Bakalov's tensor categories and WZW models applied for the chiral operad theory

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In memory of Prof. Tohru Eguchi

Beilinson-Drinfeld's chiral algebra theory – an unreachable milestone of the past 2+ decades.

- A geometric understanding of (2d) algebraic conformal field theory.
- Trials to understand in a 'normal' formalisms – Bakalov & Kac (math). Costello & Gwilliam / Lurie / Ayala & Francis (higher categories for factorization algebras) & Nekrasov & Witten (physics: chiral de Rham complex and curved beta-gamma CFT)
- Relation with (algebro-geometric understanding of) 'quantum' geometric Langlands program – which I visited around 2014.
- But, still not fully understood.

Let us restrict our goal in a realistic way.

Few people I asked have seriously read the book of Beilinson & Drinfeld. I tried its physical realization together with modern formalisms of Lurie, Gaitsgory et al. However, the pages of their books are ranging from 500-1000+ pages.

One thing that I found was that the book itself is not self-contained – it depends on prerequisites of E.Frenkel & Ben-Zvi, Kac's vertex algebras, and original papers & lecture notes.

We must stand on the shoulder of giants in order to finish understanding its mathematical physics in a limited time. Rational conformal field theory (RCFT) of Moore & Seiberg is a good starting point.

Pseudo-tensor categories [Boardman-Vogt], aka colored operads / multicategories

The axiomatic chapter 1 of Beilinson & Drinfeld explained (compound) tensor categories and pseudo-tensor category – which appered in **the subcategory of tensor category** (symmetric monoidal category). Let us focus on it.It is a long way to achieve operads [Stasheff et al, Loday-Vallette] and 'abstract-nonsense' category theory. It has its root in a long time ago of algebraic topology and algebraic geometry of homology algebras. Today, I will try not the homological mirror symmetry conjecture but the S-duality conjecture with its topological nature of homotopical algebras of difficult L_∞ algebras (also the string field theory is being tried to be in scope). But I don't know its relation to Zwiebach's theory nor the IR algebra theory. [Kapranov-Kontsevich-Soibelman2014 / Gaiotto-Moore-Witten2015]

(log-)modular tensor categories – what we knew and what we don't

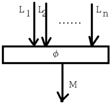
- It is closely related to the Wess-Zumino-Witten (WZW) model of extended Lie algebras and 3-dimensional topology ((extended) TQFT of Atiyah-Segal).
- Construction of WZW models was done by mathematicians quite recently. [Yi-Zhi Huang 2008]
- Verlinde formula of (dimensions of representations of) conformal blocks are done.
  $N_{ijk} = \sum_l S_i^l S_j^l S_k^l / S_0^l = dim\{< \Phi_i \Phi_j \Phi_k >\}$
- There is an $SL(2, \mathbb{Z})$-action – so 'modular' tensor category of Bakalov-Kirillov.
- We don't know how to construct its cousin in the fractional WZW models. One especially doesn't know logarithmic CFT realization of such log-modular tensor categories. [Tensor categories are still a developing area.]

Let us safely say (approximately) Bakalov's tensor categories have WZW modular functor – so what?

We need to understand coset models and central charges. We don't understand the 'derived version of' conformal blocks of Beilinson-Drinfeld chapter 4. Let us try to understand the flag variety of G/B – G: algebraic group (algebraic version of Lie group) and B: its Borel subgroup (upper triangular matrices). The 'pseudo'- tensor category is a bit of generalization of usual tensor category. It was introduced to (differential-geometrically) control the "tensor products of vector fields and differential operators."[Paugam] (together with usual functions and differential forms of de Rham) I tried to understand its fibre sum of tangent bundle and co-tangent bundles (rather, co-tangent complex) of generalized complex structure. [S. 2007] $\tilde{\beta}_a := \beta_i g^i_a + B_{ai}\partial\gamma^i$ [Ansatz of Nekrasov for coordinate transformation of curved $\beta\gamma$-CFT]

Technical details that I skipped

Pseudo-tensor categories are not enough. We need an augmentation functor h (especially, de Rham functor: $h(M) := M \otimes_{\mathcal{D}_X} \mathcal{O}_X = M/M\Theta_X$ for $M \in \mathcal{M}(X)$ of **smooth** worldsheet X and the category $\mathcal{M} = \mathcal{M}^r$ of **right** $\mathcal{D}$-modules) and compound (pseudo-)tensor categories for self-duality to make an augmented compound pseudo-tensor category of Section 1 & 2 of Beilinson & Drinfeld.



