \Delta_X\)-WLEM, we have \( \neg \varphi \lor \neg \neg \varphi \). In the former case, we have \( \neg \neg \psi \) by \( \neg (\neg \varphi \land \neg \psi) \). Thus we have \( \neg \varphi \lor \neg \psi \) as required.
(4): Assume that \( \varphi \) is \( \Delta_X\) and \( \varphi \rightarrow \exists x \xi(x) \) with \( x \not\in \text{FV}(\varphi) \). By \( \Delta_X\)-LEM, we have \( \varphi \lor \neg \varphi \). In the former case, we have \( \exists x \xi(x) \) by our assumption, and hence, a-fortiori \( \exists x (\varphi \rightarrow \xi(x)) \). In the latter case, we have \( \varphi \rightarrow \xi(x_0) \) for any \( x_0 \), and hence, a-fortiori \( \exists x (\varphi \rightarrow \xi(x)) \).
(5): Assume that \( \exists x \varphi_{qf}(x) \) is \( \Delta_X\). Applying \( \Delta_X\)-IP to \( \exists x \varphi_{qf}(x) \rightarrow \exists x \varphi_{qf}(x) \), we have \( \exists x (\exists x \varphi_{qf}(x) \rightarrow \varphi_{qf}(x)) \). Since \( \forall x (\varphi_{qf}(x) \lor \neg \varphi_{qf}(x)) \) is provable, we have \( \exists x \varphi_{qf}(x) \lor \neg \exists x \varphi_{qf}(x) \).

Theorem 3.6. The following are mutually equivalent: \( \Sigma^0_1\)-DNE, \( \Delta_c\)-LEM, \( \Delta_b\)-LEM, \( \Delta_c\)-DNE, \( \Delta_b\)-DNE, \( \Delta_c\)-IP and \( \Delta_b\)-IP.

Proof. By Proposition 3.1 and Lemma 3.5 (4) (5).

Lemma 3.7. \( \Delta_c\)-DML derives \( \Delta_c\)-WDML.
Proof. Assume \( \neg \varphi \leftrightarrow \neg \neg \varphi' \) and \( \neg \psi \leftrightarrow \neg \neg \psi' \). Assume also \( \neg (\neg \varphi \land \neg \psi) \). Then we have that \( \varphi' \) and \( \psi' \) are \( \Delta_c \) and \( \neg ((\neg \varphi \land \neg \psi')) \). Then, by \( \Delta_c\text{-DML} \), we have \( \neg \varphi' \lor \neg \psi' \), equivalently, \( \neg \neg \varphi \lor \neg \neg \psi \).

Lemma 3.8. \( \Delta_c\text{-WDM} \) derives \( \Delta_c\text{-WLEM} \).

Proof. Assume \( \neg \varphi \leftrightarrow \neg \neg \varphi' \). Then we have that \( \varphi \) and \( \varphi' \) are \( \Delta_c \) and \( \neg (\varphi \land \neg \varphi') \). By \( \Delta_c\text{-WDML} \), we have \( \neg \neg \varphi \lor \neg \neg \varphi' \), equivalently, \( \neg \varphi \lor \neg \varphi \).

Theorem 3.9. The following are mutually equivalent: \( \Sigma_1\text{-WDM} \), \( \Delta_c\text{-WLEM} \), \( \Delta_c\text{-WDM} \), \( \Delta_c\text{-WDM} \), \( \Delta_c\text{-WDM} \) and \( \Delta_c\text{-WDM} \).


Lemma 3.10. \( (\Delta_a, \Delta_b)\text{-DML} \) derives \( \Delta_a\text{-LEM} \).

Proof. Assume \( \varphi \leftrightarrow \neg \varphi' \). Then \( \varphi \) is \( \Delta_a \) and \( \varphi' \) is \( \Delta_b \). Since \( \neg (\varphi \land \varphi') \), by \( (\Delta_a, \Delta_b)\text{-DML} \), we have \( \neg \varphi \lor \neg \varphi' \), equivalently, \( \varphi \lor \varphi \).

Lemma 3.11. \( (\Delta_a, \Delta_b)\text{-WDM} \) derives \( \Delta_a\text{-LEM} \).

Proof. Assume \( \varphi \leftrightarrow \neg \varphi' \). Then \( \varphi \) is \( \Delta_a \) and \( \varphi' \) is \( \Delta_b \). Since \( \neg (\neg \varphi \land \neg \varphi') \), by \( (\Delta_a, \Delta_b)\text{-WDM} \), we have \( \neg \neg \varphi \lor \neg \neg \varphi' \), which is equivalent to \( \varphi \lor \varphi \) by our assumption and Proposition 3.4.

Theorem 3.12. The following are mutually equivalent: \( \Delta_a\text{-LEM} \), \( \Delta_a\text{-IP} \), \( \Delta_a\text{-WLEM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \), \( \Delta_a\text{-WDM} \) and \( \Delta_a\text{-WDM} \).

