Examples and possible applications of concrete compactifications of chiral algebras

Makoto Sakurai

Kaichi Gakuen

4-March-2021

The 17th Mathematics Conference for Young Researchers

The S-duality conjecture is not solved. (Physical motivation)

When I was a graduate student, I was working on loop groups, Hitchin systems, and Beilinson-Drinfeld's chiral algebra theory. It was said to be related to the Vafa-Witten S-duality. After the Strings 2005, I visited Seattle'05 of AMS. After that, E.Witten announced the gauge theoretical solution to the geometric Langlands conjecture at Luminy. It was said to be the S-duality of gauge theory.

Altough it has something to do with physicists' (type IIB's) S-duality, I still think the S-duality in general is not solved. Polchinski's textbook says the (bosonic) torus amplitude implies space-time dimensionality 26 (real dimension) by the modularity. However, the general proof of the S-duality of (super-)string theory is not available. The dilaton field is especially unavailable.

Chiral algebra theory stems from the vertex algebras.

Vertex algebras are a close cousin of VOAs (Vertex Operator Algebras) minus the Virasoro operators and $\mathbb{Z}$-grading. The Chiral algebra theory is an algebraic system which can be regarded as sheaf of vertex algebras. Let us quote the nLab:

Beilinson and Drinfel'd were unhappy with non-illuminating definition of vertex operator algebra and invented a mathematical definition of version of chiral conformal field theory on an algebraic curve, under the name **chiral algebra**; *their manuscript has being circulating from around 1995 as a long preprint and being more recently published by Amer. Math. Soc.*

It is a "holomorphic CFT" on Riemann surfaces. Like the Liouville theorem, there are poles for the operators (except for constant operators on the whole plane), which are crucial for our "OPEs".

Schemetic language to learn: (pre-)sheaf theory on space

Definition 0 [Hartshorne II.1] Let X be a topological space. A presheaf $\mathcal{F}$ of †abelian groups on X consists of the data
(a) for every open subset $U \subset X$, an abelian group $\mathcal{F}(U)$, and
(b) for every inclusion $V \subset U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,
subject to the conditions
(0) $\mathcal{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,
(1) ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
(2) if $W \subset V \subset U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 1 [Hartshorne II.1] †abelian group(s) can be replaced by "ring", "set", or "object of $\mathcal{C}$". The above definition can be rephrased that a presheaf is just a contravariant functor from the category $Top(X)$ to the category $\mathcal{A}b$ of abelian groups.

Gluing condition

Definition 2 [Hartshorne II.1] A presheaf $\mathcal{F}$ on a topological space X is a sheaf if it satisfies the following supplementary conditions:
(3) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s\,|_{V_i} = 0$ for all i, then $s = 0$;
(4) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each $i, j, s_i\,|_{V_i \cap V_j} = s_j\,|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s\,|_{V_i} = s\,|_{V_j}$ for each i.

Remark 4 One is possibly inclined to ask what kind of topology we are working on. I will work on the Zariski topology which is almost never Hausdorff. However, I dare not adopt the higher topos theory developed by Lurie. I will work in an old-fashioned style.

Malikov-Schechtman(-Vaintrob): [MSV] Definition of chiral de Rham complex

I do not recall the defition of VOAs (long axioms). I don't recall the definition of chiral algebras by Costello-Gwilliam, either. I "defined" the $\beta\gamma$-CFT's commutation relation in a brief style at the abstract. To be more precise:

Definition 5 For a smooth manifold X over $\mathbb{C}$, define

$$\begin{aligned}
[(\beta_i)_n, \gamma^j_m] &= \delta^j_i \delta_{m,-n} C \\
\gamma^i(z) &= \sum_{n\in\mathbb{Z}} \gamma^i_n z^{-n}, \\
\beta_i(z) &= \sum_{n\in\mathbb{Z}} (\beta_i)_n x^{-n-1},
\end{aligned}$$

where C is a constant that will be mapped to 1 later in the polynomial ring.

chiral dR (continued)

Definition 5 (continued) γ has conformal dimension 0. and β has 1. In addition, $\gamma^i_n(n \geqq 1)$ are the annihilation operators, and $\gamma^i_n(n \leqq 0)$ are the creation operators. Likewise, $(\beta_i)_n$ are the annihilation operators, and $(\beta_i)_n(n \leqq -1)$ are the creation operators. The normal ordered product is defined recursively

$$:xB := \begin{cases} Bx & (if \ \ x \ is \ an \ annihilation \ operator) \\ xB & (otherwise) \end{cases}$$

where $B \in End(V_N)$ for the "state space" V_N. We will regard γ's as the coordinates.

