Chiral homology and cobordism hypothesis of conformal blocks

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Duality between chiral algebras and factorization algebras

Duality beween left $\mathcal{D}$-modules and right $\mathcal{D}$-modules (chiral algebras of holomorphic vertex algebras) with additional structures. By tensoring the dualizing sheaf ω_X, the left and the right are dual. The left $\mathcal{D}$-modules (factorization algebras) are living in the Ziv Ran space $R(X)$ [finite non-empy subset of X as the element and its total as ind-scheme] of proper and connected curve X over $\mathbb{C}$.

Chiral homology versus factorization homology (1/2)

Beilinson-Drinfeld's theory: [citation] According to 3.4.10.5, the right D-module $\triangle_*^{(S)} A$ on X^S [S the category of finite non-empty sets and surjective maps] is a Lie algebra in the tensor category $M(X^S)^{ch}$. We define the Chevalley-Cousin complex $C(A)$ of A as the reduced Chevalley complex of this Lie algebra. [For a finite non-empty set Is denote by $Q(I)$ the set of all equivalence relations on I. j:embedding, $\triangle$:diagonal]

$$C(A)^{\cdot}_{X^I} = \oplus_{T \in Q(I)} \triangle_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}. \quad (1)$$

See e,g, 1.4.5 and 1.4.10 for the Chevalley complex.

This complex is obviously <u>admissible</u>, so C(A) is a $\mathcal{D}$-complex on the Ran space R(X). If A is a plain chiral algebra (degree 0), then C(A) is a Cousin complex. We define the chiral homology of X with coefficients in A or, simply, the chiral homology of A as the de Rham cohomology of $C(A)$:

$$C^{ch}(X, A) := R\Gamma_{DR}(R(X), C(A)), H_a^{ch}(X, A) := H^{-a} C^{ch}(X, A). \quad (2)$$

Chiral homology versus factorization homology (2/2)

Following the notation from 2,1,12 of [Beilinson-Drinfeld] [Maximal constant quotients and de Rham homology], for a plain chiral algebra A we can rewrite it as

$$E^1_{p,q} = H^{DR}_{p+q}(R(X)^{\circ}_{p}, \Lambda^{p}_{ext}A). \tag{3}$$

$$< A >=< A > (X) = H^{ch}_{0}(X, A)(correlator), \tag{4}$$

which are said to be the derived version of conformal blocks. (e.g. 4-point functions.)

On the other hand, the definition of factorization homology relies on the factorization algebras / sheaves and Hochschild (co)homology of associative algebras. [free loop space and differential forms / (Deligne's conjecture and E_2-algebras. Polyvector fields by Schouten bracket: Batalin-Vikovisky algebra) by Hochshild-Kostant-Rosenberg theorem]

Lian-Zuckerman's paper [CMP1993] and topological chiral algebras (revisited)

Antibraket and Batalin-Vilkovisky algebras. Closed string field theory of Zwiebach (bosonic string theory) is referred in the bibliography. For a field ϕ and its antifield ϕ^*, the antibracket $(,)$ with ghost number $+1$is

$$(A, B) \quad := \quad \frac{\delta^R A}{\delta \phi^i}\frac{\delta^L B}{\delta \phi_i^*} - \frac{\delta^R A}{\delta \phi_i^*}\frac{\delta^L B}{\delta \phi^i}. \tag{5}$$

With the Koszul-Tate differential δ and the longitudial exterior derivative d,

$$s = \delta + d + \cdots, s^2 = 0, sA = (A, S), \tag{6}$$

where S is the generator of the BRST symmetry. (Canonical transformation in the antibracket.) L_∞-algebras are related to our consideration. Let us reconsider (topological) chiral algebra theory of Beilinson-Drinfeld, in the following slides.

Review of OPE derivation of top Chern character [S.2007]

After Goubounov-Malikov-Schechtman, gerbe terms are related to characteristic classes. Its covariant refinement by Nekrasov, Beilinson-Drinfeld, and 3-dimensional generalization of 3-gerbe H-field (not B-field) $U_\alpha \to U_\beta \to U_\gamma \to U_\delta \to U_\alpha$ and its transition function g_{pq} [experimental by $\mathbb{CP}^3$: 3rd Chern character?] by myself. The OPE computations of string theory by Weyl algebras are not quasi-coherent sheaf computations of $\mathcal{D}$-modules on varieties (Spec of O_X-modules). OPE products are associative and commutative. [Beilinson-Drinfeld] We had the 2nd Chern character formula (related to Beilinson-Deligne cohomology (? univeral Chern class [Kapranov-Vasserot2]) and monopoles of Brylinski) like

$$ch_2(\mathbb{CP}^2 \bigcup \cup_{i=1}^{n} \mathbb{CP}^1) = \frac{3(n-1)}{2} \; (n = 0, 1, 2, 3). \tag{7}$$

Do cobordism, spin structures, and Kähler manifold structures exist?

