Topological chiral conformal algebras related to Batalin-Vilkovisky algebras

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BV algebras

It is a generalization of gauge theory. It consists of antifields, the antibracket, and the operator $\triangle$ in the quantum master equation. Together with the TCFT usage by Costello, there are a number of geometric formalisms including the usage by Voronov. However, it is a formalism with a homological interpretation of gauge theory and not so popular among physicists.
Schematically, it is written as follows:

$$\begin{aligned} \triangle^2 &= 0 \\ Q &= \{S, \} \; (BRST\ operator) \\ \{S, S\} - 2i\hbar\triangle S &= 0 \end{aligned}$$

by the fact of $\boxed{\triangle e^{iS/\hbar} = 0}$. Note that the BV Laplacian $\triangle$ is not a derivation w.r.t. the BV multiplication which I omit to define.

TCFT = Topological Conformal Field Theory by G.Segal

There is a Feynman diagram-like "definition" of "topological" CFT. It is like a cobordism of differential topology of Donaldson, Milnor et al.
Although it a good formalism w.r.t. the topological sigma models of Witten, we do not have a good relation with the type IIB string theory of physicists. We do not have the dilaton field that is responsible for the S-duality conjecture of physicists.
This S-duality is not equal to the S-duality of gauge theory, which mathematicians and the Kapustin-Witten papaer claimed to have proved in the last two decades.
$\exists SL(2,\mathbb{Z})$ action on the moduli τ of the upper half plane $\mathbb{H}$ and its subsequent generating function (and the free energy by logarithm).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

Malikov-Schechtman-Vaintrob's definition of $bc - \beta\gamma CFT$

For even parity ("bosons"):

$$\begin{aligned} \beta_i(z)\gamma^j(w) &\sim \frac{\delta_i^j}{z-w} \\ \beta_i(z)\beta_j(w) &\sim 0 \\ \gamma^i(z)\gamma^j(w) &\sim 0 \end{aligned}$$

Others:

$$\begin{aligned} b^i(z)c_j(w) &\sim \frac{\delta_j^i}{z-w}, \\ b^i(z)b^j(w) &\sim 0, c_i(z)c_j(w) \sim 0, \\ \gamma^i(z)b^j(w) &\sim 0, \gamma^i(z)c_j(w) \sim 0, \\ \beta_i(z)b^j(w) &\sim 0, \beta_i(z)c_j(w) \sim 0. \end{aligned}$$

$bc-\beta\gamma$ CFT and its subalgebra of curved $\beta\gamma$-CFT

A topological vertex algebra of rank D (of complex dimension for the smooth quasi-projective variety X. The quasi-projective condition is necessary for the definition of Chern classes.)

$$\begin{aligned} L &= \sum_i \left[: \partial\gamma^i(z)\beta_i(z) : + : \partial b^i(z) c_i(z) :\right]. \\ J &= \sum_i : b^i(z) c_i(z) : \\ Q &= \sum_i : \beta_i(z) b^i(z) : \\ G &= \sum_i : c_i(z) \partial\gamma^i(z) : \end{aligned}$$

The $Q^2 = 0 = G^2$ condition makes the algebra "$\mathcal{N} = 2$" supersymmetry algebra. The "new" stress-energy tensor is by the "topological half-twist" $T = L - \frac{1}{2}\partial J$, the stress-energy becomes BRST exact & the physical quantity becomes topological.

holomorphic tangent and holomorphic cotangent space

After Bressler's paper, Nekrasov introduced the Ansatz (assumption) for the coordinate change: $\beta\gamma$-system is regarded as the pair of γ^i 's (holomorphic coordinates) and β_i's (generalized fibre sum's coordinate).

$$\tilde{\beta}_a \quad := \quad \beta_i g_a^i + B_{ai}\partial\gamma^i$$

From now on, the summation of indices is omitted by the Einstein summation convention of tensor calculus. The B_a is a holomorphic one-form and g_a is a holomorphic tangent vector. The condition of topological vertex algebra is simplified and the theory is coordinate-independent (but not coordinate-free as was Beilinson-Drinfeld).

My work in progress [S.2007-]

I verified the computation of MSV, Nekrasov, and Witten in an explicit way. In particular, I dared to extend the Nekrasov and Witten paper to del Pezzo surfaces and Hirzebruch surfaces. The Chern character computation is done by the OPE regularization.

$$ch_2(F_k) = 0, ch_2(dP_n) = \frac{1}{2}(c_1^2 - 2c_2) = \frac{3(1-n)}{2}.$$

It was originally done for $n = 0, 1, 2, 3$, but now that I looked up the toric data for nef toric surfaces by Konishi-Minabe, I concluded the same formula for $n = 4, 5, 6$. The computation algorithm is the triangulation of the coordinate changing diagram and the determination of the anti-symmetric part μ (anomaly 2-forms) of B-fields by the Jacobian matrices and their determinants.

Remaining problems: 1. Kapranov-Vasserot, 2. *n*-point correlation functions, and 3. String field theory

My work was said to be a "nice review of curved $\beta\gamma$-CFT", but there were some remaining problems:
No. 1: The formal loop space methods adopted by Kapranov-Vasserot after Konsevich and Denef-Loser were highly schematic and one needed some (higher) category theory to control such a factorization monoid structure on the formal loop / arc spaces. One especially doesn't know "why" Chern characters appeared after the Atiyah class and "how" it can be generalized to other higher Chern characters by the "local Riemann-Roch". Kapranov didn't utilize Hochschild (co)homology for factorization algebras and Atiyah class [Calaque-Van den Bergh?]. Differential form-like computation implies the importance of Hochschild-Konstant-Rosenberg theorem.

No.2: My observation about the *n*-point functions

Kapranov's lectures at the Algebra Symposium said the Ziv Ran space for the factorization algebra was for the second-quantization. It is very much like the Hilbert scheme of points, but the multiplicities must be considered.

About the No. 2, I have tried to study the modular tensor categories after Bakalov. Although late Prof. Eguchi once suggested that the $\beta\gamma$-system may be obtained from path-integral of $bc - \beta\gamma$-CFT (about their bc fields), it is not obviously possible to do that. The factorization algebra structure came from topological field theory structure, rather than TCFT. The relation between TCFT and TFT is not clear-cut for the author. Holomorphic anomaly for the rational elliptic surface [BCOV, Hosono-Saito-Takahashi, Klemm] may be helpful for modular forms. It may decategorify (n-category to $(n-1)$-category) the topological string computation of generating function / free energy.

No.3: String field theory and L_∞-algebras

Although many attempts were made, the differential graded Lie algebras (dgLa) of factorization algebras are not understood in the physics community. The close cousin is the open-closed string field theory. Nevertheless, the A_∞ structure of directed Fukaya category [Kapustin-Orlov: A-model] is not manifest in the formalism of Costello-Gwilliam, yet.
The observation is that the Chern-Simons type string field theory of Witten (1986) is close to $\beta\gamma$-CFT.

$$
\begin{aligned}
S &= \frac{1}{g^2}\int\left[\frac{1}{2}\Psi Q_B\Psi+\frac{1}{3}\Psi\cdot(\Psi*\Psi)\right](action) \\
& \Psi[X^\mu(\sigma),b,c]\ (string\ \ field, 0\leqq\sigma\leqq\pi)
\end{aligned}
$$

It is especially a "bosonic" (even-parity) graded superalgebras. I don't know what about Zwiebach or HIKKO (Hata-Itoh-Kugo-Kunitomo-Ogawa) type string field interactions.