Proof. The equivalence of \( \Delta_a\text{-LEM} \) and \( \Delta_a\text{-IP} \) follows from Lemma 3.5. By Proposition 3.9, \( \Delta_a\text{-LEM} \), \( \Delta_a\text{-WEM} \) and \( \Delta_a\text{-WEM} \) are mutually equivalent. Then, by Lemma 3.5 and Proposition 3.3, \( \Delta_a\text{-LEM} \) derives \( \Delta_a\text{-WDM} \). By Lemma 3.9 and Proposition 3.3, \( \Delta_a\text{-WDM} \) and \( \Delta_a\text{-WDM} \) derive \( \Delta_a\text{-WDM} \) and \( \Delta_a\text{-WDM} \). On the other hand, \( \Delta_a\text{-WDM} \) and \( \Delta_a\text{-WDM} \) derive \( \Delta_a\text{-WDM} \) and \( \Delta_a\text{-WDM} \). Thus, all of these variation of DML and WDML are equivalent to \( \Delta_a\text{-LEM} \).

Theorem 3.13. \( \Delta_a\text{-DML} \) is equivalent to \( \Delta_b\text{-WDM} \).

Proof. We first show \( \Delta_a\text{-DML} \rightarrow \Delta_b\text{-WDM} \). Assume \( \neg \varphi \leftrightarrow \varphi' \), \( \neg \psi \leftrightarrow \psi' \) and \( \neg (\neg \varphi \land \neg \psi) \). Then \( \varphi' \) and \( \psi' \) are \( \Delta_a \). In addition, \( \neg (\varphi' \land \psi') \) follows from \( \neg (\neg \varphi \land \neg \psi) \). Then, by \( \Delta_a\text{-DML} \), we have \( \neg \varphi' \lor \neg \psi' \), equivalently, \( \neg \neg \varphi \lor \neg \neg \psi \) as required. The converse direction is similar.
For the mutual derivability between the equivalents of $\Delta_{ab}$-LEM, the following lemma is crucial, which is shown by refining the proof of $\Delta_c$-WLEM $\rightarrow \Sigma^0_1$-WDML in [4, Proposition 2].

**Lemma 3.14.** $\Delta_{ab}$-DML derives $\Delta_a$-WDML.

**Proof.** Note that $\Delta_a$-WDML is equivalent to

\[
\left( \exists x \varphi_{qf}(x) \leftrightarrow \neg \exists x' \varphi'_{qf}(x') \right) \land \left( \exists y \psi_{qf}(y) \leftrightarrow \neg \exists y' \psi'_{qf}(y') \right) \land \neg (\neg \exists x \varphi_{qf}(x) \land \neg \exists y \psi_{qf}(y))
\]

$\rightarrow \exists x \varphi_{qf}(x) \lor \exists y \psi_{qf}(y)$.

We assume

\[\begin{align*}
\exists x \varphi_{qf}(x) & \leftrightarrow \neg \exists x' \varphi'_{qf}(x'), \\
\exists y \psi_{qf}(y) & \leftrightarrow \neg \exists y' \psi'_{qf}(y'), \\
\neg (\neg \exists x \varphi_{qf}(x) \land \neg \exists y \psi_{qf}(y)) &
\end{align*}\]

(1)

(2)

(3)

to show $\exists x \varphi_{qf}(x) \lor \exists y \psi_{qf}(y)$. Define

\[\begin{align*}
\xi(x) & \equiv \forall z < x (\neg \varphi_{qf}(z) \land \neg \psi_{qf}(z)) \land \varphi_{qf}(x); \\
\chi(y) & \equiv \forall z < y (\neg \varphi_{qf}(z) \land \neg \psi_{qf}(z)) \land \neg \varphi_{qf}(y) \land \psi_{qf}(y).
\end{align*}\]

Then we have

\[\neg (\exists x \xi(x) \land \exists y \chi(y))\]

(4)

by definition straightforwardly. Define

\[\begin{align*}
\zeta & \equiv \exists x' \varphi'_{qf}(x') \lor \exists y \chi(y); \\
\eta & \equiv \exists y' \psi'_{qf}(y') \lor \exists x \xi(x).
\end{align*}\]

We first claim

\[-(\zeta \land \eta).\]

(5)

Suppose $\zeta \land \eta$, equivalently, $\exists x' \varphi'_{qf}(x') \lor \exists y \chi(y)$ and $\exists y' \psi'_{qf}(y') \lor \exists x \xi(x)$. If $\exists x' \varphi'_{qf}(x')$ and $\exists y' \psi'_{qf}(y')$, then we have $\neg \neg \exists x' \varphi'_{qf}(x') \land \neg \neg \exists y' \psi'_{qf}(y')$, equivalently, $\neg \exists x \varphi_{qf}(x) \land \neg \exists y \psi_{qf}(y)$ by (1) and (2). This contradicts (3). If $\exists x' \varphi'_{qf}(x')$ and $\exists x \xi(x)$, since $\exists x \xi(x)$ derives $\exists x \varphi_{qf}(x)$, we have a contradiction by (1). If $\exists y \chi(y)$ and $\exists y' \psi'_{qf}(y')$, since $\exists y \chi(y)$ derives $\exists y \psi_{qf}(y)$, we have a contradiction by (2). If $\exists y \chi(y)$ and $\exists x \xi(x)$, then we have a contradiction by (3). Thus we have shown the our claim.