: (Pseudo-tensor category of multiple inputs)

Also, I didn't talk about the ribbon category and quantum groups that were mentioned in the book of Bakalov & Kirillov. I didn't explain if the rigid neutral tensor category (Tannakian categories) comes to our consideration, either. "An X-local structure on A provides a flat projective (KZ/Gauss-Manin) connection ∇ on the chiral homology sheaves" [BD §4]

TFT construction of RCFT correlators [Fuchs-Runkel-Schweigert] may be helpful, but...

It is a surprising fact that a (topologically-twisted) CFT is a kind of TFT. This fact is used in the work of Lurie's Higher Algebra. It might be helpful in the understanding of AGT conjecture of 2d/4d correspondence.

Bakalov's lecture notes have the Moore-Seiberg data for a semisimple abelian category $\mathcal{C}$. **Conformal blocks** $\mathcal{C}^{\boxtimes n} \to Vec_f(n \geqq 0)$ / **Rotational isomorphism / A symmetric object R / Gluing isomorphism / Commutativity isomorphism** satisfying axioms $MS1 - MS7$ [Bakalov-Kirillov]. Axioms: Non-degeneracy / Normalization / Associativity / Rotation axiom / Symmetry / Hexagon axiom / Dehn twist. Zhu's c_2-cofiniteness might be important for extending such axioms for non-semisimple case.

My work [S.2007-] of chiral de Rham complex on smooth (pseudo-)projective algebraic surfaces [4 real dimension if the base field is $\mathbb{C}$.]

- I don't know about its relation to the AGT correspondence. [It's similar, but I think it is a different story.]
- I finished $\mathbb{CP}^2$ as well as its toric blowups of $0, 1, 2, 3$ generic points [year 2007]. I extended its results by nef-toric surfaces by looking up Konishi-Minabe. Now, the blowups are non-toric $4, 5, 6$ generic points.

$$ch_2(dP_n) = \frac{3(1-n)}{2}$$

- It turned out that the other minimal model of $\mathbb{CP}^1 \times \mathbb{CP}^1$ or Hirzebruch surfaces have *zero Pontryagin anomaly*.
- Technically, Witten's Hopf surface is not projective.

What should we do, next?

My Ph.D. work was a rather mathematical physics, and not so much on algebraic geometry / algebraic topology for representation theory. It was based on the (holomorphic) OPE calculation [locality axiom] of phycisists. But we know the flag variety G/B is more preferable in view of chiral de Rham complex theory of Malikov-Schechtman-Vaintrob. (also, elliptic curves [Ekeren-Heluani2018] for chiral homology & configuration space.)

What about trying $\mathbb{CP}^1$ model? [Ueno2008] It's easy & simple but has 2 affine neighborhoods – which is enough for a toy model of our consideration. The link between WZW / coset models & chiral de Rham complex for flag varieties will be clear-cut. It is an $\mathbf{sl}_2$-modules on the critical level [MSV]. Its quantum cohomology may be interesting. [Arkhipov-Kapranov] Batyrev's results might be extended to non-toric cases. $HQ(G/B) = H^*(\mathcal{F}) \otimes_{\mathbb{C}[A_+]} \mathbb{C}[A]$ for $A := H_2(G/B)$, $\mathcal{F} \approx \mathcal{L}G$.

Appendix: some definitions in depth.

Definition 1 [Bakalov-Kirillov] A monoidal category is a category $\mathcal{C}$ with the following data and axioms:

- (i) a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- (ii) a functorial isomorphism $\alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$ (associativity isomorphism) of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- (iii) a unit object $\mathbb{1} \in Ob\mathcal{C}$ and functorial isomorphisms $\lambda_V : \mathbb{1} \otimes V \xrightarrow{\sim} V$, $\rho_V : V \otimes \mathbb{1} \xrightarrow{\sim} V$ for $V \in Ob\mathcal{C}$.
- (iv) Associativity axiom Suppose X_1, X_2 are two expressions obtained from $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ by inserting $\mathbb{1}'s$ & brackets. Then all isomorphisms $\phi : X_1 \xrightarrow{\sim} X_2$, composed of $\alpha's, \lambda's, \rho's$ & their inverses, must be equal.
- (v) ('additive' monoidal categories) $\mathbb{1}$ is a simple object (not isomorphic to zero & every subobject is isomorphic to zero / to $\mathbb{1}$) in $\mathcal{C}$ & $End_{\mathcal{C}}(\mathbb{1}) = k.(ch(k) = 0)$

