Extending toric chiral-factorization algebra theory

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Introduction

The chiral & factorization algebra theory of Beilinson & Drinfeld is yet to be understood in physical applications. It also lacks some dimension and singularity analysis. Let's try such a thing in dimension 2 – where geometric Langlands program does not have a quick application by Arinkin. "Toric" examples were done by the author at the year 2008, but not the cases of non-toric del Pezzo surfaces and so on.

What's chiral-factorization algebra?

The physical origin was in the conformal field theory and Vertex (Operator) Algebra. But Belavin-Pokyakov-Zamolodchikov and Knizhnik-Zamolochikov equation are not minifest in this approach. We do not like the level matching condition and $z, \bar{z}$ dependence of string theory – we are doing almost the same compuatation twice for left and right movers of (anti-)holomorphic coordinates, and Polchinski does not have **a good complex-analytic interpretation** of modular transformation of 1-loop of (complex) torus amplitude after moduli dependence.

Chern-Simons anomaly's 2-form (S., 2007) The holomorphic closed 2-form (chiral de Rham complex's canonical form) has the form of "quantum anomaly" in the following form

$$ch_2(X) = \frac{3(1-n)}{2} \quad (n = 0, 1, 2, \cdots, 6)$$

for n-point generic blow-up of complex projective plane $\mathbb{CP}^2$.

In more general, a quantum BV complex or Beilinson-Drinfeld algebra is
a differential graded-commutative algebra A (whose differential we denote by $\triangle$) over the ring $\mathbf{R}[[\hbar]]$ of formal power series over the real numbers in a formal constant $\hbar$, equipped with a Poisson bracket $\{-,-\}$ of the same degree as the differential such that the following equation holds for all elements $a, b \in A$ of homogeneous degree $|a|, |b| \in \mathbb{Z}$

$$\triangle(a \cdot b) = (\triangle a) \cdot b + (-1)^{|a|} a \triangle b + \hbar\{a, b\}$$

Setup – toric data and Cox rings of total coordinate rings

Vertices $e_i \in \mathbb{Z}^2$, and vectors v_i from the origin to the vertices. $(i = 1, 2, \cdots, N \bmod N)$ The dual basis w_i^1, w_i^2 for affine neighbourhood U_i spanned by the 2 vectors v_i, v_{i+1} is as follows:

$$(v_i, w_i^1) = 1, \quad (v_i, w_i^2) = 0,$$
$$(v_{i+1}, w_i^1) = 0, \quad (v_{i+1}, w_i^2) = 1.$$

The affine neighbourhood $U_i (i = 1, 2, \cdots, N \bmod N)$ is described by

$$Spec\ \mathbb{C}[x^{(w_i^1, E_1)} y^{(w_i^1, E_2)}, x^{(w_i^2, E_1)} y^{(w_i^2, E_2)}],$$

where $E_1 = (1, 0)$, and $E_2 = (0, 1)$ are the standard bases of $\mathbb{Z}^2$.

Coordinate transformation and transition function [gerbes]

Via the Jacobian transformation functions g, g', g'', the anomaly with respct to $U_0 \to U_1 \to U_2 \to U_0$ is

$$d\mu = tr(gdg^{-1})^3, d\mu' = tr(g'dg'^{-1})^3,$$
$$d\mu'' = tr(g''dg''^{-1})^3.$$

The total "anomaly" is

$$\psi = \mu + \mu' + \mu'' + tr(g''dg \wedge dg').$$

The dimension 2 is preferable to eliminate the undetermined freedom μ, μ', μ''.
Remark The μ-term was originally from the anti-symmetric part of B-field of curved $\beta\gamma$ CFT by the following Ansatz of Nekrasov.