Next we claim

\[-\zeta \leftrightarrow \exists x \xi(x)\]

(6)

and

\[-\eta \leftrightarrow \exists y \chi(y).\]

(7)
Theorem 3.17. The following are mutually equivalent:

Lemma 3.16.

Lemma 3.15.

Since $\neg \zeta$ and $\neg \eta$ are equivalent to $\exists x \varphi_{qf}(x) \land \neg \exists y \chi(y)$ and $\exists y \psi_{qf}(y) \land \neg \exists x \xi(x)$ respectively, it suffices to show $\exists x \xi(x) \iff \exists x \varphi_{qf}(x) \land \neg \exists y \chi(y)$ and $\exists y \chi(y) \iff \exists y \psi_{qf}(y) \land \neg \exists x \xi(x)$. In the following, we show them in parallel (note that the difference between the definitions of $\xi$ and $\chi$ causes the difference of the inequalities in the following proofs). First assume $\exists x \xi(x)$ (resp. $\exists y \chi(y)$). Then we have $\exists x \varphi_{qf}(x)$ (resp. $\exists y \psi_{qf}(y)$) and also have $\neg \exists y \chi(y)$ (resp. $\neg \exists x \xi(x)$) by (4). For the converse direction, assume $\exists x \varphi_{qf}(x) \land \neg \exists y \chi(y)$ (resp. $\exists y \psi_{qf}(y) \land \neg \exists x \xi(x)$). Taking the least $x$ (resp. $y$) such that $\exists x \varphi_{qf}(x)$ (resp. $\exists y \psi_{qf}(y)$) by bounded search, which is available in our context, one can show that there exists $x$ (resp. $y$) such that $\forall z < x \varphi_{qf}(x)$ (resp. $\forall z < y \psi_{qf}(y)$) and $\varphi_{qf}(x)$ (resp. $\psi_{qf}(y)$). On the other hand, if there exists $z < x$ (resp. $z \leq y$) such that $\psi_{qf}(z)$ (resp. $\varphi_{qf}(z)$), then we have $\chi(z)$ (resp. $\zeta(z)$) for the least such $z$. This contradicts $\neg \exists y \chi(y)$ (resp. $\neg \exists x \xi(x)$). Thus we have $\forall z < x \neg \varphi_{qf}(z)$ (resp. $\forall z \leq y \neg \varphi_{qf}(z)$). Then we have $\zeta(x)$ (resp. $\chi(y)$), and hence, a-fortiori $\exists x \xi(x)$ (resp. $\exists y \chi(y)$).

Eventually we claim

$$\zeta \leftrightarrow \neg \eta \quad (8)$$

and

$$\eta \leftrightarrow \neg \zeta. \quad (9)$$

By (5), we have $\zeta \rightarrow \neg \eta$ and $\eta \rightarrow \neg \zeta$. On the other hand, by (6), (7) and the definitions of $\zeta$ and $\eta$, we have $\neg \eta \leftrightarrow \exists y \chi(y) \rightarrow \zeta$ and $\neg \zeta \leftrightarrow \exists x \xi(x) \rightarrow \eta$.

Since $\xi(x)$ and $\chi(y)$ are equivalent to some quantifier-free formulas respectively in our context, $\zeta$ and $\eta$ are equivalent to some formulas in $\Sigma_0$ respectively (see [8] Lemma 4.4)). Then, by (6), (7), (8), (9) and Remark 2.2, we have that both of $\zeta$ and $\eta$ are $\Delta_{ab}$. Applying $\Delta_{ab}$-DML to (5), we have $\neg \zeta \lor \neg \eta$. Then, by (6) and (7), $\exists x \varphi_{qf}(x) \lor \exists y \psi_{qf}(y)$ follows. \hfill \Box

Lemma 3.15. $\Delta_{ab}$-WDML derives $\Delta_{ab}$-DML.

Proof. Assume $\neg \varphi \leftrightarrow \varphi'$ and $\neg \psi \leftrightarrow \psi'$. Assume also $\neg (\varphi \land \psi)$. Then we have $\neg (\neg \varphi' \land \neg \psi')$. Applying $\Delta_{ab}$-WDML, we have $\neg \neg \varphi' \lor \neg \psi'$, and hence, $\neg \varphi \lor \neg \psi$. \hfill \Box

Lemma 3.16. $\Delta_{ab}$-WDML derives $\Delta_{ab}$-LEM.