Braids, symmetric tensor category

Definition 2 [Bakalov-Kirillov] A braid in n strands is an isotopy class of a union of n non-intersecting segments of smooth curves (strands) in $\mathbb{R}^3$ with end points $\{1, 2, \cdots, n\} \times \{0\} \times \{0, 1\}$, such that for each of these strands the third coordinate z is strictly increasing from 0 to 1. All braids form a group B_n called the braid group in n strands.
Definition 3 [Bakalov-Kirillov] A braided tensor category (BTC) is a category $\mathcal{C}$ with $\otimes, \mathbb{1}, \alpha, \lambda, \rho, \sigma$ (functorial isomorphism of monoidal cateogry) as above, such that for any two expressions X_1, X_2 of the form $((V_{i_1} \otimes V_{i_2}) \otimes (\mathbb{1} \otimes V_{i_3})) \otimes \cdots \otimes V_{i_n}$ and any $\phi : X_1 \xrightarrow{\sim} X_2$ obtained by composing $\alpha's, \lambda's, \rho's, \sigma's$ and their inverses, ϕ depends only on its image in the braid group B_n. A BTC $\mathcal{C}$ is called a symmetric tensor cateogry (STC) if all isomorphism σ satisfy $\sigma_{WV}\sigma_{VW} = id_{V\otimes W}$.

Conformal blocks τ^* & the space of coinvariants τ

Definition 4 [Bakalov-Kirillov] The space of coinvariants is the vector space

$$\tau(C, \vec{p}; V) \;\; := \;\; V_{\mathfrak{g}(C-\vec{p})} = V/\mathfrak{g}(C-\vec{p})V,$$

for a compact non-singular complex curve C and the Lie algebra $\mathfrak{g}(C-\vec{p}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}(C-\vec{p})$ where $\vec{p} = (p_1, \cdots, p_n)$ are n distinct points. We used the theory of tensor products by $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$, where $V_1, \cdots, V_n \in Ob\mathcal{O}_k$ and $\mathcal{O}_k$ for **positive integer level** k (the category of $\hat{\mathfrak{g}}$-modules of level k that have weight decomposition with finite-dimensional weight subspaces, such that the action of $\hat{\mathfrak{g}}^+$ is locally nilpotent and the action of $\mathfrak{g}$ is integrable.) Apart from the Weyl module ($k \in \mathbb{C}$), the integrable subcategory $\mathcal{O}_k^{int}$ of integrable modules $\mathcal{O}_k$ has a structure of a modular tensor category. The space of conformal blocks is the dual vector space.

Pseudo-tensor category in the original form of Beilinson-Drinfeld [BD2004]

Definition 5 [BD] A pseudo-tensor category is a class of objects $\mathcal{M}$ together with the following datum:
(a) For any $I \in \mathcal{S}$, an I-family of objects $L_i \in \mathcal{M}, i \in I$, and an object $M \in \mathcal{M}$ one has **the set** $P_I^{\mathcal{M}}(\{L_i\}, M) = P_I(\{L_i\}, M)$. (set of I-operations) ($\mathcal{S}$: the category of finite non-empty sets and surjective maps. (Indices))
(b) For any map $\pi : J \twoheadrightarrow I$ in $\mathcal{S}$, families of objects $\{L_i\}_{i\in I}, \{K_j\}_{j\in J}$, and an object M one has the composition map

$$P_I(\{L_i\}, M) \times \Pi_I P_{J_i}(\{K_j\}, L_i) \to P_J(\{K_j\}, M), (\phi, (\psi_i)) \mapsto \phi(\psi_i)$$

Here in the notation $P_{J_i}(\{K_j\}, L_i)$ we assume that $j \in J_i$. The composition is associative. An identity element $id_M \in P_\cdot(\{M\}, M)$ exists.