Embedding Hilbert spaces into indefinite inner product spaces based on operator inequalities and its applications

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Introduction

$\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$

Several years ago, in joint work with Wu and Yang,

I found a family $\mathcal{P}_{2,1}$ of triplets of holomorphic functions on $\mathbb{D}^2$:

$(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{P}_{2,1} \Rightarrow T_{\varphi_1} T^*_{\varphi_1} + T_{\varphi_2} T^*_{\varphi_2} - T_{\varphi_3} T^*_{\varphi_3}$ is a projection,

where T_φ denotes a Toeplitz operator on the Hardy space over $\mathbb{D}^2$.

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where T_φ denotes a Toeplitz operator on the Hardy space over $\mathbb{D}^2$.

Trivial example

$T_{z_1} T_{z_1}^* + T_{z_2} T_{z_2}^* - T_{z_1 z_2} T_{z_1 z_2}^* (= I - (I - T_{z_1} T_{z_1}^*)(I - T_{z_2} T_{z_2}^*))$.

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Part 1: operator theory derived from $(*)$

Operator inequality

$$0 \leq T_1 T_1^* + T_2 T_2^* - T_3 T_3^* \leq I$$

is discussed. This is joint work with A. Uchiyama.

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Part 2: application of $(*)$ to complex analysis

Indefinite Schwarz-Pick inequalities are given.

Part 1: operator theory

Setting

$\mathcal{H}$: a Hilbert space,

T_1, T_2, T_3: bounded linear operators on $\mathcal{H}$

We deal with T_1, T_2 and T_3 satisfying

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My question

- How do T_1, T_2 and T_3 relate?
- To approach this problem from my point of view,

  we need de Branges-Rovnyak theory.

A quick review of de Branges-Rovnyak theory

- $T : \mathcal{H} \to \mathcal{K}$ any bounded linear operator,
- $\langle Tx, Ty \rangle_T := \langle Px, Py \rangle_{\mathcal{H}}$ $(P := P_{(\ker T)^\perp})$,
- $\mathcal{M}(T) := (T\mathcal{H}, \langle \cdot, \cdot \rangle_T)$ is a Hilbert space

  $(\because T\mathcal{H} \cong \mathcal{H}/\ker T \cong (\ker T)^\perp)$.

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Fundamental Theorem

Set

$$\mathbb{T} : \mathcal{M}(T_1) \oplus \mathcal{M}(T_2) \to \mathcal{H}, \quad (T_1x_1, T_2x_2) \mapsto T_1x_1 + T_2x_2.$$

Then

$$\mathcal{M}(T_1) + \mathcal{M}(T_2) := \mathcal{M}(\mathbb{T}) = \mathcal{M}(\sqrt{T_1T_1^* + T_2T_2^*})$$

(See Ando's lecture notes).

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Setting

Suppose

$$0 \leq T_1 T_1^* + T_2 T_2^* - T_3 T_3^* \leq I.$$

We set

$$T = \sqrt{T_1 T_1^* + T_2 T_2^* - T_3 T_3^*}.$$

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Observation

By the fundamental theorem of de Branges-Rovnyak theory,

$$\mathcal{M}(T_1) + \mathcal{M}(T_2) = \mathcal{M}(\sqrt{T_1 T_1^* + T_2 T_2^*}) = \mathcal{M}(T) + \mathcal{M}(T_3).$$

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Problem

$$\mathcal{M}(T) = \mathcal{M}(T_1) + \mathcal{M}(T_2) - \mathcal{M}(T_3) \ ?$$

Indefinite range inclusion

Theorem 1 (S-Uchiyama)

Suppose

$$0 \leq T_1 T_1^* + T_2 T_2^* - T_3 T_3^* \leq I.$$

If $u \in \mathcal{M}(T)$, then,

$$\forall \varepsilon > 0 \quad \exists \mathbf{z}_\varepsilon = (z_1(\varepsilon), z_2(\varepsilon), z_3(\varepsilon)) \in \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \quad \text{s.t.}$$

(i) $T_1 z_1(\varepsilon) + T_2 z_2(\varepsilon) - T_3 z_3(\varepsilon) \to u \quad (\varepsilon \to 0)$

(ii) $0 \leq \|z_1(\varepsilon)\|_{\mathcal{H}}^2 + \|z_2(\varepsilon)\|_{\mathcal{H}}^2 - \|z_3(\varepsilon)\|_{\mathcal{H}}^2 \uparrow \|u\|_{\mathcal{M}(T)}^2 \quad (\varepsilon \downarrow 0).$

- We may write $\mathcal{M}(T) \hookrightarrow \mathcal{M}(T_1) + \mathcal{M}(T_2) - \mathcal{M}(T_3)$ (?).

