Factorization algebras and the quasi-modular form conjecture of mirror symmetry

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BV algebras and factorization algebras

After Beilinson-Drinfeld, Costello-Gwilliam in a sense generalized [but a direct relation is not clear] the factorization algebras (categorically equivalent to chiral algebras) from complex algebraic curves to other differentiable manifolds. It is after the "nice" cosheaves of vector spaces or cochain complexes on a manifolds M, while the functor $Sym\ F : U \mapsto Sym(F(U))$ is a factorization algebra. [Cite: Definition 6.1.3 of CG in short.] A prefactorization algebra on M with values in a multicategory $\mathcal{C}$ is a factorization algebra if it has the following property: for every open subset $U \subset M$ and every Weiss cover $\{U_i \mid i \in I\}$ of U, the sequence

$$\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_k \mathcal{F}(U_k) \to \mathcal{F}(U) \to 0$$

is exact. [Weiss topology] Let U be an open set. A colletion of open sets $\mathfrak{U} = \{U_i \mid i \in I\}$ is a Weiss cover of U if for any finite collection of points $\{x_1, x_2, \cdots, x_k\}$ in U, there is an open set $U_i \in \mathfrak{U}$ such that $\{x_1, \cdots, x_k\} \subset U_i$.

Si Li's work at String-math 2021 after Ekeren-Heluani

The chiral index of elliptic curve E_τ

$$Ind^{BCOV}(E_\tau) = Tr\ q^{L_0 - 1/24} \exp[\frac{1}{\hbar} \sum_{k \geqq 0} \oint_A \eta_k \frac{W^{(k+2)}}{k+2}],$$

where
$W^k(\partial_z \phi) = (\partial_z \phi)^k + O(\hbar)$
are the bosonic realization of $W_{1+\infty}$-algebra.

Si Li suggested that the chiral index above coincides with the "stationary" Gromov-Witten invariants on the mirror elliptic curve. The details are not accessible until now.

My work on the $(bc-)\beta\gamma$ CFT's chiral algebras

Even parity ("bosons") OPEs are

$$\beta_i(z)\gamma^j(w) \sim \frac{\delta_i^j}{z-w}, \beta_i(z)\beta_j(w) \sim 0, \gamma^i(z)\gamma^j(w) \sim 0.$$

(where δ_i^j is the Kronecker delta) and similar OPEs with bc ghost ("odd parity") fields.

The coordinate transformations give rise to the obstruction 2−form of the chiral de Rham complex. In the cases of (toric) del Pezzo surfaces and Hirzebruch surfaces, 2nd Chern characters are in agreement with the Hirzebruch-Riemann-Roch theorem

$$\begin{aligned} ch_2(dP_n) &= \frac{3(1-n)}{2} \; (n = 0, 1, 2, \cdots, 6) \\ ch_2(F_k) &= 0. \end{aligned}$$

Gwilliam on factorization algebras & perturbative QFTs.

Gorbounov-Gwilliam-Williams published an Astérisque article (2020) that is on the "holomorphic σ-model" (infinite volume limit) giving rise to the above-mentioned $\beta\gamma$-CFT (after Nekrasov and Witten). This treatment of BV/BRST method is not equal to but similar to my article.

The L_∞-algebras and observables of QFTs are central ideas of him. The Stasheff-like argument may suggest the link to Zwiebach's string field theory – which I know only a little.
The deformation quantization is likely to be in discussion, but not the geometric quantization of symplectic manifolds (espacially hyperkähler) by physicists (and some geometers).

Local mirror symmmetry on Hirzebruch or rational elliptic surface (and E-string theory) [1/2]

For K3 surfaces [Bryan-Leung] and genus g curves,

$$\begin{aligned} F_g(q) &:= \sum_{n=0}^{\infty} N_g(n) q^n \\ &= (\frac{d}{dq} G_2(q))^g \frac{q}{\triangle(q)} \end{aligned}$$

and rational elliptic surface [Bryan-Leung]

$$F_g(q) = (\frac{d}{dq} G_2(q))^g (\frac{q}{\triangle(q)})^{1/2},$$

the Gromov-Witten potential is written as the quasi-modular form. $\triangle(q) = \eta(\tau)^{24}$ is the weight 12 modular form and G_2 is the Eisenstein series (quasi-modular form)
$G_2(q) = -\frac{1}{24} + \sum_{k=1}^{\infty} \sigma(k) q^k$ (where $\sigma(k) = \sum_{d|k} d$).

Local mirror symmmetry on Hirzebruch or rational elliptic surface (and E-string theory) [2/2]

After Bryan-Leung, I conjectured the generating functions (Gromov-Witten potential) of the target space of topological string theory are in certain good conditions quasi-modular forms. I talked about it several times since 2005. The origianl Bryan-Leung was curve counting and mirror symmetry of Yau-Zaslow conjecture (not so much on CFTs). The power 24 and 12 are said to be related to the Euler number of the target space-time manifold.

The cases of (local) $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2, \mathbb{C}^3/\mathbb{Z}_3$ are conjectured by Aganagic-Bouchard-Klemm as well (2008, 2006 preprint). Coates-Iritani [Fock sheaf] (2018) and Fang-Ruan-Zhang-Zou (2018-19 $K_{\mathbb{F}_1}$ and $K_{\mathbb{WP}(1,1,2)}$ as well) gave some affirmative answers.

Observation

The generating function of mirror symmetry (Gromov-Witten potential) itself is diffuclt to compute. It is especially not easy to introduce the "target space-time" to the chiral algebra theory of Beilinson-Drinfeld. The infinite volume limit of holomorphic sigma-model is a rare case where we can introduce such a chiral algebra theory. I don't know how to compute the partition function of curved beta-gamma CFTs in such a case.

The chiral de Rham complex on Hirzebruch surfaces and $\mathbb{P}^1 \times \mathbb{P}^1$ are well-defined, but the Chevalley-Cousin complex of Beilinson-Drinfeld chiral algebra is a global structure and may be different from just a chiral de Rham complex. In other words, I don't know whether chiral homology is defined directly from chiral de Rham complex.

Distinction between source space (Riemann surface as world-sheet) and the target space-time manifold

In dimension 1 of the work by Si Li, the distinction between "world-sheet" Riemann surface and the "space-time" target space is relatively unnecessary. Nevertheless, in the practical situations of target space with complex dimension 2 or 3, this distinction must be mandatory. Unfortunately, the work by Beilinson-Drinfeld is not sharp at the target space-time. The BD algebras are mainly on the dimension 1 case and some "sprouts" of "derived" math, though the infinite dimensional Grassmann is introduced.

In the work of Ekeren-Heluani, there was a global coordinate for elliptic curves and it was the key. I might try the case of orbifolds $\mathbb{C}^2/\mathbb{Z}_n, \mathbb{C}^3/\mathbb{Z}_n$ as the global coordinate cases, but one still does not know how to compute the chiral index in general. Some familiarity with both Si Li's ongoing work and Coates-Iritani's Fock sheaf is necessary to solve such a circumstance.