Dimensions of chiral conformal fields and quasi-coherent sheaves

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Dimensions – old and new

Einstein speculated on the dimensionality of the space (not space-time). [physics] Later on, he invented his theory of special / general relativity of "4-(real) dimensional space-time". [Lorentzian signature (-,+,+,+) or (+,-,-,-)] But physicists know (and most of mathematicians do not know) Einstein at the end of his time dreamt of "unification of forces" with an attempt of Kaluza-Klein theory for "4+1" dimensional space-time plus an inner space of electro-magnetic fields. [His time was not related to electro-weak or strong force of non-abelian gauge theories.]
(I don't know the details, but mathematicians were more advanced in the sense that Cantor [and Dedekind] once argued the [discontinuous] one-to-one correspondence between "I (interval $I \subset \mathbb{R}$)" and "I^2".)

4-manifolds and monopoles [(Donaldson,) Witten 1994-]

In this talk, the target space-time is always "4-dimensional" (defined over real numbers) rather than "5-dimensions" or higher (like physicists' Calabi-Yau 3-folds). [I don't know the explicit relation to the AGT correspondence, though.] The confusion comes from the conventions of algebra, geometry, and physics.

But let us recall other notions of dimensions very briefly. Mass dimension, conformal weight / dimensions [physics], topological / cohomological / Kodaira dimension, dimension of vector space / representations. Dimensions are ubiquitous, today. [Though linear algebra as a theory is said to be (relatively) "free from" dimensions.]

I don't intend to refuse Witten's argument of $(2,0)$-SCFT [super-conformal field theory] on 6-dimension.

However, in his approach to geometric Langlands conjecture, Witten argued (together with gerbe and Hitchin fibration of Higgs bundles) the "6-dimensional origin" of physics. I don't really understand his claim; but whatever interpretation one might utilize, the resultant space-time should be 4-dimensions. My approach to conformal field theories (and gauge theories) is conservative. The reason is because I cannot formulate "dimensional reduction / local Calabi-Yau" of physicists in a coherent manner.

Footnote: old textbooks like Green-Schwarz-Witten and Polchinski claim by ghost-number-anomaly or modularity, the space-time of (super-)string theory must be 26 (or 10) "critical"-dimension; or one extra for 11-dim "M(atrix)"-theory of physicists. Note old papers by Vafa, and Witten claim topological Landau-Ginzburg theory can be non-critical dimensional.

Quantum anomalies and topological invariants

There are several kinds of "anomalies" related to the process of quantization of physical systems. They include gauge anomalies, gravitational anomalies, and mixed anomalies. My trial was on the "mixed anomalies" – but it is not directly related to the traditional mixed anomalies of Green-Schwarz mechanism for 10-dimensional space-time.

It was said the quantum anomaly can be best understood in the path-integral approach of the measure Jacobian of infinite-dimensional functional spaces. It is the so-called **Fujikawa's methods of path-integral derivation** via heat-kernel regularization and Dirac operators. But I don't try to recall the works of Bismut, Beilinson-Manin, and Paycha on the Polyakov measure of string theory.

Spin$^{(c)}$ structures and Seiberg-Witten theory / Donaldson theory

My trial to Dirac's magnetic monopoles is not directly related to the monopoles of supersymmetric theories like Seiberg-Witten theory or electro-magnetic duality of Kapustin-Witten's paper.

I just recall the fact that the first Pontryagin class [which can be caluculated for complex surfaces if we regard the second Chern characters as such] is nothing but the "spin structure's obstruction class" of the space-time.

My work in progress [new: $7, 8$-point blowups]

In 2007, I computed the anomaly 2-forms and obstruction to the global existence of chiral de Rham complex via holomorphic OPE computations.

$$ch_2(dP_n) = \frac{3(1-n)}{2} (n = 0, 1, 2, 3),$$
$$ch_2(F_k) = 0. \text{ [Hirzebruch surfaces]}$$

I some years ago updated this result by utilizing the toric data borrowed from the paper of Konishi-Minabe. This is for the cases $n = 4, 5, 6$ (dP_4, dP_5, dP_6) as the topological deformation of nef toric surfaces.