$$\tilde{\beta_a} := \beta_i g_a^i + B_{ai} \partial \gamma^i,$$

where $B_a \in \Omega_U^1, g_a \in T_U$ are summed by the Einstein summation of indices. In this model of $\beta\gamma$ CFT, γ is identified with the target space coordinate, and β is identified with the vector field of generalized complex manifold of Hitchin school. The OPE (Operator Product Expansion) is preserved

$$\beta_i(z)\gamma^j(w) \sim +\frac{\delta_i^j}{z-w} + (regular)$$

w.r.t. the Kronecker delta $\delta_j^i = \begin{cases} 1 (i = j) \\ 0 (i \neq j) \end{cases}$

Mathematica Program

Mathematica doesn't have an official package of differential forms. A third-party package of differential forms is already obsolete. We can manually write a code like the following

```
g0[n_] := {D[u[n], x], D[u[n], y]};
g[m_, n_] := Inverse[g0[m]].g0[n];
dgdg[m_, k_, n_] := D[g[m, k], x].D[g[k, n], y] −
D[g[m.k], y].D[g[k, m], x];
anom[m_, k_, n_] := xyTr[g[m, k].dgdg[k, n, m]];
tot[i_, j_, k_] :=
−anom[i, j, k] + m[i, j] − m[i, k] + m[j, k];
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The total Chern-Simons anomaly is in good agreement with the prediction of Hirzebruch-Riemann-Roch theorem via the intersection theory. [summation over triangle decompostions]

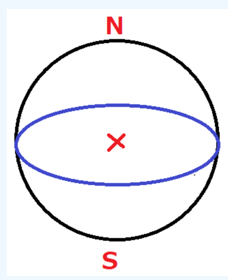
Scope of computational algebraic geometry – smooth and toric?

The "smooth" condition was thought to be important – especially because of differential form-like compuatation. However, Malikov-Schechtman (2014) proposed an analogue of "derived de Rham complex" [Bhatt2012] after the cotangent complex of Illusie (1971,1972) and the homotopical algebra of Beilinson-Drinfeld. It works for singular varieties. In addition, the "toric" condition can be relaxed by introducing the Cox ring for Mori dream space [Hu, Keel 2000] and quasitorus actions, which includes the non-toric Fano cases. My computation was by the topological deformation of non-toric del Pezzo surfaces to nef toric surfaces.

Derived algebraic geometry and deformation theory

The Beilinson-Drinfeld chiral algebra was something like the factorization algebra of observables (later studied by Costello-Gwilliam) plus the sheaves on ind-scheme of "Ziv Ran space" R(X). The Ran space geometry was first studied in the context of contractivity for smooth projective curve X. But the dimension of X can be higher by Francis-Gaitsgory, like the higher-dimensional sigma model of physicists.
We now would like to construct some duality theory of topological field theory and topological conformal field theory – like the so-called AGT correspondence of 2d / 4d spaces. The "class S" chiral algebra theory of Beem, Rastelli et al. is in this direction of physics of Gaiotto curve.
I still would like to understand the higher Kodaira-Spencer classes of Esnault-Vieweg [1994] as an application of such recent results on Ran space and conformal blocks.

Dirac monopole and Brylinski's text



The non-trivial topology and electro-magnetic duality were started from Dirac's monopole and its quantization condition. It can reproduce the level 1 Wess-Zumino-Witten term of Lagrangian

$$L_{WZW}(g) = \frac{1}{4\pi} tr(g^{-1}\partial g g^{-1}\bar{\partial} g)$$
$$+ \frac{1}{12\pi} d^{-1} tr\left[(g^{-1}dg)^3\right]$$

for the Nekrasov's Ansatz

$$\frac{1}{2\pi}\tilde{\beta}\bar{\partial}\tilde{\gamma} = \frac{1}{2\pi}\beta\bar{\partial}\gamma + L_{WZW}(g).$$

Unlike the Dirac monopole (and 't Hooft-Polyakov monopole of QFT), we do not have a fibre bundle formalism. Rather, we have (co)sheave formalisms with the K-theory or algebraic topology with a relation to the algebro-geometric reformulation of Vertex algebroids and geometric qunatization.

References

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