Proof. Assume $\varphi \leftrightarrow \neg \varphi'$ and $\neg \varphi \leftrightarrow \varphi'$. Then both of $\varphi$ and $\varphi'$ are $\Delta_{ab}$. Since $\neg (\neg \varphi \land \neg \varphi')$, by $\Delta_{ab}$-WDML, we have $\neg \neg \varphi \lor \neg \neg \varphi'$. In the former case, we have $\varphi$ by Proposition 3.1. In the latter case, we have $\neg \varphi$ by our assumption. \hfill \Box

Theorem 3.17. The following are mutually equivalent: $\Delta_{ab}$-LEM, $\Delta_{ab}$-IP, $\Delta_{ab}$-WLEM, $(\Delta_{ab}, \text{Fml})$-DML, $(\Delta_{ab}, \Delta_c)$-DML, $(\Delta_{ab}, \Delta_d)$-DML, $(\Delta_{ab}, \Delta_b)$-DML, $\Delta_{ab}$-DML, $(\Delta_{ab}, \text{Fml})$-WDML, $(\Delta_{ab}, \Delta_c)$-WDML, $(\Delta_{ab}, \Delta_d)$-WDML, $(\Delta_{ab}, \Delta_b)$-WDML, $\Delta_{ab}$-WDML, $\Delta_{ab}$-WML, $\Delta_{ab}$-DML, $\Delta_{ab}$-DML.
Proof. The equivalence of $\Delta_{ab}$-LEM and $\Delta_{ab}$-IP follows from Lemma 3.5(4)(5). By Lemma 3.5(1), $\Delta_{ab}$-LEM, derives $\Delta_{ab}$-WLEM. By Lemma 3.5(2)(3), $\Delta_{ab}$-WLEM derives $(\Delta_{ab}, \text{Fml})$-DML and $(\Delta_{ab}, \text{Fml})$-WDML, and hence, $(\Delta_{ab}, \Delta_c)$-DML, $(\Delta_{ab}, \Delta_a)$-DML, $(\Delta_{ab}, \Delta_b)$-DML, $(\Delta_{ab}, \Delta_c)$-WDML, $(\Delta_{ab}, \Delta_a)$-WDML and $(\Delta_{ab}, \Delta_b)$-WDML. By Lemma 3.14, all of them derive $\Delta_{ab}$-WDML, which derives $\Delta_{ab}$-LEM by Lemma 3.16. In addition, by Lemmata 3.14 and 3.15 and the fact that $\Delta_b$-DML derives $\Delta_{ab}$-DML, we have that $\Delta_a$-WDML and $\Delta_b$-DML are also equivalent to $\Delta_{ab}$-LEM. 

3.2 Summary

The strength of our principles is summarized in Tables 1, 2 and 3 (cf. Theorems 3.6, 3.9, 3.12, 3.13 and 3.17)

<table>
<thead>
<tr>
<th>$\Gamma$-LEM</th>
<th>$\Delta_{ab}$-LEM</th>
<th>$\Delta_a$-LEM</th>
<th>$\Delta_b$-LEM</th>
<th>$\Delta_c$-LEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$-WLEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td>$\Delta_b$-LEM</td>
<td>$\Sigma_1^\text{WDML}$</td>
</tr>
<tr>
<td>$\Gamma$-DNE</td>
<td>$\perp$</td>
<td>$\perp$</td>
<td>$\Sigma_1^\text{WDML}$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma$-IP</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td>$\Sigma_1^\text{WDML}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The strength of the variations of $\Gamma$-LEM, $\Gamma$-WLEM, $\Gamma$-DNE and $\Gamma$-IP for $\Gamma \in \{\Delta_{ab}, \Delta_a, \Delta_b, \Delta_c\}$

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$\Delta_{ab}$</th>
<th>$\Delta_a$</th>
<th>$\Delta_b$</th>
<th>$\Delta_c$</th>
<th>Fml</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td></td>
</tr>
<tr>
<td>$\Delta_a$-DML</td>
<td>$\Delta_a$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta_b$-DML</td>
<td>$\Delta_b$-LEM</td>
<td>$\Delta_b$-LEM</td>
<td>$\Delta_b$-LEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma_1^\text{WDML}$</td>
<td>$\Sigma_1^\text{WDML}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: The strength of the variations of $(\Gamma, \Theta)$-DML for $\Gamma, \Theta \in \{\Delta_{ab}, \Delta_a, \Delta_b, \Delta_c\}$

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>$\Delta_{ab}$</th>
<th>$\Delta_a$</th>
<th>$\Delta_b$</th>
<th>$\Delta_c$</th>
<th>Fml</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td>$\Delta_{ab}$-LEM</td>
<td></td>
</tr>
<tr>
<td>$\Delta_a$-DML</td>
<td>$\Delta_a$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td>$\Delta_a$-LEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta_b$-DML</td>
<td>$\Delta_b$-LEM</td>
<td>$\Delta_b$-LEM</td>
<td>$\Delta_b$-LEM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Sigma_1^\text{WDML}$</td>
<td>$\Sigma_1^\text{WDML}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: The strength of the variations of $(\Gamma, \Theta)$-WDML for $\Gamma, \Theta \in \{\Delta_{ab}, \Delta_a, \Delta_b, \Delta_c\}$
By Remark 2.2 it is trivial that $\Delta_a$-LEM, equivalently $(\Delta_a, \text{Fml})$-DML, implies $\Delta_a$-DML, which implies $\Delta_{ab}$-DML, equivalently $\Delta_{ab}$-LEM. Thus the weak logical principles studied in this paper consist of the following hierarchy over HA or EL:

$$\Sigma_1^0\text{-DNE} \Rightarrow \Sigma_1^0\text{-WDML} \Rightarrow \Delta_a\text{-LEM} \Rightarrow \Delta_a\text{-DML} \Rightarrow \Delta_{ab}\text{-LEM}.$$ 

As shown in Tables 1, 2 and 3 except $\Delta_a$-DNE and $\Delta_{ab}$-DNE, every $\Delta_X$-variant of LEM, WLEM, DNE, DML, WDML and IP where $X \in \{a, b, c, ab\}$ is equivalent to one of the above five principles. The properness of the first and second implications are shown in [1] (essentially) and [4,11] respectively. We show the properness of the third and fourth implications in Section 4.