This month, I updated the above-mentioned result by new toric data borrowed from Coates et al. $n = 7, 8$ (dP_7, dP_8) are now possible and I finished all del Pezzo surfaces (including the data from toric del Pezzo orbifolds and toric log del Pezzo surfaces).

Sketch of what I recently did. [toric data]

By my Mahtematica code, I computed the total anomaly for the del Pezzo surfaces dP_7, dP_8.

1. Orbifold del Pezzo $S_2^2 \cong dP_7$

Toric data: 12-gon **[not a quadrilateral]**

$v =$
$\{\{2,1\},\{1,1\},\{0,1\},\{-1,1\},\{-2,1\},\{-2,0\},\{-2,-1\},\{-1,-1\},$
$\{0,-1\},\{1,-1\},\{2,-1\},\{2,0\}\}$

2. Orbifold del Pezzo $S_1^2 \cong dP_8$

Toric data: 15-gon **[not a triangle]**

$v =$
$\{\{3,2\},\{0,1\},\{-3,0\},\{-6,-1\},\{-5,-1\},\{-4,-1\},\{-3,-1\}$
$,\{-2,-1\},\{-1,-1\},\{0,-1\},\{1,-1\},\{2,-1\},\{3,-1\},\{3,0\},\{3,1\}\}$

The number of vertices, which I once misunderstood, should be carefully redefined when the edges include some lattice points. We have to separate the edges when the angle is 180° degree.

Sketch of what I recently did. [the result]

Up to the proportional constant $-1/2$, it is

1. **[n=7]**

$$\boxed{\begin{array}{l}\mathbf{18} + m[1,2] - m[1,12] + m[2,3] + m[3,4] + m[4,5] + m[5,6] \\ +m[6,7] + m[7,8] + m[8,9] + m[9,10] + m[10,11] + m[11,12]\end{array}}$$

2. **[n=8]**

$$\boxed{\begin{array}{l}\mathbf{21} + m[1,2] - m[1,15] + m[2,3] + m[3,4] + m[4,5] \\ +m[5,6] + m[6,7] + m[7,8] + m[8,9] + m[9,10] + m[10,11] \\ +m[11,12] + m[12,13] + m[13,14] + m[14,15]\end{array}}$$

Since all the $m[i,j]$ (anti-symmetric μ-term as the moduli of the theory) above can be read as 0 from the lattice point analysis of power counting, the result is $3(n-1)$. This summarized result can be deduced from the linear system of divisors and Hirzebruch-Riemann-Roch theorem of surfaces.

The assumptions for the computations

Conformal dimensions of γ, β are $0, 1$. The γ-fields are regarded as the (projective) coordinates of the "target" space-time 4-manifolds (complex [quasi-]projective surfaces) M. And β-fields are regarded as the fields for the direct sum of (holomorphic) tangent bundle and co-tangent bundle. Coordinate transforms preserve the OPEs.

From the "world-sheet" Riemann surfaces Σ to the space-time M, we have the "holomorphic" sigma model, which can be obtained by the topological half-twist **[see e.g. Kapustin]**. I used the Nekrasov Ansatz, $\boxed{\beta_i \to \tilde{\beta}_a = \beta_i g^i_a + B_{ai} \partial \gamma^i,}$ **[coordinate transform]** where $B_a \in \Omega^1_U, g_a \in T_U$ and U : coordinate neighbourhood of M with the Einstein summation notation of tensor calculus. $B_a = \frac{1}{2}(\sigma_{ab} - \mu_{ab}) d\tilde{\gamma}^b, d\mu =$ tr $(g^{-1} dg)^3$**[wedge product]**, g: symmetric tensor regarded as the Jacobian matrices and μ: anti-symmetric tensor.

Conclusion

I propose the claim that we should make a dimension theory related to the (topologial holomorphic) sigma model interpretation of the physical systems, rather than just a dimensional reduction and/or ad-hoc compactifications.

The Dolbeault-like filtration with the cotangent complexes (rather than cotangent bundles) and conformal fimensions of curved $\beta\gamma$-systems should be understood accordingly as a "Legendre transform" between the canonical quantization of chiral Poisson algebras and path-integral quantization for the non-linear factorization spaces. The Verlinde algebra and dimensionality analysis should be supplemented to the book of Beilinson-Drinfeld.