I was anxious about whether the infinite-dimensionality will violate the Čech cohomology arguments. This argument is not perfect, but there is some benefit in this formalism.

Stress-energy tensor L (if it exists) and OPEs among them

Remark 6 (OPE relations) If it is assumed $|z| > |w|$,

$$\beta_i(z)\gamma^j(w) = \frac{\delta_i^j}{z-w} + (regular)$$

Similarly,

$$\begin{aligned} \beta_i(z)\beta_j(w) &= (regular), \\ \gamma^i(z)\gamma^j(w) &= (regular). \end{aligned}$$

The stress energy tensor is given by

$$L(z) = :\partial_z\gamma(z)\beta(z):.$$

The OPE of the stress-energy tensor is

$$L(z)L(w) \sim \frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}.$$

$bc - \beta\gamma$-CFT is defined (Chiral fermions)

The conformal dimension of b_i and c^j are 0 and 1 (details abbreviated, but "fermionic ghosts").

$$\beta_i(z)\gamma^j(w) \sim \frac{\delta_i^j}{z-w}, \gamma^i(x)\gamma^j(w) \sim 0, \beta_i(z)\beta_j(w) \sim 0.$$

In addition,

$$\begin{aligned} b^i(z)c_j(w) &\sim \frac{\delta_j^i}{z-w}, b^i(z)b^j(w) \sim 0, c_i(z)c_j(w) \sim 0, \\ \gamma^i(z)b^j(w) &\sim 0, \gamma^i(z)c_j(w) \sim 0, \\ \beta_i(z)b^j(w) &\sim 0, \beta_i(z)c_j(w) \sim 0. \end{aligned}$$

Topological Vertex Algebras of rank D
(D is the complex dimension of the "target space" X)

Definition 7

$$
\begin{aligned}
L &= \sum_i [: \partial\gamma^i(z)\beta_i(z) : + : \partial b^i(z)c_i(z) :] \\
J &= \sum_i : b^i(z)c_i(z) :, \\
Q &= \sum_i : \beta_i(z)b^i(z) :, \\
G &= \sum_i : c_i(z)\partial\gamma^i(z) : .
\end{aligned}
$$

The conformal dimension of Q, G are $1, 2$.

OPEs without considerations to supersymmetry algebras

Remark 8 (OPEs of topological vertex algebras (1/2))

$$
\begin{aligned}
L(z)L(w) &\sim \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}, \\
J(z)J(w) &\sim \frac{D}{(z-w)^2}, \\
L(z)J(w) &\sim -\frac{D}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w}.
\end{aligned}
$$

Cohomology w.r.t. Q, G. (Supersymmetry algebras)

Remark 9 (OPEs of topological vertex algebras (2/2))

$$\begin{aligned}
G(z)G(w) &\sim 0, L(z)G(w) \sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w}, \\
J(z)G(w) &\sim -\frac{G(w)}{z-w}, \\
Q(z)Q(w) &\sim 0, L(z)Q(w) \sim \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{z-w}, \\
J(z)Q(w) &\sim \frac{Q(w)}{z-w}, \\
Q(z)G(w) &\sim \frac{D}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w)}{z-w}.
\end{aligned}$$

Since $Q^2 \sim 0 \sim G^2$, we can consider the cohomology theory. By the mode expansion $Q(z) = \sum_{n\in\mathbb{Z}} Q_n z^{n-1}$, we obtain $[Q_0, G(w)]_+ = L(w)$ and the theory becomes topological.

Redefinition of chiral algebras by Nekrasov's lecture [*After Bressler and the Hitchin school*]

The definition of MSV was physics-oriented and not coordinate-independent. One will try to define the chiral de Rham complex by regarding γ's as coordinates and β's as generalized vector fields.

Definition 10 For a holomorphic tangent vector $V \in T_U$, and a holomorphic 1-form $B \in \Omega^1_U$ (the sections at a coordinate neighbourhood U)

$$\begin{aligned} J_V &= \; :\beta_i V^i(\gamma)(z): \\ &:= \lim_{\epsilon\to 0}[\beta_i(z+\epsilon)V^i(\gamma(z)) - \frac{1}{\epsilon}\partial_i V^i(\gamma(z))], \\ C_B &= B_i(\gamma(z))\partial\gamma^i. \end{aligned}$$

Then C_B and J_V are conformal dimension 1, but J_V has an extra $(z-w)^{-3}$ term.

Nekrasov's Ansatz (assumption) for coordinate transformation(s).