### 4 Underivability results

To show our underivability results with using the technique developed in [5], we consider the following propositional formulas:

- $\text{LEM}(\Delta_a) : (p \leftrightarrow \neg p') \rightarrow p \lor \neg p$;
- $\text{LEM}(\Delta_{ab}) : (p \leftrightarrow \neg p') \land (\neg p \leftrightarrow p') \rightarrow p \lor \neg p$;
- $\text{DML}(\Delta_a) : (p \leftrightarrow \neg p') \land (q \leftrightarrow \neg q') \rightarrow (\neg (p \land q) \rightarrow \neg p \lor \neg q)$.

As in [5], we sometimes write $\varphi[p_1, \ldots, p_n]$ for a propositional formula $\varphi$ consisting of propositional variables $p_1, \ldots, p_n$. For a propositional formula $\varphi[p_1, \ldots, p_n]$ and HA-formulas $\psi_1, \ldots, \psi_n$, $\varphi[\psi_1, \ldots, \psi_n]$ denotes the HA-formula obtained from $\varphi[p_1, \ldots, p_n]$ by replacing each $p_i$ with $\psi_i$. Under this notation, $\Delta_a$-LEM, $\Delta_{ab}$-LEM and $\Delta_a$-DML can be written as $\text{LEM}(\Delta_a)[\psi_1, \psi_2]$, $\text{LEM}(\Delta_{ab})[\psi_1, \psi_2]$ and $\text{DML}(\Delta_a)[\psi_1, \psi_2, \psi_1', \psi_2']$ respectively with $\psi_1, \psi_2, \psi_1', \psi_2' \in \Sigma_1^0$. For the detailed description of an IPC-Kripke model and an extended frame, we refer the reader to see [5, Section 2]. Here we recall only some definitions and a result which we need. In this section, we use classical logic at the meta-level.

**Definition 4.1** (cf. [5] Definition 2, Remark 2 and Definition 4]). An extended frame $\mathcal{E} = ((K, \le), f, (I, \le_I))$ is a triple of frames $(K, \le)$ and $(I, \le_I)$, and a monotone mapping $f$ between them, that is, $k \le k'$ implies $f(k) \le_I f(k')$ for each $k, k' \in K$. In particular, $\mathcal{E}$ is locally directed if $f^{-1}(\uparrow i) \cap \uparrow k$ is directed for all $i \in I$ and $k \in K$, that is, for each $i \in I$ and $k \in K$, if $l, l' \in f^{-1}(\uparrow i) \cap \uparrow k$, then there exists $l'' \in f^{-1}(\uparrow i) \cap \uparrow k$ such that $l'' \le l$ and $l'' \le l'$. Each IPC-Kripke model $\mathcal{I} = (I, \le_I, \models)$ induces an IPC-Kripke model $\mathcal{K}_{\mathcal{E}, \mathcal{I}} = (K, \le, \models_{\mathcal{E}, \mathcal{I}})$ by defining

$$k \models_{\mathcal{E}, \mathcal{I}} p :\iff f(k) \models p.$$
for each \( k \in K \) and propositional variable \( p \). A propositional formula \( \varphi \) is valid on \( E \) if \( K, I, \models E, \varphi \) for each IPC-Kripke model \( I = (I, \leq, \models) \), that is, for each valuation \( \models \) on \( (I, \leq) \); we then write \( E \models \varphi \). For an extended frame \( E \), define \( T(E) := \{ \varphi \mid E \models \varphi \} \).

**Definition 4.2** (cf. [5, Example 1]). Let \( K = (K, \leq, \models) \) be an IPC-Kripke model, and define a set \( \Phi_K \) of upward closed subsets of \( K \) by

\[
\Phi_K := \{ \{ k \in K \mid k \models p \} \mid p \in V \}.
\]

Define binary relations \( \preceq_K \) and \( \sim_K \) on \( K \) by

\[
k \preceq_K k' :\Leftrightarrow \forall U \in \Phi_K (k \in U \Rightarrow k' \in U),
k \sim_K k' :\Leftrightarrow k \preceq_K k' \land k' \preceq_K k.
\]

Then \( \preceq_K \) is a preorder and \( \sim_K \) is an equivalence relation on \( K \). Let

\[
I_K := K / \sim_K, \quad [k]_K \preceq_K [k']_K :\Leftrightarrow k \preceq_K k', \quad f_K(k) := [k]_K,
\]

where \([k]_K\) (we suppress the subscript \( K \) when it is clear from the context) is the equivalence class of \( k \) with respect to \( \sim_K \). Then

\[
E_K := ((K, \preceq), f_K, (I_K, \preceq_K))
\]

is an extended frame, and we call it the extended frame generated by the IPC-Kripke model \( K \).