Appendix: My Mathematica code's macro (1/5)

The definition is as follows:

$$
\boxed{
\begin{array}{l}
g0[n\_] := \{D[u[[n]], x], D[u[[n]], y]\}; \\
g[m\_, n\_] := Inverse[g0[m]].g0[n]; \\
dgdg[m\_, k\_, n\_] := \\
\quad D[g[m, k], x].D[g[k, n], y] - D[g[m, k], y].D[g[k, n], x]; \\
anom[m\_, k\_, n\_] := xyTr[g[m, k].dgdg[k, n, m]]; \\
tot[i\_, j\_, k\_] := -anom[i, j, k] + m[i, j] - m[i, k] + m[j, k];
\end{array}
}
$$

TotalAnomaly[v_] := Module[{NN, i}, {NN = Length[v];
Print[NN, " Vertices"];
v2 = v; v2 = Append[v2, v[[1]]];
Print[v2]; Print[Length[v2]];
u = {};
SumOfAnomalies = 0;

My Mathematica code's macro (2/5)

(Continued) For[i = 1, i <= NN, i++,
ans1general =
Solve[{v2[[i]].{w11x, w11y} == 1, v2[[i]].{w12x, w12y} == 0,
v2[[i + 1]].{w11x, w11y} == 0,
v2[[i + 1]].{w12x, w12y} == 1}, {w11x, w11y, w12x, w12y}];
If[i == 1,
u = ({{x^w11x y^w11y, x^w12x y^w12y}} /. ans1general[[1]]);
Print[u[[i]]];,
u = Append[
u, ({x^w11x y^w11y, x^w12x y^w12y} /. ans1general[[1]])]];
Echo[u[[i]]];]};
For[i = 2, i + 1 <= NN, i++, SumOfAnomalies += tot[1, i, i +
1]];
Print[SumOfAnomalies];]

My Mathematica code's macro (3/5)

```
FigureGeneration[NN_, Title_] :=
For[i = 1, i <= NN, i++,  Coordinate =
Solve[{p == u[[i]][[1]], q == u[[i]][[2]]}, {x, y}][[1]];
Regular = (x^L y
^ M (D[x /. Coordinate, p] D[y /. Coordinate, q] -
D[x /. Coordinate, q] D[y /. Coordinate, p]) /.
Coordinate) // Expand // Simplify;
Region =
RegionPlot[{Evaluate[(p D[Log[Regular], p] // Expand //
Simplify) >=
0 && (q D[Log[Regular], q] // Expand // Simplify) >=
0]}, {L, -7, 7}, {M, -7, 7}, Axes -> True,
AxesLabel -> {L, M}];
```

My Mathematica code's macro (4/5)

```
For[j = i + 1, j <= NN, j++,
Coordinate2 =
Solve[{p == u[[j]][[1]], q == u[[j]][[2]]}, {x, y}][[1]];
Regular2 = (x^L y
^ M (D[x /. Coordinate2, p] D[y /. Coordinate2, q] -
D[x /. Coordinate2, q] D[y /. Coordinate2, p]) /.
Coordinate2) // Expand // Simplify;
Region2 =
RegionPlot[{Evaluate[(p D[Log[Regular2], p] // Expand //
Simplify) >=
0 && (q D[Log[Regular2], q] // Expand // Simplify) >=
0]}, {L, -7, 7}, {M, -7, 7}, Axes -> True,
AxesLabel -> {L, M}];
```

My Mathematica code's macro (5/5)

```
Figure =
Show[RegionPlot[{L2 < 0}, {L, -7, 7}, {M, -7, 7}, Axes -> True,
AxesLabel -> {L, M},
FrameLabel -> {ToString[i] <> " and " <> ToString[j]}],
Graphics[{Green, PointSize[0.03], Point[{{-1, -1}}]}], Region,
Region2];
Export[
Title <>
ToString[i] <> "and" <> ToString[j] <> ".pdf", Figure];];]
```