Topological chiral algebras and characteristic classes

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Beilinson-Drinfeld chiral algebra [AMS 2004] and physics

The chiral half (z dependence of $z = x + yi, \bar{z}$ function) of string theory's operator algebras were axiomatized as "VOA" (Vertex-Operator-Algebra) theory long ago. The relation to Riemann surfaces with some genus was studied as Tsuchiya-Ueno-Yamada [1989]. It is related to vector bundles on moduli spaces and chiral conformal blocks (and KZ/Gauss-Manin connections on them). Relations to gauge theory were studied (not AdS/CFT correspondence of physical string) from beginning, but the so-called AGT correspondence of 4d/2d theories (and Schur polynomials) has become popular quite recently.

This talk is on 4-dimensional target spaces – smooth & with orbifold singularities – that are regarded as complex quasi-projective surfaces. It's before AGT & Costello-Gwilliam / their Astérisque, but the author started it as an incarnation of Beilinson-Drinfeld theory of chiral-factorization algebras.

Digression: Relation to deformation / geometric quantization [revisited]

The author was trying to seek for the relation between (Fedosov) quantization and geometric quantization before he wrote the Ph.D. thesis. The author thought it was related to Soibelman's works, but he was not sure about the explicit formalism.

On the other hand, Gaiotto-Witten [2008: Branes And Quantization, 2021: Probing Quantization Via Branes] suggested that physics has the relation between deformation and geometric quantization by assuming D-brane languages. It seems that Sharpe et al. has nice works to translate physicists' D-brane theory (especially topological A-/B-models) into some sheaf theory.

The author would like to understand such a circumstance by use of intersection homology and perverse sheaves, but it is not possible until now.

Chiral de Rham complex as the large volume limit [formal geometry?] according to the BV-BRST justification by Gorbounov-Gwilliam-Williams [Astérisque 2020]

As compared to Malikov-Schechtman-Vaintrob [CMP 1999]'s chiral de Rham complex theory, one must change the $bc - \beta\gamma$ system to $\beta\gamma$ system. Some notational problem occurs when one compares its formalism to (a kind of) physicists' holomorphic $\sigma-$model.

The author will try to write down the "Atiyah class" – an obstruction to constructing a global connection for a complex vector bundle. Nowadays its Atiyah class is widely used, but one concentrates the cases of "2nd Chern character" interpretation via the conformal (total) anomaly after Nekrasov's lectures.

Its 2nd Chern character can be compared to the spin structure. [See Donaldson's review of Atiyah's work at Bulletin of the AMS 2021] Like the (algebraic) index theorem – one can define topological invariants even if there are obstructions / anomalies.

Definitions of generalized complex geometry after Hitchin school [Nekrasov's lectures / Bressler's works]

Definition 1 Let X be a smooth complex quasi-projective surface which has the generalized complex structure. Then by use of Courant-Dirac brackets, one can define the inner product of generalized holomorphic vector fields.

Definition-Assumption 2 Let $\gamma^i (i = 1, 2)$ be the local complex coordinates of above-mentioned surface X. Let $\beta_i (i = 1, 2)$ be generalized holomorphic vector fields. Then the Nekrasov Ansatz is a relation, for coordinate neighborhood U of the intersection of γ^i and $\tilde{\gamma}^a$: $\tilde{\beta}_a := \beta_i g_a^i + B_{ai} \partial \gamma^i$,
where $B_a \in \Omega_U^1, g_a \in T_U$ are summed by the Einstein summation of indices.

Axiomatic assumption for Operator-Product-Expansions

Assumption 3 The OPE is preserved under the above-mentioned coordinate change(s): $\gamma^i \to \tilde{\gamma}^a$

$$\begin{aligned} \beta_i(z)\gamma^j(w) &\sim +\frac{\delta_i^j}{z-w} + (regular) \\ \tilde{\beta}_a(z)\tilde{\gamma}^b(w) &\sim +\frac{\delta_a^b}{z-w} + (regular), \end{aligned}$$

w.r.t. the Kronecker delta. Note that original Beilinson-Dinfeld (coordinate-free and $\mathcal{D}$-modules / pseudo-tensor categories are introduced) is more abstract about the definition this OPEs. [Like the blow-up of non-quasi-coherent sheaf] Kapranov-Vasserot's theory of motivic integration-like argument is more preferable in view of coordinate-independent theories, but the author thinks differential form-like theory is enough for the justifications.