**Definition 4.3** (cf. [5, Definition 12]). For each propositional formula \( \varphi[p_1, \ldots, p_n] \), \( \Sigma \varphi \) denotes the following schema of HA:

\[
\forall x_1(\psi_1(x) \lor \neg \psi_1(x)) \land \ldots \land \forall x_n(\psi_n(x) \lor \neg \psi_n(x)) \rightarrow \varphi[\exists x_1\psi_1(x), \ldots, \exists x_n\psi_n(x)],
\]

where \( \psi_1, \ldots, \psi_n \) are HA-formulas. For an extended frame \( E \), \( \Sigma(T(E)) \) is the schema (of HA) consisting of \( \Sigma \varphi \) where \( \varphi \in T(E) \).

**Theorem 4.4** (cf. [5, Theorem 29]). Let \( K = (K, \leq, \models) \) be a finite IPC-Kripke model such that \( (I_K, \preceq_K) \) is a rooted and finite tree, and let \( \varphi[p_1, \ldots, p_n] \) be a propositional formula. If \( K \not\models \varphi \) and \( E_K \) is locally directed, then HA + \( \Sigma(T(E_K)) \) does not prove some \( \Sigma_1^0 \) instance of \( \varphi \), namely, HA + \( \Sigma(T(E_K)) \) \( \not\models \varphi[p_1, \ldots, p_n] \) for some HA-formulas \( \psi_1, \ldots, \psi_n \in \Sigma_1^0 \).

In what follows, we show our underivability results with using Theorem 4.4.

**Theorem 4.5.** HA \( \not\models \Delta_{ab} \) LEM.
Proof. It is easy to see that $\text{LEM}(\Delta_{ab})[p, p']$ is not valid in the following Kripke model from [5, Example 5]:

\[
\begin{array}{c}
p
\end{array} \quad \begin{array}{c}
p'
\end{array}
\quad \begin{array}{c}
1 \\
2
\end{array}
\quad \begin{array}{c}
0
\end{array}
\]

The extended frame generated by this Kripke model is locally directed. By Theorem 4.4, we have that $\text{HA}$ does not prove some $\Sigma_1^0$ instance of $\text{LEM}(\Delta_{ab})$, and hence, $\text{HA} \not\vdash \Delta_{ab}$-LEM.

**Proposition 4.6.** $\text{DML}(\Delta_a)[p, p', q, q']$ is not valid in the following IPC-Kripke model $K^*$:

\[
\begin{array}{c}
p'
\end{array} \quad \begin{array}{c}
q'
\end{array}
\quad \begin{array}{c}
p, q
\end{array} \quad \begin{array}{c}
p', q
\end{array} \quad \begin{array}{c}
4
\end{array}
\quad \begin{array}{c}
p, q
\end{array} \quad \begin{array}{c}
p', q
\end{array} \quad \begin{array}{c}
3
\end{array}
\quad \begin{array}{c}
2
\end{array} \quad \begin{array}{c}
1
\end{array} \quad \begin{array}{c}
0
\end{array}
\]

Proof. Since $\text{DML}(\Delta_a)[p, p', q, q']$ is intuitionistically equivalent to

\[
(p \leftrightarrow \neg p') \land (q \leftrightarrow \neg q') \land \neg (p \land q) \rightarrow \neg p \lor \neg q,
\]

it suffices to show $0 \not\models [10]$. It is trivial that $0 \models \neg (p \land q)$. In addition, one can verify routinely that $i \models p \rightarrow \neg p'$, $p' \rightarrow p$, $q \rightarrow \neg q'$, $q' \rightarrow q$ for all $i \in \{0, 1, 2, 3, 4\}$ (note that $3 \not\models p', q'$ does not cause any trouble for our purpose). Thus we have also $0 \models (p \leftrightarrow \neg p') \land (q \leftrightarrow \neg q')$. On the other hand, since $1 \models p$ and $2 \models q$, we have $0 \not\models \neg p \lor \neg q$. \qed

**Lemma 4.7.** For the Kripke model $K_a$ in Proposition 4.6, $\text{LEM}(\Delta_{ab})$ is contained in $T(\mathcal{E}_{K_a})$.

Proof. Note that the Kripke frame $(I_{K_a}, \leq_{K_a})$ is as follows:

\[
\begin{array}{c}
[1]
\end{array} \quad \begin{array}{c}
[2]
\end{array} \quad \begin{array}{c}
[4]
\end{array} \quad \begin{array}{c}
[0] = [3]
\end{array}
\]