It is said that the existence of cobordism is equivalent to Chern classes' existence. Spin structure and Kähler manifold structure exist for $\mathbb{CP}^1$. What about the case of 4-manifolds? $\mathbb{CP}^1 \times \mathbb{CP}^1$ is OK. The signature number (and Rohlin-like argument) says the connected sum $\mathbb{CP}^2 \sharp \mathbb{CP}^2$ seems to a candiate. The Pontryagin class or Chern-Simons 3-form seems to be the condition for the existence of Witten genus ($2k$-modular form). Del Pezzo surfaces cannot kill such a term. [Matsushima's obstruction also says we cannot include such a case in the Kähler-Einstein manifolds.] $\mathbb{CP}^1 \times \mathbb{CP}^1$ is the simple and nice example of a good Hirzebruch surface.

G.Segal before Atiyah: Definition of CFT and cobordism.

Much before the Baez-Dolan and Jacob Lurie's (extended) TFTs, we have the modular category of Segal. We would like to identify the chiral algebra theory as a kind of rational conformal field theory of finite-dimensioinal chiral primary fields (with their moduli theory).

We respect the work of Ayala-Francis 2017, but what we deal with is the chiral homology rather than factorization homology (aka topological chiral homology). Higher algebras, higher categories, and higher topos theory. There is a simplicial decomposition of such a manifold. For Segal's sense of pointed Riemann surfaces, we have identified the incoming strings and outgoing strings as the insertion of vertex algebras. The chiral homology was defined for de Rham functor h, DR up to homotopy, so we would like to think of its relation with homotopy algebras of open-closed string field theory.

Eguchi-Sugawara's textbook 2015 of conformal field theory and Mathieu moonshine phenomena [Physics]

Rational conformal field theory $A \otimes \bar{A}$ of a kind of "chiral algebra" A includes Wess-Zumino-Witten model and coset model. But we do not know whether this chiral algebra can be regarded as Beilinson-Drinfeld chiral algebras. It has a good review by Moore-Seiberg. It seems that the factorization sheaves and quantum groups by Bezrukavnikov-Finkelberg-Schechtman is written after the geometric understanding of Ziv Ran space and conformal blocks.

The pants decomposition of string worldsheets exists and we have something like the Verlinde formula for the conformal blocks (4-point correlation functions $< \phi_1 \phi_2 \phi_3 \phi_4 >$). The topology of pants diagram is like the coproduct of Hopf algebras.

Well, today we do not talk about the Verlinde formula and elliptic genus from twisted chiral de Rham complex. Let us skip the detail.

Lurie's conjecture and Moore-Seiberg's braids & Tannakian arguments

The chiral algebra story includes Francis-Gaitsgory's generalization and some of related story of geometric Langlands conjecture. We would like to think of Gaitsgory's article 2008 (Twisted Whittaker model and factorizable sheaves) including Lurie's conjecture for "Whittaker sheaf".

$$Whit^c(Gr_G) \quad \cong \quad Rep(U_q(\check{G})), q = exp(\pi ic) \tag{8}$$

Although there is a folklore identification of Whittaker functions as the origin of S-duality conjeture, I do hesitate to state something about it, today. I only looked up some related topological objects including Nahm monopoles and instantons. But, I don't read Braverman-Dobrovolska-Finkelberg on the superpotential of Gaiotto-Witten, yet.

Factorization algebras of perturvative quantum field theory by Costello-Gwilliam

The cochain computation of myself was computations in depth, but the work of [CG] is rather more mathematical physics. We would like to understand the factorization of tensor products by "observables". It is like the constructive / algebraic quantum field theory of causal nets of observables. The interacting L_∞-algebras are not written as the volume 2, yet.
It is cosheaves rather than sheaves of vertex algebras. It might be meaningful to look for an interesting relation between Beilinson-Drinfeld theory of chiral / factorization algebras and Costello-Gwilliam's theory of factorization algebras. The Maurer-Cartan equation is one of the yet-to-be-understood keynotes of such a kind of homological mirror symmetry.