We can make a formula for the OPEs of operators of the form $J_v + C_\xi, J_W + C_\eta$. Let us omit such computation details. (See the sections 2.5, 2.6 of my paper.)
Ansatz 11 For $\gamma \in U \cap U' \neq \varnothing$ of two coordinate neighborhoods U, U', assume that the coordinate transformation is written as

$$\tilde{\beta}_a \quad = \quad \beta_i g_a^i + B_{ai}\partial\gamma^i = J_{g_a} + C_{B_a}.$$

where the indices are summed by the Einstein summation convention of tensor calculus. By the compatibility w.r.t. the $\gamma\gamma \sim 0$ of the tatget space coordinates and $\tilde{\beta}_a(z)\tilde{\gamma}^b(w) \sim \frac{\delta_a^b}{z-w}$ we obtain the Jacobian matrix $g_a^i = \frac{\partial \gamma^i}{\partial \tilde{\gamma}^a}$.

Consequence of the Nekrasov's Ansatz

Remark 12 For the remaining equation $\tilde{\beta}_a\tilde{\beta}_b \sim 0$, one considers the decomposition of the "B-field" $B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab})d\tilde{\gamma}^b$ into symmetric part $\sigma_{ab} = \sigma_{ba} = \sum_{i,j} \partial_i g^j_a \partial_j g^i_b$ (fixed) and the antisymmetric part μ by

$$d\mu \;\; = \;\; tr(g^{-1}dg)^3$$

for the coordinate transformation $U \to U'$ and their Jacobian matrix g.
Note that this computation is somewhat heavy and one uses the †Maurer-Cartan equation and monopole analogy of level 1 Wess-Zumino-Witten term. About the †MC equation, group manifolds, and the quantum master equation, we have a study by string field theory, which I will omit in this talk.

Witten's paper and its generalization while using Nekrasov's Ansatz

Ansatz 13 Witten's paper of $\mathcal{N}=(0,2)$ model (Do not confuse it with the 6-dim $\mathcal{N}=(2,0)$ theory of another context) Witten make a pole analysis and under $U_0 \to U_1 \to U_2 \to U_3$

$$\begin{aligned} \gamma^j &\to \gamma^j \\ \beta_i &\to \beta_i' = \beta_i + f_{ij}\partial\gamma^j, \end{aligned}$$

where $f_{ij} = -f_{ji}$ is an anti-symmetric tensor field. Witten computed $\mathbb{CP}^2$ and Hopf surfaces as the example.

Result 14 [S. 2007-] For the generic $n-$point blowups of complex plane $\mathbb{CP}^2$, we have the 2nd Chern chracter formula:

$$ch_2(dP_n) = \frac{1}{2}(c_1^2 - 2c_2) = \frac{3(1-n)}{2},$$

where n is $0, 1, 2, 3$ for the original paper and $4, 5, 6$ for the update.

Sketch of the algorithm (1/2)

By a somewhat heavy computation,

$$\gamma^i \quad \to \quad \gamma^i, \beta_j \to \beta_j - \frac{1}{2}(\psi_{\alpha\beta\gamma})_{lj}\partial\gamma^l,$$

and the "total anomaly" $\psi_{\alpha\beta\gamma}$

$$\psi_{\alpha\beta\gamma} \quad = \quad \mu_{\alpha\beta} + \mu_{\beta\gamma} + \mu_{\gamma\alpha} - tr(g'' dg' \wedge dg)$$

can be computed by the triagulation of the coordinate neighbourhoods' changing diagram.

Toric Data 15 ($n = 0, 1, 2, 3$) For dP_n

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix},$$

Sketch of the algorithm (2/2)

Toric Data 16 ($n = 4, 5, 6$) There are some nef toric surfaces W_n that are isomorphic to dP_n: [See Konishi-Minabe's paper of topological vertex]

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 & -2 & -1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \end{pmatrix},$$

Discussion: Non-perturbative definition of Chrial algebras

By the consideration of Witten index, Tan-Yagi (2008) derived a non-perturvative version of chiral algebras and instanton effects for $\mathbb{CP}^1$ target space. The author also knew something about the loop space technique among Arkhipov-Kapranov, Kapranov-Vasserot, and Malikov-Schechtman for the small quantum cohomology ring QH and the Dirac operator (and heat kernels by Bismut) on the loop space and index theorem. The generalization to toric Fano manifolds are by Batyrev rings.

The author thinks, together with the D-module technology of Givental et al, the logarithmic CFT will be important for the open Gromov-Witten invariants together with the holomorphic anomaly of BCOV. In addition to Hosono-Saito-Takahashi and Klemm, Bryan-Leung has a modular form conjecture (of Yau-Zaslow) for 1/2 K3 (rational elliptic surfaces). I would like to obtain the chiral-factorization algebra theory derivation of such results.