Fix a valuation $\models$ on $(I_{K_a}, \leq_{K_a})$. Let $\mathcal{I}_a$ denote the Kripke model $(I_{K_a}, \leq_{K_a}, \models)$. Fix propositional variables $p$ and $p'$. We show $0 \models_{\mathcal{I}_a} \text{LEM}(\Delta_{ab})[p, p']$. If $[0] \models p$, then we have $0 \models_{\mathcal{I}_a} p$, and hence, we are done. Assume
[0] \not\models p \ (equivalently, [3] \not\models p). Since 1, 2, 4 \models p \lor \neg p \ for \ any \ valuation \ \models \cdot, \ it \ suffices \ to \ show \ i \models \xi_{\mathcal{K}_*} \cdot, \ p \lor \neg p \ or \ i \models \xi_{\mathcal{K}_*} \cdot, \ (p \leftrightarrow p') \land (\neg p \leftrightarrow p') \ for \ i \in \{0,3\}. \ If \ [3] \not\models p' \ and \ [4] \not\models p, \ since \ 3 \not\models \xi_{\mathcal{K}_*} \cdot, \ p \rightarrow p', \ we \ have \ 0,3 \not\models \xi_{\mathcal{K}_*} \cdot, \ (p \rightarrow p') \land (\neg p' \rightarrow p). \ If \ [3] \not\models p' \ and \ [4] \models p, \ since \ 3 \not\models \xi_{\mathcal{K}_*} \cdot, \ (p \rightarrow p') \land (\neg p' \rightarrow p), \ we \ have \ 0,3 \not\models \xi_{\mathcal{K}_*} \cdot, \ (p \leftrightarrow p') \land (\neg p \leftrightarrow p').

Remark 4.8. \ Since \ quantifier-free \ formulas \ are \ definable \ in HA, \ the \ instances \ of \ \Delta_{ab-LEM} \ (resp. \ \Delta_{a-DML}) \ are \ derived \ from \ some \ instances \ of \ \Sigma-LEM(\Delta_{ab}) \ (resp. \ \Sigma-DML(\Delta_a)) \ in HA.

Theorem 4.9. HA + \Delta_{ab-LEM} \not\models \Delta_{a-DML}.

Proof. \ Note \ that \ the \ extended \ frame \ generated \ by \ \mathcal{K}_* \ in \ Proposition 4.6 \ is \ locally \ directed. \ Applying \ Theorem \ 4.4 \ to \ Proposition \ 4.6 \ we \ have \ that \ HA + \Sigma-T(\xi_{\mathcal{K}_*}) \ does \ not \ prove \ some \ \Sigma_0^e-\instance \ of \ DML(\Delta_a), \ and \ hence, \ HA + \Sigma-T(\xi_{\mathcal{K}_*}) \not\models \Delta_{a-DML}. \ By \ Lemma 4.7 \ and \ Remark \ 4.8, \ we \ have \ HA + \Delta_{ab-LEM} \not\models \Delta_{a-DML}. \ \Box

Proposition 4.10. \ LEM(\Delta_a)[p,p'] \ is \ not \ valid \ in \ the \ following \ IPC-Kripke \ model \ \mathcal{K}_*:

\[\begin{array}{c}
p' \\
3 \\
2 \\
1 \\
0 \\
\end{array}\]

Proof. \ Recall \ LEM(\Delta_a)[p,p'] \equiv (p \leftrightarrow \neg p') \rightarrow p \lor \neg p. \ \ It \ is \ trivial \ that \ 0 \not\models p \lor \neg p. \ In \ addition, \ one \ can \ verify \ routinely \ that \ i \models p \rightarrow p', \ \neg p' \rightarrow p \ for \ all \ i \in \{0,1,2,3\} \ (note \ that \ 2 \not\models p' \ does \ not \ cause \ any \ trouble \ for \ our \ purpose). \ Then \ we \ have \ 0 \not\models (p \leftrightarrow \neg p') \rightarrow p \lor \neg p. \ \Box

Lemma 4.11. \ For \ the \ Kripke \ model \ \mathcal{K}_* \ in \ Proposition 4.10, \ DML(\Delta_a) \ is \ contained \ in T(\xi_{\mathcal{K}_*}).

Proof. \ Note \ that \ the \ Kripke \ frame \ (I_{\mathcal{K}_*}, \leq_{\mathcal{K}_*}) \ is \ as \ follows:

\[\begin{array}{c}
[1] \\
[3] \\
[0] = [2] \\
\end{array}\]
Fix a valuation $\vDash$ on $(I_{K_0}, \leq_{K_0})$. Let $I_0$ denote the Kripke model $(I_{K_0}, \leq_{K_0}, \vDash)$. Fix propositional variables $p, p', q$ and $q'$. We show $0 \vDash_{\varepsilon_{K_0}, I_0} \text{DML}(\Delta_a)[p, p', q, q']$. If $[0] \vDash p$ or $[0] \vDash q$, then we have $0 \vDash\varepsilon_{K_0, I_0} p$ or $0 \vDash\varepsilon_{K_0, I_0} q$ respectively.