A hidden Kreĭn space

For $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, we set

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{K}} := \langle x_1, y_1 \rangle_{\mathcal{H}} + \langle x_2, y_2 \rangle_{\mathcal{H}} - \langle x_3, y_3 \rangle_{\mathcal{H}},$$

and

$$\mathbb{T} : \mathcal{K} \to \mathcal{H}, \quad (x_1, x_2, x_3) \mapsto T_1 x_1 + T_2 x_2 - T_3 x_3.$$

Then

$$\langle Tx, Tx \rangle_{\mathcal{H}} = \| T_1^* x \|_{\mathcal{H}}^2 + \| T_2^* x \|_{\mathcal{H}}^2 - \| T_3^* x \|_{\mathcal{H}}^2 = \langle \mathbb{T}^\sharp x, \mathbb{T}^\sharp x \rangle_{\mathcal{K}}.$$

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathbb{T}^\sharp} & \mathcal{K}_0 \\ {\scriptstyle T}\downarrow & \nearrow {\scriptstyle \widetilde{V}} & \\ \mathcal{L} & & \end{array} \qquad (\mathcal{L} := (\ker T)^\perp)$$

where $\mathcal{K}_0$ is the complition of $\mathbb{T}^\sharp \mathcal{L}$ w.r.t. the norm $\langle \mathbb{T}^\sharp x, \mathbb{T}^\sharp x \rangle_{\mathcal{K}}$.

Part 2: application to complex analysis

Schur class

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Schwarz lemma

$$f \in \mathcal{S}(\mathbb{D}) \text{ and } f(0) = 0 \Rightarrow |f(z)| \leq |z| \quad (z \in \mathbb{D}).$$

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Schwarz lemma

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Schwarz-Pick inequality

$$f \in \mathcal{S}(\mathbb{D}) \Rightarrow \left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \leq \left| \frac{z - w}{1 - \overline{w} z} \right| \quad (z, w \in \mathbb{D}).$$

- $d(z, w) = \left| \dfrac{z - w}{1 - \overline{w} z} \right|$ is a distance on $\mathbb{D}$.

Hilbert space proof of Schwarz lemma

1. Suppose $f \in \mathcal{S}(\mathbb{D})$ and $f(0) = 0$.
2. Then $T_f H^2(\mathbb{D}) \subset T_z H^2(\mathbb{D})$.
3. Hence $T_f T_f^* \leq T_z T_z^*$ by Douglas range inclusion theorem.
4. Therefore

$$\frac{|f(\lambda)|^2}{1 - |\lambda|^2} = \langle T_f T_f^* k_\lambda, k_\lambda \rangle \leq \langle T_z T_z^* k_\lambda, k_\lambda \rangle = \frac{|\lambda|^2}{1 - |\lambda|^2}.$$

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Remark

- This proof can not be used in undergraduate complex analysis.
- However, it can be applied to several variables.

Indefnite Schwarz lemma

Theorem 2

$\psi = (\psi_1, \psi_2)$: $\mathbb{D}^2 \to \mathbb{D}^2$ holomorphic.

If $\psi(0,0) = (0,0)$, then

$$0 \leq |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \leq 2(|z_1|^2 + |z_2|^2 - |z_1 z_2|^2)$$

$$(\forall z = (z_1, z_2) \text{ in } \mathbb{D}^2).$$

Further, if

$$0 \leq T_{\psi_1} T^*_{\psi_1} + T_{\psi_2} T^*_{\psi_2} - T_{\psi_1\psi_2} T^*_{\psi_1\psi_2} \leq I$$

then

$$0 \leq |\psi_1(z)|^2 + |\psi_2(z)|^2 - |\psi_1(z)\psi_2(z)|^2 \leq |z_1|^2 + |z_2|^2 - |z_1 z_2|^2.$$

$|z_1|^2 + |z_2|^2 - |z_1 z_2|^2$?

For $z = (z_1, z_2, z_3),\ w = (w_1, w_2, w_3) \in \mathbb{C}^3$, we set

$$\langle z, w\rangle_{\mathcal{K}} := z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}.$$

$\mathcal{K}$: the Kreĭn space $(\mathbb{C}^3, \langle\cdot,\cdot\rangle_{\mathcal{K}})$.

Set

$$\Phi : \mathbb{D}^2 \to \mathcal{K}, \quad (z_1, z_2) \mapsto (z_1, z_2, z_1 z_2)$$

and

$$\begin{aligned}\Omega &= \{(z_1, z_2) \in \mathbb{C}^2 : 0 \le |z_1|^2 + |z_2|^2 - |z_1 z_2|^2 < 1\} \\ &= \{(z_1, z_2) \in \mathbb{C}^2 : 0 \le \langle\Phi(z), \Phi(z)\rangle_{\mathcal{K}} < 1\}.\end{aligned}$$

Then, $\mathbb{D}^2$ is the bounded connected open subset of Ω.

Indefinite Schwarz-Pick inequality

For $z = (z_1, z_2)$, $w = (w_1, w_2) \in \mathbb{D}^2$, we set

$$d(z, w) = \sqrt{\left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1}\right|^2 + \left|\frac{z_2 - w_2}{1 - \overline{w_2} z_2}\right|^2 - \left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1} \cdot \frac{z_2 - w_2}{1 - \overline{w_2} z_2}\right|^2}.$$

Indefinite Schwarz-Pick inequality

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Theorem 3

$\psi = (\psi_1, \psi_2)$: $\mathbb{D}^2 \to \mathbb{D}^2$ holomorphic.

Then

$$d(\psi(z), \psi(w)) \leq \sqrt{2} d(z,w) \quad (z, w \in \mathbb{D}^2).$$

Further, if

$$0 \leq T_{\psi_1} T_{\psi_1}^* + T_{\psi_2} T_{\psi_2}^* - T_{\psi_1\psi_2} T_{\psi_1\psi_2}^* \leq I$$

then

$$d(\psi(z), \psi(w)) \leq d(z,w).$$