Anti-symmetric μ-terms as the moduli of the theory.

Proposition 4 If we decompose the B-field B_a (of the Nekrasov Ansatz) as the symmetric part σ and the anti-symmetric part μ like $B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab})d\tilde{\gamma}^b$, then σ is determined by the OPEs and μ is also constrained to the equations: $d\mu = \mathrm{tr}(g^{-1}dg)^3$ [wedge products]. Here the operator g (also of the Nekrasov Ansatz) is also determined as the Jacobian matrices by use of the OPEs.
Sketch of proof Some lengthy computations of generalized complex manifolds by use of the Maurer-Cartan equation.

Remark 5 For X surfaces, $d\mu = 0$ can be deduced, but this fact is not used later on. It is because of the toric diagram's techniques of exponents allow us to set (a suitable number of) $\mu_{ij} = -\mu_{ji} = c_{ij}$. This fact is implemented to the Mathematica code. In other words, three dimensional (or higher) X can be implemented if we deal with the toric diagrams accordingly.

Some sample results [S. 2007-] and remarks.

For smooth (toric) del Pezzo surfaces dP_n (of n-point blow-up(s) of $\mathbb{CP}^2$) and Hirzebruch surfaces F_k, one can compute the 2nd Chern character (in agreement with Riemann-Roch) or Pontryagin class as the obstruction or "anomaly" for $U_0 \to U_1 \to U_2 \to U_0$ to the global existence of holomorphic conformal field theories.
$ch_2(dP_n) = \frac{3(1-n)}{2} (n = 0, 1, 2, \cdots, 6)$ / $ch_2(F_k) = 0$.

Note also that the $n = 0, 1, 2, 3$ cases are straightforward by toric diagrams and $n = 4, 5, 6$ cases are by use of nef toric surfaces by deforming the original dP_n, whose toric date the author borrowed from the paper of Konishi-Minabe. For $n = 1$ blow-up case & Hirzebruch surfaces, the obstruction vanishes and one might expect some global structure. The author does not know how to compute the corresponding conformal blocks, though. The $n = 7, 8, 9$ cases must be interesting. Does Ran spaces [prestacks] / factorization spaces [Paugam] have something to do with configuration spaces?

Appendix: Is this work just a review?

The author hopes some lessons might be learnt from his trials of these two decades.

Physicists must stand on the shoulders of giants and – the author believes – so do mathematicians. It is like the BCOV theory of holomorphic anomalies after quantum Kodaira-Spencer theory.

The author started from the imitation of Eguchi-Yang matrix models / SCFTs (topological twists and topological sigma models of Witten) as well as motivic integration of crepant resolution after Kontsevich theory of mirror symmetry (and Auroux-Katzarkov-Orlov). It included the so-called rational elliptic surfaces (1/2 K3 surfaces and E-string theory in physicists' jargon). Now it is possible to study such "non-toric topological string theory and anomalies" / local mirror symmetry due to Klemm and Kazuhiro Sakai. Explicit computation codes are worth trying.

Some ongoing works: stringy Chern classes [or "stringy Chern characters"] for toric orbifolds by Batyrev?

After MacPherson, Chern classes for smooth varieties can be generalized to "some singular varieties" for a certain constructible function related to their resolution of singularities by use of motivic integration. Such singular varieties in the sense of Batyrev-Schaller were normal projective $\mathbb{Q}$-Gorenstein varieties with at worst log-terminal singularities.

By the author's Mathematica code (whose initial form was suggested by Yosuke Imamura), he computed the Chern characters obtained by the OPE calculation of curved $\beta\gamma$ systems. Experimentally, for $S_7^2, S_6^2, S_5^2, S_4^2, S_3^2, S_2^2, S_1^2$ [toric orbifold del Pezzo surfaces' paper by Akhtar-Coates-Corti-Heuberger-Kasprzyk-Oneto-Petracci-Prince-Tveiten 2015]
$ch_2^{\beta\gamma}(S_n^2) = -\frac{3}{2}, -3, -\frac{3}{2}, 0, \frac{1}{2}, 0, \frac{1}{3}$?. Evaluation of logarithmic derivatives before Plot[] was useful for simplification of exponents.