Then we have that $i \vDash \neg(p \land q)$ implies $i \vDash \neg p \lor \neg q$ for all $i \in \{0, 1, 2, 3\}$. Thus we have $0 \vDash\varepsilon_{K_0, I_0} (p \land q) \rightarrow \neg p \lor \neg q$, and hence, we are done. Assume $[0] \not\vDash p, q$. Since $1, 3 \not\vDash \neg (p \land q) \rightarrow \neg p \lor \neg q$ for any valuation $\vDash$, it suffices to show $i \not\vDash\varepsilon_{K_0, I_0} (p \land q) \rightarrow \neg p \lor \neg q$ or $i \not\vDash\varepsilon_{K_0, I_0} (p \leftrightarrow \neg p') \land (q \leftrightarrow \neg q')$ for $i \in \{0, 2\}$. Assume also $[3] \vDash p$. Then $2 \not\vDash\varepsilon_{K_0, I_0} p$ and $3 \not\vDash\varepsilon_{K_0, I_0} p$. If $2 \not\vDash\varepsilon_{K_0, I_0} p \leftrightarrow \neg p'$, then we have $2 \vDash\varepsilon_{K_0, I_0} p'$ and $3 \vDash\varepsilon_{K_0, I_0} p'$, which is a contradiction.

Finally assume $[3] \not\vDash p, q$. Since $2 \not\vDash\varepsilon_{K_0, I_0} \neg p$, we have $2 \vDash\varepsilon_{K_0, I_0} \neg p \lor \neg q$. Now we show $0 \not\vDash\varepsilon_{K_0, I_0} \neg(p \land q) \rightarrow \neg p \lor \neg q$. If $[1] \not\vDash p$ or $[1] \not\vDash q$, then $0 \vDash\varepsilon_{K_0, I_0} \neg(p \land q)$, and hence, $0 \vDash\varepsilon_{K_0, I_0} \neg(p \land q) \rightarrow \neg p \lor \neg q$. If $[1] \not\vDash p$ and $[1] \not\vDash q$, then we have $1 \not\vDash\varepsilon_{K_0, I_0} \neg(p \land q)$, and hence, $0 \not\vDash\varepsilon_{K_0, I_0} \neg(p \land q)$, and hence, $0 \not\vDash\varepsilon_{K_0, I_0} \neg(p \land q) \rightarrow \neg p \lor \neg q$.

**Theorem 4.12.** $\text{HA} + \Delta_a\text{-}\text{DML} \not\vDash \Delta_a\text{-}\text{LEM}$.

**Proof.** Note that the extended frame generated by $K_0$ in Proposition 4.10 is locally directed. Applying Theorem 4.4 to Proposition 4.10 we have that $\text{HA} + \Sigma-T(\varepsilon_{K_0})$ does not prove some $\Sigma^0_1$-instance of $\text{LEM}(\Delta_a)$, and hence, $\text{HA} + \Sigma-T(\varepsilon_{K_0}) \not\vDash \Delta_a\text{-}\text{LEM}$. By Lemma 4.11 and Remark 4.8 we have $\text{HA} + \Delta_a\text{-}\text{DML} \not\vDash \Delta_a\text{-}\text{LEM}$.

As mentioned in [6] Remark 2.5, the second-order versions of our principles, except $\Delta_X$-variants of IP, are equivalent to the first-order counterparts in the presence of Church’s thesis

$$CT : \forall a \exists x \forall y \exists z (T(x, y, z) \land a(y) = U(z)),$$

where $T$ and $U$ are the standard primitive recursive predicate and function from the Kleene normal form theorem (see [13] Section 1.11.7). On the other hand, it is known that $\text{EL} + \text{CT}$ is a conservative extension of $\text{HA}$ (cf. [13] Theorem 3.6.2)). Therefore, for each principles $P$ and $Q$ (except $\Delta_X$-variants of IP), if $\text{HA} + P$ does not proves $Q$ in the language of $\text{HA}$, it follows that $\text{EL} + \text{CT} + P$ does not proves $Q$ in the language of $\text{EL}$. Then we have the following underivability results from Theorems 4.5 4.9 and 4.12

**Corollary 4.13.** $\text{EL} + \text{CT} \not\vDash \Delta_{ab}\text{-}\text{LEM}$.

**Corollary 4.14.** $\text{EL} + \text{CT} + \Delta_{ab}\text{-}\text{LEM} \not\vDash \Delta_a\text{-}\text{DML}$.

**Corollary 4.15.** $\text{EL} + \text{CT} + \Delta_a\text{-}\text{DML} \not\vDash \Delta_a\text{-}\text{LEM}$.
5 Questions

In this paper, we have studied extremely weak principles but non-provable in $\text{HA}$ or $\text{EL}$. Then it is interesting to see the relation of our principles and other weak principles in the literature. One of such principles is so-called weak Markov’s principle $\text{WMP}$ (see [9]). It is known that $\Sigma^0_1$-$\text{WDML} + \text{WMP}$ is equivalent to $\Sigma^0_1$-$\text{DNE}$ [9, Proposition 1.1]. On the other hand, it has been already shown in [4] that $\text{WMP}$ is independent with $\Delta_a$-$\text{LEM}$ over $\text{EL}$. Then an interesting question is whether $\text{WMP}$ derives $\Delta_a$-$\text{DML}$ or $\Delta_{ab}$-$\text{LEM}$ which we haven’t known. Other considerable examples are the double negated variants of our principles. Some of them are already studied in [6] as mentioned in Section 2. In fact, it is possible to show $\text{HA} + \Delta_{ab}$-$\text{LEM} \not\vdash \neg\neg \Delta_a$-$\text{LEM}$ as for [6, Remark 4.13]. However, we don’t know the entire structure of the interrelation between them.

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